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PROJECTIVELY FLAT FOLIATIONS

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Abstract We describe the structure of regular codimension 1 foliations with numerically projectively flat tangent bundle on complex projective manifolds of dimension at least 4. Along the way, we prove that either the normal bundle of a regular codimension 1 foliation is pseudo-effective, or its conormal bundle is nef.

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1. Introduction

Let X be a smooth projective algebraic variety over the field of complex numbers, of dimension $n \geqslant 2$. If T_X is numerically flat, then X is a finite étale quotient of an abelian variety as a classical consequence of Yau's theorem on the existence of a Kähler–Einstein metric. In [42], Jahnke and Radloff proved that if the normalized tangent bundle $\mathrm{S}^n T_X \otimes \mathscr{O}_X(K_X)$ is numerically flat, then $K_X \equiv 0$, and hence T_X is numerically flat. Thus, X is again a finite étale quotient of an abelian variety. Let $\mathscr{E}$ be a vector bundle of rank $r \geqslant 1$ on X. Recall from [42, Theorem 1.1] that the normalized vector bundle $\mathrm{S}^r\mathscr{E} \otimes \det(\mathscr{E}^*)$ is numerically flat if and only if $\mathscr{E}$ is semistable with respect to some ample divisor H on X and equality holds in the Bogomolov–Gieseker inequality,

$$c_2(\mathscr{E}) \cdot H^{n-2} = \frac{r-1}{2r} c_1(\mathscr{E})^2 \cdot H^{n-2}.$$

The aim of this paper is to generalize the theorem of Jahnke and Radloff to codimension 1 regular foliations.

Theorem 1.1. *Let X be a complex projective manifold of dimension $n \geqslant 4$, and let $\mathscr{F} \subset T_X$ be a regular codimension 1 foliation. Suppose that the normalized vector bundle $\mathrm{S}^{n-1}\mathscr{F} \otimes \det(\mathscr{F}^*)$ is numerically flat. Then one of the following holds.*

(1) *There exists a $\mathbb{P}^1$-bundle structure $\varphi\colon X \to Y$ onto a finite étale quotient of an abelian variety and $\mathscr{F}$ induces a flat Ehresmann connection on φ.*

(2) *There exists an abelian variety A as well as a finite étale cover $\gamma\colon A \to X$ such that $\gamma^{-1}\mathscr{F}$ is a linear foliation.*

(3) *There exists an abelian scheme $f\colon B \to C$ onto a smooth complete curve of genus at least 2 as well as a finite étale cover $\gamma\colon B \to X$ such that $\gamma^{-1}\mathscr{F}$ is induced by f.*

(4) *The divisor K_X is ample and $\kappa(K_{\mathscr{F}}) = \dim X$. Moreover, the vector bundles $\mathscr{N}^*$ and Ω^1_X are nef.*

Notice that no example of foliations satisfying the conclusion of Theorem 1.1 (4) is known to the author.

If $\dim X = 2$, then the conclusion of Theorem 1.1 is false. Indeed, turbulent and Riccati foliations do not fit in the description of Theorem 1.1 above (see [11]).

The assumption $\dim X \geqslant 4$ is only used to prove Theorem 5.1. Nevertheless, we expect the above result to be true if $\dim X = 3$ as well.

The proof of Theorem 1.1 makes use of the following result, which might be of independent interest.

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Theorem 1.2. *Let X be a complex projective manifold of dimension $n \geqslant 1$, and let $\mathscr{F} \subset T_X$ be a regular codimension 1 foliation with normal bundle $\mathscr{N}$. Then, either $\mathscr{N}$ is pseudo-effective or $\mathscr{N}^*$ is nef.*

In fact, a more general statement is true (see Theorem 4.4).

Previous results

Greb, Kebekus and Peternell generalized the theorem of Jahnke and Radloff to projective varieties with klt singularities in [31]. More precisely, they proved that torus quotients are the only klt varieties with semistable tangent sheaf and extremal Chern classes. In [26], the structure of projective log smooth pairs with numerically flat normalized logarithmic tangent bundle is described. Passing to an appropriate finite cover and up to isomorphism, these are projective spaces and log smooth pairs with numerically flat logarithmic tangent bundles blown up at finitely many points away from the boundary. In addition, it has been shown in [17] that log smooth pairs with numerically flat logarithmic tangent bundle are toric fiber bundles over finite étale quotients of abelian varieties.

