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Simulation studies of tip tracking control of a single-link flexible robot based on a lumped model

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SUMMARY
In this paper, tip tracking control of a single-link flexible robot is studied through computer simulations. The controller investigated is of a two-loop PD type, which is deduced from a control approach originally developed for elastic joint robots. A very simple model, with the flexible link of the robot being lumped to a spring-mass unit, is employed for the controller design. Bounded Input and Bounded Output (BIBO) stable tip tracking performance is guaranteed. The controller is robustly stable under system parameter uncertainties, and is comparatively easy to implement. In the simulations, different payloads are considered, and the system performance is compared with the traditional Singular Perturbation (SP) method. The simulation results show that the PD type controller designed based on the simple lumped model is able to provide quite satisfactory performance.

KEYWORDS: Flexible link robot; Lumping; Tracking control.

1. INTRODUCTION

Conventional rigid-link robots are widely used in industrial production. Inevitably, the links of the robots are usually built for rigidity, with commensurate weight penalties. As an undesirable result, large joint drives are necessary to drive such robots, however, only at a comparatively low speed. These drawbacks greatly limit the application of these robots in the fields where high speed, high accuracy and low energy support are required.

A flexible robot, whose links are made of some light-weight materials, and are considerably more rigid in compression than in flexure, is free of these drawbacks. The light-weight flexible-link robot needs lower power consumption for high speed operations, which makes it more feasible than the conventional rigid-link robots in many applications, such as space applications. Motivated by the advantages, much research work has been carried out on modelling and control of flexible robots in the literature.[1–16]

A price we must pay for the advantages is that the dynamic models of flexible-link robots are much more complicated than that of the rigid-link ones. The original dynamics of a flexible-link robot, being described by partial differential equations and thus possessing an infinite dimensionality, are not easy to be directly used in both system analysis and control design. Most commonly, the dynamic equations are truncated to some finite dimensional models with either the Assumed Modes Method (AMM)[2] or the Finite Element Method (FEM).[1] Much work on control design based on AMM/FEM models have been reported in the literature, both for single-link flexible robots[2–11] and multi-link ones.[12–15] A tutorial on modelling and control of flexible robots can be found in reference 7.

The control objective of a flexible robot system is often set as the point-to-point position control or regulation, in which the main task is to suppress the residue vibrations. Many approaches have been reported in the literature for controlling residue vibrations. Among them, an important approach is the Input Shaping Method,[11] in which the control input is pre-shaped such that the excitation of the vibrations on certain frequencies are reduced to a minimum. Although it is true that the point-to-point position control is the only requirement in many applications, such as the automatic manufacturing assembly, it is more desirable to be able to drive the robot along a pre-defined trajectory. This requirement leads to the problem of tip tracking control design. Tip tracking control of manipulators with flexible links is difficult, because of the non-minimum phase property of such systems due to the finite speed of propagation of the mechanical wave along the links.[17] Generally, the tracking problem of flexible robots is solved by transferring the problem into two sub-problems: (i) tracking of the joint motion (the dynamic behaviour of the system is of minimum phase, if the output is the joint angular position and the input is the torque at the joint), and (ii) suppression of the elastic vibrations of the flexible links. Such a consideration directly leads to the Singular Perturbation (SP) method.[3–12] This method is able to decouple the rigid motion and the flexible vibrations of the robot, and generate a composite control law. In particular, as shown in references 3–12, the system dynamics are firstly decoupled into a fast sub-system, and a slow sub-system. The fast states are selected to be the elastic forces and their time derivatives, and the slow states to be that of the equivalent rigid motion. Then, a fast control is designed to stabilize the fast sub-system around the equilibrium trajectory defined by the slow sub-system under the effect of the slow control. As stated in reference 3, the controllers designed by SP approach are based on the state-space models and the full states measurements/estimates are required. This leads to

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high computational cost and is undesirable for engineering implementation. At the same time, the SP method requires that the robot link cannot be too flexible in order to keep the perturbation parameter sufficiently small. Another approach to achieve tip trajectory tracking is the inverse dynamics (computed torque) method, which has been used in control of rigid-link manipulators and is also investigated for the flexible-link case.[4,5] This method seems to be able to result in better tip motions over other techniques. However, the successful application of this method depends heavily on highly accurate models and efficient computational algorithms/powerful computing facilities.

From the engineering point of view, the dynamic model should be (i) accurate enough to capture the dominant dynamical behaviour, while (ii) simple enough for controller design, which will lead to easy implementation and tuning. Davis and Hirschorn[6] have formulated an appropriate hybrid lumped/distributed model of a flexible robot to achieve tracking control using acceleration feedback. Their approach can overcome the difficulties arising from the truncation procedure in modelling an actually distributed system, however, the design procedure is quite complicated. Moreover, the approach requires the undesirable acceleration measurement, which is usually very noisy, and it is difficult to obtain accurate measurement by accelerometer feedback. In this paper, we achieve the two objectives stated above by lumping the distributed-parameter flexible link into a simple spring-mass unit. Based on the lumped model, a very simple two-loop PD type controller is applied here to achieve BIBO stable tip tracking performance. This PD controller is applied here to achieve BIBO stable tip tracking performance. This PD controller was originally designed for a single-link flexible joint robot in reference 18. The controller is robustly stable under system parameter uncertainties, and is comparatively easy for engineering implementation since only position and velocity signals are needed for feedback (tip sensing may be required to obtain tip deflection and its velocity). Moreover, the controller requires only four feedback variables (the position and velocity signals at the hub and the tip), and thus the computational burden is low. Extensive computer simulations are presented to test the proposed tracking controller under different unknown payloads, and the results are compared with a traditional SP controller. It is concluded that the simple PD type tracking controller is able to provide quite satisfactory performance.

The rest of the paper is organized as follows: the lumped dynamic model of a single-link flexible robot is presented in Section 2, the design of the PD tracking controller is discussed in Section 3, followed by simulation results in Section 4 and conclusions in Section 5.

2. LUMPED DYNAMIC MODEL

This section considers lumping the flexible robot into a simple spring-mass system. Lumping is a commonly used method which can reasonably simplify the dynamic model of flexible robots. In reference 15, an adaptive tip position controller was presented for a multi-link flexible robot based on a lumped model, in which each flexible link is lumped into two rigid sub-links connected by a fictitious joint. The lumping process which we shall introduce in this paper is different from that in [16] but similar to that in reference 20. The method introduced here can be described by (i) lumping the distributed mass of the flexible beam to a point mass located at its tip, and (ii) representing the flexibility of the beam by a weightless linear "bending" spring.

Figure 1(a) represents the original flexible manipulator, which is constrained at the rotor of the motor and moves in the horizontal plane, while Figure 1(b) is the lumped spring-mass system. In Figure 1, *XOY* is the fixed base frame and *xOy* is the local reference frame rotating with the hub. System parameters and variables are defined as: L, the length of the flexible beam: EI., the uniform flexural rigidity of the beam; M_t, the concentrated mass tip payload; M_e, the equivalent lumped mass of the beam located at the tip; k, the equivalent bending spring stiffness; ρ, the uniform weight per unit length of the beam; I_h, the hub inertia; τ, the control torque; θ, the rotation angle of the hub; y_t, tip deflection measured from the undeformed beam, i.e., $y_t = y(L, t)$; p, arc approximation of the tip position, $p = L\theta + y_t$.

Firstly, the distributed mass of the flexible link is lumped to an equivalent concentrated mass M_e, at the tip. In doing this, we shall assume that the flexible beam undergoes no rigid motion, i.e., the motor is locked such that $\dot{\theta} \equiv 0$. This assumption eliminates the effect of I_h in the lumping

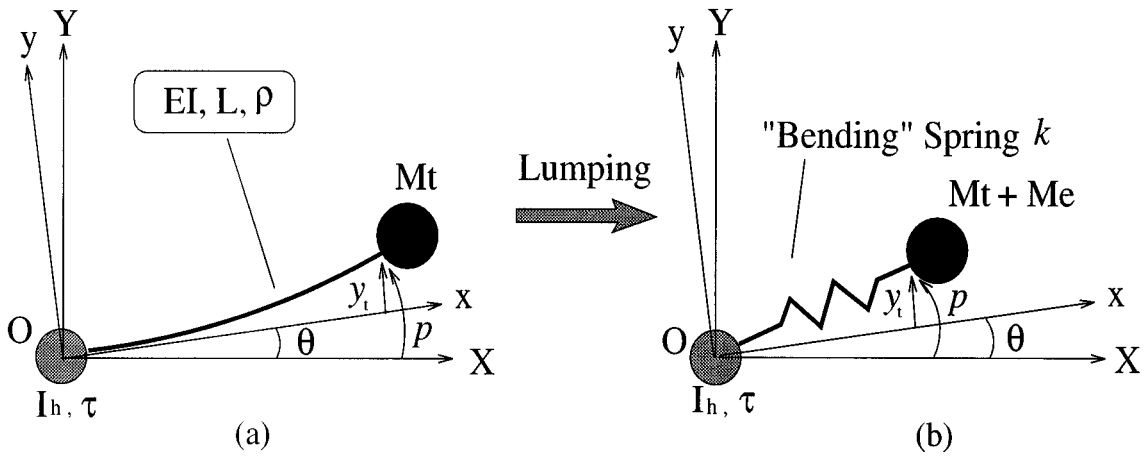


Fig. 1. Lumping of a Flexible Beam.

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procedure, but the advantage is that the modes shape functions of the vibrating beam, and subsequently the equivalent mass M_e and spring stiffness k can be analytically determined, by using the Constrained Assumed Modes method,[7] as will be shown later. If rigid motion is permitted, the flexible modes (Unconstrained Assumed Modes)[7] can be identified experimentally by sinusoidally exciting the joint motors over a range of frequencies.[8] However, the analytical form of the modes shape functions and hence M_e and k are difficult to obtain. Most important, it has been shown in reference 9 that the natural frequencies calculated by the Constrained Assumed Modes method are very close to the experimental values. Thus, the lumped model obtained from the Constrained Assumed Modes in this paper is expected to give a good approximation of the original distributed-parameter system.