Structure of the paper

Section 2 gathers notation, known results and global conventions that will be used throughout the paper. In Section 3, we provide technical tools for the proof of our main results. Section 4 is mostly taken up by the proof of Theorem 1.2. The proof of Theorem 1.1 is long and therefore subdivided into numerous steps: Sections 5 to 9 prepare for it. In Section 5, we describe regular codimension 1 foliations with numerically projectively flat tangent bundle on nonminimal complex projective manifolds. Next, we prove abundance for K_X in Section 6 following the strategy employed in [42]: We use a Shafarevich map construction to prove that it reduces to the special case where $\mathscr{F} \cong \mathscr{L}^{\oplus n}$ for some line bundle $\mathscr{L}$. In Section 7, we address regular codimension 1 foliations with numerically projectively flat tangent bundle and nef conormal bundle on minimal varieties. Section 9 describes regular codimension 1 foliations with numerically projectively flat tangent bundle and pseudo-effective normal bundle still on minimal varieties. To this end, we show in Section 8 that the nef reduction map for $K_{\mathscr{F}}$ is an abelian scheme up to a finite étale cover. With these preparations at hand, the proof of Theorem 1.1 which we give in Section 10 becomes short.

2. Notation, conventions and used facts

2.1. Global conventions

Throughout the paper, all varieties are assumed to be defined over the field of complex numbers. We will freely switch between the algebraic and analytic context.

2.2. Projective space bundles

If $\mathscr{E}$ is a locally free sheaf of finite rank on a variety X, we denote by $\mathbb{P}_X(\mathscr{E})$ the variety $\mathrm{Proj}_X(\mathrm{S}^\bullet \mathscr{E})$.

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2.3. Stability

The word *stable* will always mean *slope-stable with respect to a given ample divisor*. Ditto for *semistable* and *polystable*. We refer to [37, Definition 1.2.12] for their precise definitions.

2.4. Numerically flat vector bundles

For the reader's convenience, we recall the definition of numerically flat vector bundles following [22, Definition 1.17].

Definition 2.1. A vector bundle $\mathscr{E}$ of rank $r \geqslant 1$ on a smooth projective variety is called *numerically flat* if $\mathscr{E}$ and $\mathscr{E}^*$ are nef vector bundes.

Remark 2.2. Let X be a smooth projective variety, and let $\mathscr{E}$ be a vector bundle of rank $r \geqslant 1$ on X with $c_1(\mathscr{E}) \equiv 0$. Then $\mathscr{E}$ is numerically flat if and only if $\mathscr{E}$ is nef.

Theorem 2.3. *Let X be a smooth projective variety, and let $\mathscr{E}$ be a vector bundle of rank $r \geqslant 1$ on X. Let H be an ample divisor on X. Then the following conditions are equivalent.*

(1) *The vector bundle $\mathscr{E}$ is numerically flat.*

(2) *The vector bundle $\mathscr{E}$ is flat and semistable with respect to H.*

(3) *The vector bundle $\mathscr{E}$ admits a filtration*

$$\{0\} = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_m = \mathscr{E}$$

by vector subbundles such that the quotients $\mathscr{Q}_k := \mathscr{E}_k/\mathscr{E}_{k-1}$ are Hermitian flat.

Proof. This follows easily from [22, Theorem 1.18] and [54] together (see [26, Theorem 2.9]). □

Remark 2.4. Setting and notation as in Theorem 2.3. If $\mathscr{E}$ is numerically flat, then it is semistable with respect to any ample divisor.

Lemma 2.5 [22, Corollary 1.19]. *Let $\mathscr{E}$ be a vector bundle on a smooth projective variety. If $\mathscr{E}$ is numerically flat, then $c_i(\mathscr{E}) \equiv 0$ for any integer $i \geqslant 1$.*

2.5. Projectively flat and numerically projectively flat vector bundles

Next, we recall the definition of projectively flat vector bundles.

Definition 2.6. A vector bundle $\mathscr{E}$ of rank $r \geqslant 1$ on a smooth projective variety X is called *projectively flat* if $\mathbb{P}(\mathscr{E})$ comes from a representation $\pi_1(X) \to \mathrm{PGL}(r,\mathbb{C})$.