In the Constrained Assumed Modes method, the deflection of the beam is represented by:

$$y(x, t) = \sum_{i=1}^{\infty} F_i(x) q_i(t),\ 0 \le x \le L$$

where $q_i(t)$ are the assumed flexible modes, and $F_i(x)$ are the corresponding time-independent modes shape functions. It should be pointed out that different types of boundary conditions can be used to derive the modes shape functions, and the choice of proper boundary conditions depends on the configuration of the practical flexible robot system.[16] Suppose the clamped-free boundary conditions are used here, which are given in the following[2]

$$F_i(x) = \frac{1}{A_i}\left[\cosh\left(\frac{\beta_i}{L}x\right) - \cos\left(\frac{\beta_i}{L}x\right) - \gamma_i\left(\sinh\left(\frac{\beta_i}{L}x\right) - \sin\left(\frac{\beta_i}{L}x\right)\right)\right] \quad (1)$$

where A_i are constants, and

$$\gamma_i = \frac{\cosh(\beta_i) + \cos(\beta_i)}{\sinh(\beta_i) + \sin(\beta_i)}$$

with β_i being positive solutions of the following equation

$$1 + \cosh(\beta_i)\cos(\beta_i) + \frac{M_t\beta_i}{\rho L}[\sinh(\beta_i)\cos(\beta_i) - \cosh(\beta_i)\sin(\beta_i)] = 0 \quad (2)$$

It is known that the first mode $q_1(t)$ (with the lowest natural frequency) is most dominant in describing the vibration of the beam. Normally, lumping of a mechanical system is carried out under the principle that the lumped system should be able to catch the dynamic behavior of the original system on the most dominant natural frequency.[19] We therefore approximate $y(x, t)$ by

$$y(x, t) = F(x)q(t) \quad (3)$$

where $q(t) = q_1(t)$ and $F(x) = F_1(x)$. This approximation of $y(x, t)$ will lead to a lumped model whose natural frequency is very close to the most dominant frequency calculated from the Constrained Assumed Modes method. Further, as stated in reference 9, the calculated frequency is a very good approximation of the experimental (true) value. Therefore, the lumped model is used to approximate the dominant dynamic behavior of the original flexible robot system.

The equivalent mass M_e is, under the assumption of no rigid motion, obtained by equating the kinetic energy of M_e and that of the original flexible vibrating beam, i.e.,

$$\frac{1}{2}M_e y_t^2 = \frac{\rho}{2}\int_0^L y^2(x, t)dx \quad (4)$$

Substituting (3) into (4) and recalling that $y_t = y(L, t)$, we arrive at

$$M_e = \frac{\rho\int_0^L F^2(x)dx}{F^2(L)}$$

Next, let us deal with the flexibility of the beam. In reference 15, this is done by introducing a fictitious joint between the two rigid pieces of each link. Considering that y_t actually represents the elastic deformation of the beam, we shall introduce an idea of linear "bending" spring. Such a spring is very similar to the flexible beam itself, except that its elastic potential energy is assumed to be

$$E_{p-spring} = ky_t^2/2 \quad (5)$$

with k being the equivalent spring stiffness. By equating the potential energy of the "bending" spring $E_{p-spring}$ and that of the flexible beam, we have

$$\frac{1}{2}EI\int_0^L [y''(x, t)]^2 dx = \frac{1}{2}ky_t^2$$

or

$$\frac{1}{2}EIq^2(t)\int_0^L [F''(x)]^2 dx = \frac{1}{2}kq^2(t)F^2(L)$$

Thus, the equivalent spring stiffness k can be calculated by

$$k = EI\frac{\int_0^L [F''(x)]^2 dx}{F^2(L)}$$

Let $p = L\theta + y_t$ be the approximation of the tip position under small deflection. The total kinetic energy E_k and the total potential energy E_p of the system are given by

$$E_k = \frac{1}{2}I_h\dot{\theta}^2 + \frac{1}{2}(M_t + M_e)p^2 \quad (6)$$

$$E_p = \frac{1}{2}ky_t^2 \quad (7)$$

By using the well-known *Lagrange Equations*, we obtain the following lumped dynamic model

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$$M\ddot{p} = -k(p - L\theta) \quad (8)$$

$$I_h\ddot{\theta} - kL(p - L\theta) = \tau \quad (9)$$

where $M = M_t + M_e$.

The resulting simple model (8)–(9) is used to describe the system's dominant dynamic behaviour and gives some important physical insights into the original distributed-parameter system. The simple form of the model will result in simple controllers for the benefit of easy implementation. It is noted that the model (8)–(9) is very similar to that of a single-link flexible joint robot given in reference 18.

3. CONTROLLER DESIGN

In this section, a PD type controller, which has been successfully applied to flexible joint robots in reference 18, is used to achieve BIBO stable tip tracking control of the flexible link robot based on the lumped model (8)–(9).

In most cases of robot control, we do not know the exact value of the tip payload M_t, except its range $[M_{t-min}, M_{t-max}]$. Let $\hat{M}_t$ be the estimate of the tip payload, and define

$$\hat{M} = \hat{M}_t + M_e \in [M_{min}, M_{max}]$$

where $M_{min} = \hat{M}_{t-min} + M_e$ and $M_{max} = M_{t-max} + M_e$. We further define $\alpha = (\hat{M} - M)/M$ and tip tracking error $e_1 = p_d - p$, where p_d is the desired tip trajectory with its time derivatives up to the 4th order are bounded. From (8), we can obtain the estimated desired joint angle trajectory θ_d

$$\theta_d = \frac{\hat{M}}{kL}\ddot{p}_d + \frac{1}{L}p_d \quad (10)$$

Let $e_2 = \theta_d - \theta$ be the joint angle tracking error, then we can construct a two-loop PD type feedback controller as

$$\tau = \hat{M}L(\ddot{p}_d + K_{1v}\dot{e}_1 + K_{1p}e_1) + I_h(\ddot{\theta}_d + K_{2v}\dot{e}_2 + K_{2p}e_2) \quad (11)$$

The feedback of e_1 and $\dot{e}_1$ corresponds to the outer loop of the tip, and that of e_2 and $\dot{e}_2$ corresponds to the inner loop of the hub. Note that $\ddot{\theta}_d$ and $\dot{\theta}_d$ can be obtained from (10), the controller is therefore easy to implement because only position and velocity measurements at the tip and hub are needed for feedback. The stability of closed-loop system is stated in the following theorem:

Theorem 1: The closed-loop system (8), (9) and (11) is BIBO stable if the following conditions are satisfied:

C1. $K_{1v}, K_{1p}, K_{2v}, K_{2p} > 0$

C2. $K_{2v} \geq (1 + \alpha)K_{1v}$

C3. $K_{2p}/K_{2v} > K_{1p}/K_{1v}$

C4. $1 + \alpha > 0$

which are independent of system parameters.

Proof: See Appendix A.

Remarks:

(i) If $\alpha = 0$, i.e., $\hat{M} = M$, then the RHS of (15) is zero and the error system is asymptotically stable.

(ii) The four conditions **C1–C4** are independent of the system parameters. Although α depends on M, checking of the four conditions does not require the value of M. As a matter of fact, since both $\hat{M}$ and M belong to $[M_{min}, M_{max}]$, we are able to obtain the maximum value and the minimum value of $1 + \alpha$, say M_{max}/M_{min} and M_{min}/M_{max}. Therefore, condition **C2** can be satisfied if we choose $K_{2v} \geq K_{1v}M_{max}/M_{min}$. Moreover, $\hat{M}$, $I_h > 0$ in (11) always hold. Subsequently, the controller is robustly stable under system parameter uncertainties.

(iii) Since $\alpha = (\hat{M} - M)/M = \hat{M}/M - 1$, and a negative estimate payload $\hat{M}$ makes no sense in this case, we always have $1 + \alpha = \hat{M}/M > 0$. Therefore, condition **C4** is always true. Thus, actually only conditions **C1–C3** are necessary to guarantee BIBO stability for the lumped system.

(iv) The controller (11), in which only position and velocity measurements are needed, is easy to tune and implement in practice.

(v) The proposed two-loop PD controller is independent of the equivalent spring stiffness k. Consequently, the controller may be used to handle a robot with very flexible link, unlike the SP method in which the robot link cannot be too flexural in order to guarantee the perturbation parameter to be sufficiently small.

With the help of the lumped model (8)–(9), we obtain a simple PD type controller for the very difficult tip tracking control of the flexible link robot. Due to the simple form and the advantages stated in the Remarks, the PD type tracking controller is quite attractive to engineering applications. However, since the lumped model is only an approximation of the original flexible robot system, the effectiveness of the controller (11) needs to be further verified. This is done in the next section by means of numerical simulations.

4. SIMULATION TESTS

Numerical simulations have been carried out to verify the effectiveness of the controller. A set of simulation results by using the SP method for the single-link flexible robot[3] are also provided for comparison. The reason to choose this method is that it has been compared with several other control approaches in end-effector control of a single-link flexible robot, and it was concluded that it gave the best experimental tracking performance.[10] It should be noted that in SP controller design, only the slow variable (i.e., the joint motion) is controlled to track a pre-defined trajectory, and the fast variables associated with flexible vibration are regulated along the equilibrium trajectory defined by the slow sub-system[3] to increase the tip positioning accuracy.

The controller presented here can be easily implemented as described by the block-diagram in Figure 2.

The plant is simulated with a 4-mode assumed modes model obtained by the modelling procedure presented in reference 2. The desired tip trajectory p_d is generated by the following linear system with zero initial states