Lemma 2.7 [42, Proof of Proposition 1.1 (2)]. *Let $\mathscr{E}$ be a vector bundle on a smooth projective variety. If $\mathscr{E}$ is projectively flat, then $c_i(\mathscr{E}) \equiv \frac{1}{r^i}\binom{r}{i} c_1(\mathscr{E})^i$ for any integer $i \geqslant 1$.*

One key notion is that of a numerically projectively flat vector bundle. We recall the definition.

Definition 2.8. A vector bundle $\mathscr{E}$ of rank $r \geqslant 1$ on a smooth projective variety X is called *numerically projectively flat* if $\mathrm{S}^r\mathscr{E} \otimes \det \mathscr{E}^*$ is numerically flat.

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Remark 2.9. Setting and notation as in Definition 2.8. Then $\mathscr{E}$ is numerically projectively flat if and only if $\mathrm{S}^r\mathscr{E} \otimes \det \mathscr{E}^*$ is nef since $\det(\mathrm{S}^r\mathscr{E} \otimes \det \mathscr{E}^*) \cong \mathscr{O}_X$ (see Remark 2.2).

Remark 2.10. Setting and notation as in Definition 2.8. Then $\mathscr{E}$ is numerically flat if and only if $\mathscr{E}$ is numerically projectively flat and $c_1(\mathscr{E}) \equiv 0$.

We will need the following characterization of numerically projectively flat vector bundles.

Theorem 2.11. *Let X be a smooth projective variety of dimension $n \geqslant 2$, and let $\mathscr{E}$ be a vector bundle of rank $r \geqslant 1$ on X. Let H be an ample divisor on X. Then the following conditions are equivalent.*

(1) *The vector bundle $\mathscr{E}$ is numerically projectively flat.*

(2) *The vector bundle $\mathscr{E}$ is projectively flat and semistable with respect to H.*

(3) *The vector bundle $\mathscr{E}$ admits a filtration*

$$\{0\} = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_m = \mathscr{E}$$

by vector subbundles such that the quotients $\mathscr{Q}_k := \mathscr{E}_k/\mathscr{E}_{k-1}$ are Hermitian projectively flat with

$$\frac{1}{\operatorname{rank} \mathscr{Q}_k} c_1(\mathscr{Q}_k) \equiv \frac{1}{r} c_1(\mathscr{E}).$$

(4) *The vector bundle $\mathscr{E}$ is semistable with respect to H and equality holds in the Bogomolov–Gieseker inequality,*

$$c_2(\mathscr{E}) \cdot H^{n-2} = \frac{r-1}{2r} c_1(\mathscr{E})^2 \cdot H^{n-2}.$$

(5) *For any morphism $C \to X$ from a smooth complete curve C, the vector bundle $\mathscr{E}|_C$ is semistable.*

Proof. This follows easily from [42, Theorem 1.1] and [42, Proposition 1.1] together (see also [26, Theorem 2.13]). □

Remark 2.12. Setting and notation as in Theorem 2.11. If $\mathscr{E}$ is numerically projectively flat, then it is semistable with respect to any ample divisor.

Example 2.13. Let $\mathscr{L}$ be a line bundle on a smooth projective variety X. Then $\mathscr{L}^{\oplus m}$ is numerically projectively flat for any integer $m \geqslant 1$.

The following is an easy consequence of Theorem 2.11.

Lemma 2.14. *Let X be a smooth projective variety of dimension $n \geqslant 1$, and let $\mathscr{E}$ be a vector bundle of rank $r \geqslant 1$ on X.*

(1) *If $\mathscr{E}$ is numerically projectively flat, then $\mathscr{E} \otimes \mathscr{L}$ is numerically projectively flat as well for any line bundle $\mathscr{L}$ on X.*

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(2) *Let Y be a smooth projective variety, and let $f\colon Y \to X$ be a morphism. If $\mathscr{E}$ is numerically projectively flat, then $f^*\mathscr{E}$ is also numerically projectively flat. In addition, the converse statement is true if f is surjective.*

(3) *Let $\mathscr{G}$ be a vector bundle on X of rank $s \geqslant 1$. Then $\mathscr{E} \oplus \mathscr{G}$ is numerically projectively flat if and only if $\mathscr{E}$ and $\mathscr{G}$ are numerically projectively flat and*

$$\frac{1}{r}c_1(\mathscr{E}) \equiv \frac{1}{s}c_1(\mathscr{G}) \equiv \frac{1}{r+s}c_1(\mathscr{E} \oplus \mathscr{G}).$$

We will need the following easy observation.