$$p_d = \frac{b}{(s + a)^4}$$

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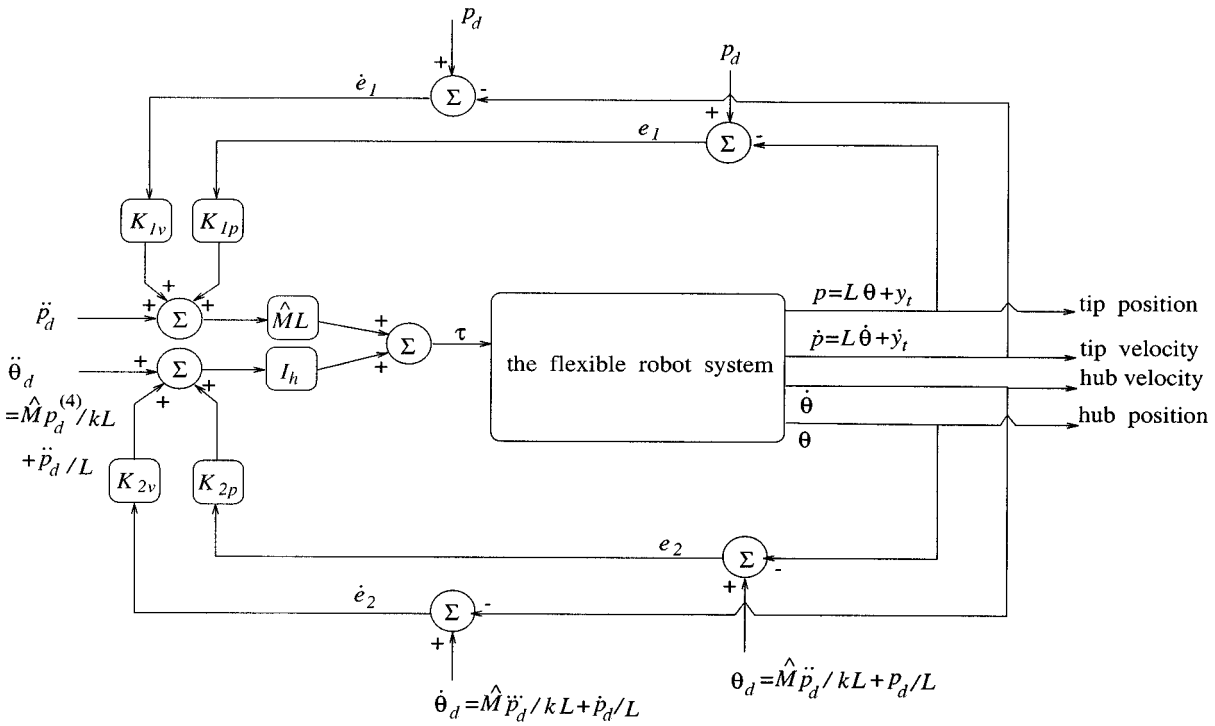


Fig. 2. The system diagram.

with $a = 6.0$ and $b = a^4\pi/2$. The p_d is also used as the desired joint trajectory in SP control. Clearly, up to the 4th-order derivatives of p_d are all bounded, and a larger a gives a faster response and the final position is

$$p_d(\infty) = \frac{b}{a^4} = \pi/2$$

The system parameters are given in Table I.

The singular perturbation controller is constructed according to the procedure in reference 3. The slow variable (i.e. the joint motion θ) is controlled to follow p_d, and the fast control of flexible modes is achieved by LQR method.

Suppose we do not know the exact value of the tip payload except for its changing range: $[M_{t-min} = 0.05, M_{t-max} = 0.15]$ kg with nominal value $M_{t-nom} = 0.1$ kg. To test the system's robustness to payload variations, controller (11) and the SP controller are always constructed using the nominal payload M_{t-nom} (i.e., $\hat{M}_t = M_{t-nom}$ in (11)) for different payload variations. According to the lumping procedure in Section 2, we have $\beta = 1.248$ from (2), the equivalent lumped mass $M_e = 0.0238\ kg$, and $\hat{M} = \hat{M}_t + M_e = 0.1238\ kg$.

Firstly, let the actual payload $M_t = M_{t-nom}$ (i.e., the estimated payload is equal to the true payload, thus $\alpha = 0$ and the closed-loop system is asymptotically stable according to Remark 1), and set control parameters to be $K_{1p} = 0.005$, $K_{2p} = 39.0$, $K_{1v} = 0.01$ and $K_{2v} = 9.05$. It is easy to check that the four conditions are all satisfied and the four poles of error system (15) are located at

$$-2.26 \pm 8.05\text{i},\ -2.26 \pm 4.68\text{i}$$

The tip tracking performance is shown in Figure 3. The tip trajectory $L\theta + y_t$ using SP control is given by the dashed line. In comparison with the performance of PD type controller (11) (solid line), the tip motion of SP exhibits a little larger overshoot and longer settling time.

Then, the above controller (i.e., with the same k_{iv}, k_{ip} $(i = 1, 2)$) and the same nominal-payload SP controller are

Table I. System Parameters.

ρ	L	I_h	EI
0.1 kg/m	1.0 m	0.2 kgm^2	2.0 Nm^2

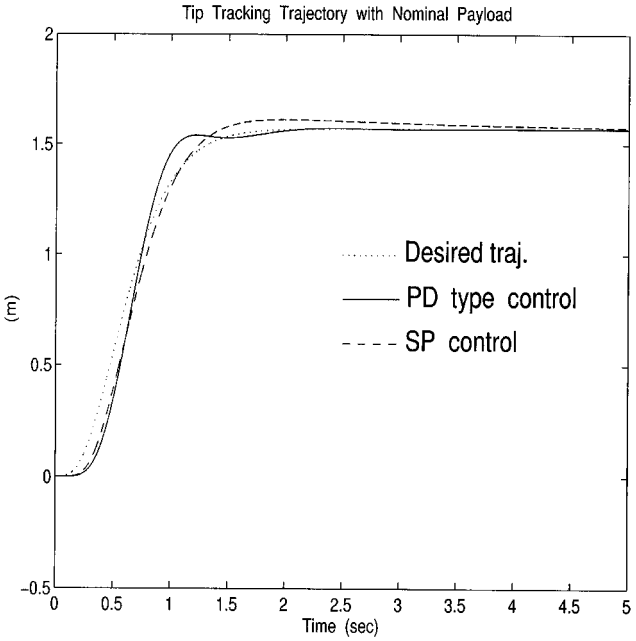


Fig. 3. Nominal payload case: $M_t = M_{t-nom} = 0.1\ kg$.

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tested for the minimum and the maximum payload, i.e. $M_t = M_{t-min} = 0.05$ kg and $M_t = M_{t-max} = 0.15$ kg. Clearly, in these two cases, the closed-loop systems under the PD type controller (11) are BIBO stable instead of asymptotically stable because $\alpha \neq 0$. The tip tracking performances are given in Figures 4–5. It can be seen that both controller (11) and the SP controller show good robustness to payload variations. Controller (11) gives a closer tracking in the minimum payload case, but exhibits a little larger overshoot in the maximum payload case. In both cases, controller (11)

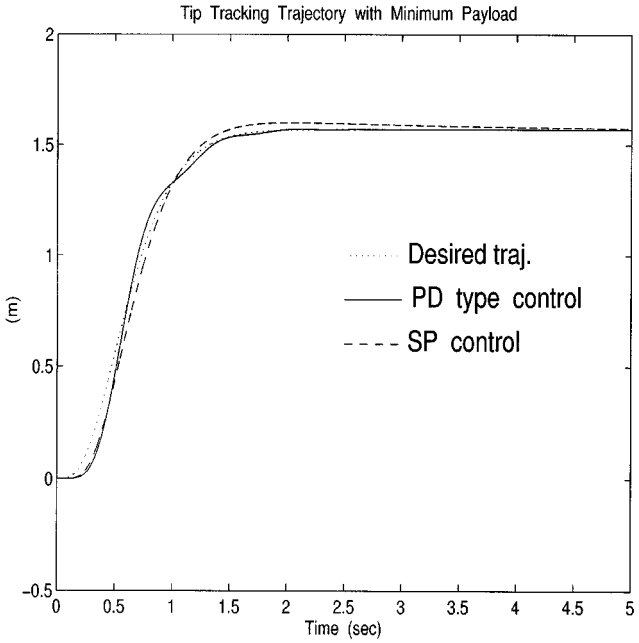


Fig. 4. Minimum payload case: $M_t = M_{t-min} = 0.05\ kg$.

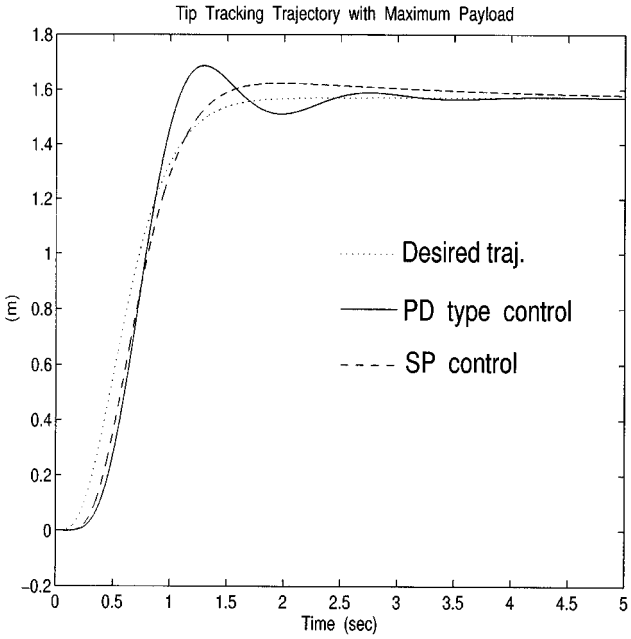


Fig. 5. Maximum payload case: $M_t = M_{t-max} = 0.15\ kg$.

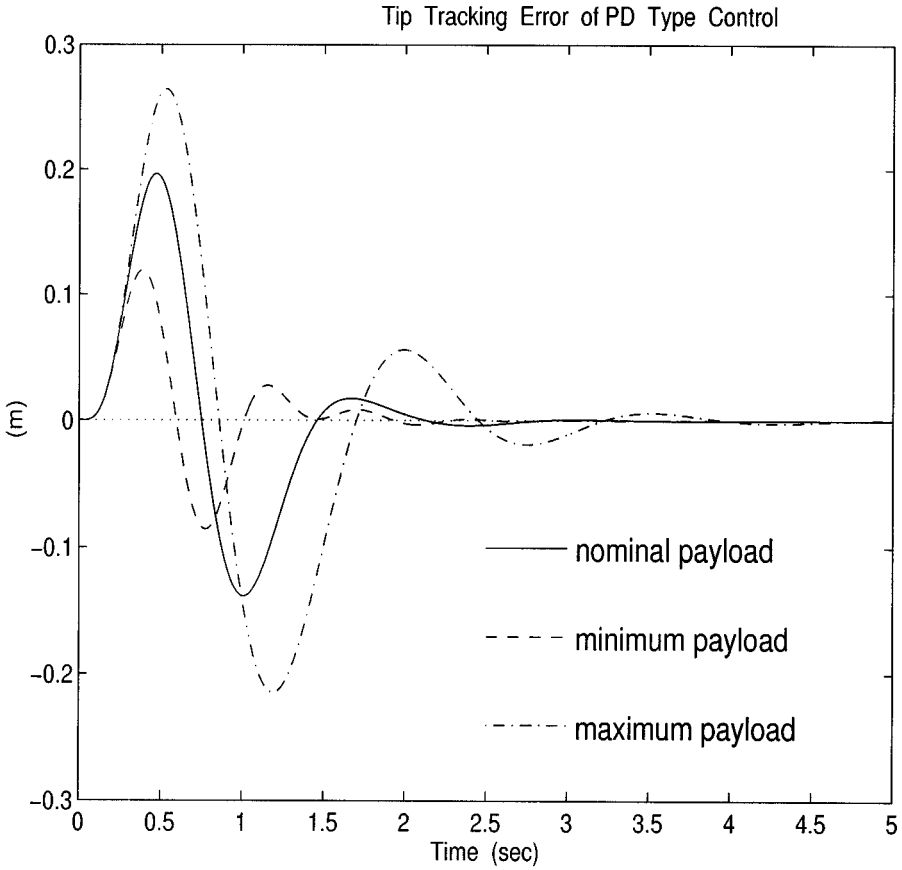


Fig. 6. Tip tracking error $e_1 = p_d - p$ of the PD type controller.

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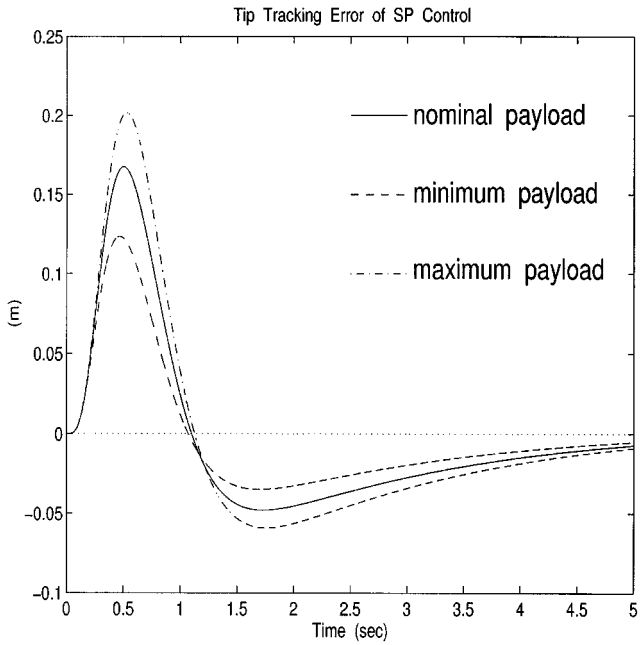


Fig. 7. Tip tracking error $p_d - (L\theta + y_t)$ of SP control.

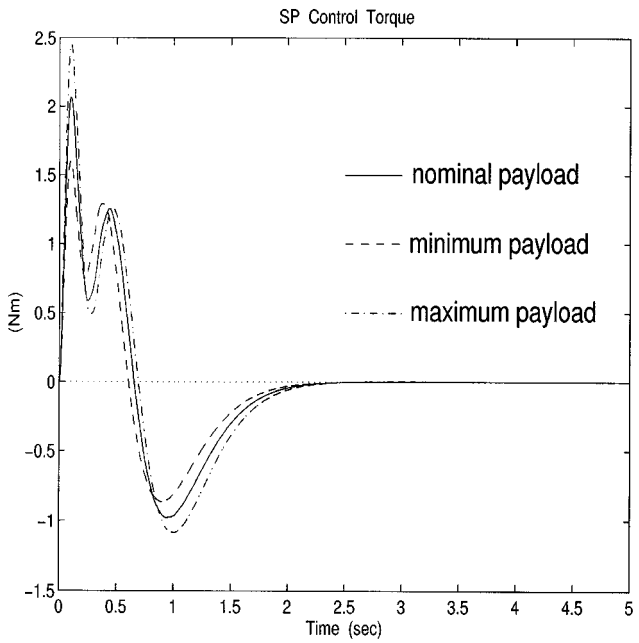


Fig. 9. Control torque of the SP controller.

shows faster convergence of the tip of the final position.

In Figures 3–5, it can be further seen that for both the controller (11) and the SP controller, a smaller payload leads to a better tracking performance. This is clearly shown in Figures 6–7, in which the tip tracking error in the three payload cases are plotted for controller (11) and the SP controller, respectively. The minimum payload case exhibits the smallest tracking error, while the maximum payload case suffers from the largest tracking error. This is expected since a flexible robot with heavier payload is in general more difficult to control. Through intensive computer simulations, it was found that for controller (11), generally $\hat{M}_t > M_t$ gives better results than $\hat{M}_t < M_t$, however, it is not necessarily true in all situations. The control torques are plotted in Figures 8–9 for controller (11) and SP control, respectively. One can see that a larger control effort is needed for a heavier tip payload.

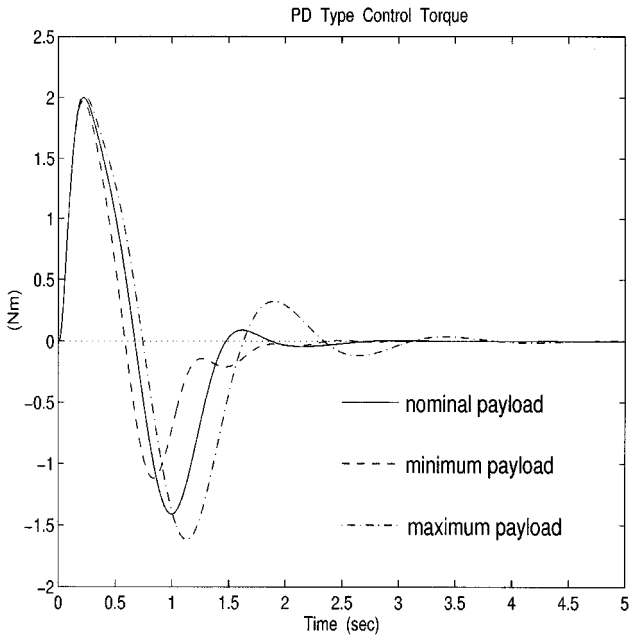


Fig. 8. Control torque of the PD type controller.

The comparisons above show that the system performance using the simple PD type tracking controller (11) is comparable with that of the traditional SP controller. This verifies the effectiveness of the proposed approach. Further, the PD type controller is quite attractive to engineering practice because of its simplicity, good robustness and very low computational cost.

5. CONCLUSIONS

In this paper, a lumped dynamic model of a single-link flexible robot is firstly obtained. Then, a PD type controller with its tuning parameters independent of system parameters is developed to achieve BIBO stable tip tracking control. Only position and velocity measurements are needed to realize it. Simulation results are provided and compared with the traditional SP controller to show the effectiveness of the controller and its good robustness in handling tip payload variations. It should be pointed out that the PD type controller presented in this paper is only applicable to the single-link case.

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APPENDIX A: PROOF OF THEOREM 1

Detailed procedures for obtaining the four conditions **C1**∼**C4** are given in reference 18. Only proof checking is provided here.

Subtracting (8) from (10) and noting $\hat{M}=(1+\alpha)M$, we have

$$M\ddot{e}_1+\alpha M\ddot{p}_d=-k(e_1-Le_2) \tag{12}$$

Taking second-order time derivative on both sides of equation (8), we obtain

$$M_p^{(4)}=-k(\ddot{p}-L\ddot{\theta}) \tag{13}$$

Substituting (9) for $\ddot{\theta}$ into (13) gives

$$Mp^{(4)}=-k\ddot{p}+\frac{kL}{I_h}[\tau+kL(p-L\theta)] \tag{14}$$

Substitution of controller (11) into (14), and using equations (10) and (12), we arrive at the following fourth-order error equation of e_1:

$$\begin{aligned}&I_hMe_1^{(4)}+I_hK_{2v}Me_1^{(3)}+(MkL^2+I_hk+I_hK_{2p}M)\ddot{e}_1+\\&(MkL^2K_{1v}+I_hK_{2v}k+\alpha kML^2K_{1v})\dot{e}_1+(MkL^2K_{1p}+\\&I_hK_{2p}k+\alpha kML^2K_{1p})e_1\\=&-\alpha M[I_hp_d^{(4)}+I_hK_{2v}p_d^{(3)}+(I_hK_{2p}+kL^2)\ddot{p}_d]\end{aligned} \tag{15}$$

in which the RHS can be considered as an input to the error system. The Routh table is as follows:

s^4	I_hM	$MkL^2+I_hk+I_hK_{2p}M$	$MkL^2K_{1p}+I_hK_{2p}k+\alpha kML^2K_{1p}$
s^3	$I_hK_{2v}M$	$MkL^2K_{1v}+I_hK_{2v}k+\alpha kML^2K_{1v}$	0
s^2	η_1	$MkL^2K_{1p}+I_hK_{2p}k+\alpha kML^2K_{1p}$	
s^1	η_2	0	
s^0	η_3		

where

$$\begin{aligned}\eta_1&=MkL^2[1-(\alpha+1)K_{1v}/K_{2v}]+I_hK_{2p}M\\\eta_2&=MkL^2K_{1v}+I_hK_{2v}k+\alpha kML^2K_{1v}\\&\quad-I_hK_{2v}M[MkL^2(1+\alpha)K_{1p}+I_hK_{2p}k]/\eta_1\\&\geq MkL^2K_{1v}+I_hK_{2v}k+\alpha kML^2K_{1v}-K_{2v}[MkL^2(1+\alpha)K_{1p}\\&\quad+I_hK_{2p}k]/K_{2p}\\&=MkL^2(1+\alpha)(K_{1v}K_{2p}-K_{1p}K_{2v})/K_{2p}\\\eta_3&=MkL^2(1+\alpha)K_{1p}+I_hK_{2p}k\end{aligned}$$

From the above four conditions **C1–C4**, it is easy to check that η_1; η_2; $\eta_3>0$, such that the error system (15) has all its poles on the LHS plane. Further, because up to the 4th-order derivatives of p_d are all bounded, we can conclude that the RHS (input) of (15) is bounded and thus its output is bounded, i.e. the error system is BIBO stable. (QED)

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