Lemma 2.15. *Let $f\colon X \to Y$ be a smooth projective morphism with connected fibers between smooth algebraic varieties, and let $\mathscr{E}$ be a vector bundle of rank $r \geqslant 1$ on X. Suppose that the restriction of $\mathscr{E}$ to every fiber of f is numerically projectively flat. Suppose in addition that $\mathscr{E}|_F \cong \mathscr{O}_F^{\oplus r}$, where F is a general fiber of f. Then there exists a vector bundle $\mathscr{G}$ on Y such that $\mathscr{E} \cong f^*\mathscr{G}$.*

Proof. Let F be any fiber of f. Because the function $y \mapsto h^0(X_y, \mathscr{E}|_{X_y})$ is upper semicontinuous in the Zariski topology on Y (see [36, Theorem 12.8]), we have $h^0(F, \mathscr{E}|_F) \geqslant r$. By [22, Proposition 1.16], the evaluation map $H^0(F, \mathscr{E}|_F) \otimes \mathscr{O}_F \to \mathscr{E}|_F$ is an injective morphism of vector bundles. This implies that $\mathscr{E}|_F \cong \mathscr{O}_F^{\oplus r}$. By [36, Corollary 12.9] together with the base change theorem [36, Theorem 12.11], the sheaf $\mathscr{G} := f_*\mathscr{E}$ is locally free of rank r and the formation of $f_*\mathscr{E}$ commutes with arbitrary base change. This easily implies that $\mathscr{E} \cong f^*\mathscr{G}$. □

2.6. Foliations

In this section, we have gathered a number of results and facts concerning foliations which will later be used in the proofs.

Definition 2.16. A *foliation* on a smooth variety X is a coherent subsheaf $\mathscr{F} \subseteq T_X$ such that

(1) $\mathscr{F}$ is closed under the Lie bracket, and

(2) $\mathscr{F}$ is saturated in T_X. In other words, the quotient $T_X/\mathscr{F}$ is torsion-free.

The *rank* r of $\mathscr{F}$ is the generic rank of $\mathscr{F}$. The *codimension* of $\mathscr{F}$ is defined as $q := \dim X - r$.

The *canonical class* $K_{\mathscr{F}}$ of $\mathscr{F}$ is any Weil divisor on X such that $\mathscr{O}_X(-K_{\mathscr{F}}) \cong \det \mathscr{F}$.

Let $X^\circ \subseteq X$ be the open set where $\mathscr{F}$ is a subbundle of T_X. The foliation is called *regular* if $X^\circ = X$.

A *leaf* of $\mathscr{F}$ is a maximal connected and immersed holomorphic submanifold $L \subseteq X^\circ$ such that $T_L = \mathscr{F}_{|L}$.

The *normal sheaf* of $\mathscr{F}$ is $\mathscr{N} := (T_X/\mathscr{F})^{**}$. The q-th wedge product of the inclusion $\mathscr{N}^* \subseteq \Omega^1_X$ gives rise to a nonzero global section $\omega \in H^0(X, \Omega^q_X \otimes \det \mathscr{N})$ whose zero locus has codimension at least 2 in X. Moreover, ω is *locally decomposable* and *integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point

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of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1,\ldots,q\}$. The integrability condition for ω is equivalent to the condition that $\mathscr{F}$ is closed under the Lie bracket. Conversely, let $\mathscr{L}$ be a line bundle on X, and let $\omega \in H^0(X,\Omega_X^q \otimes \mathscr{L})$ be a global section whose zero locus has codimension at least 2 in X. Suppose that ω is locally decomposable and integrable. Then the kernel of the morphism $T_X \to \Omega_X^{q-1} \otimes \mathscr{L}$ given by the contraction with ω defines a foliation of codimension q on X. These constructions are inverse of each other.

Let X and Y be smooth complex algebraic varieties, and let $f\colon X \to Y$ be a dominant morphism that restricts to a smooth morphism $f^\circ\colon X^\circ \to Y^\circ$, where $X^\circ \subseteq X$ and $Y^\circ \subseteq Y$ are dense open subsets. Let $\mathscr{F}$ be a foliation on Y. *The pull-back* $f^{-1}\mathscr{F}$ *of* $\mathscr{F}$ *via* f is the foliation on X whose restriction to X° is $(df^\circ)^{-1}\big(\mathscr{F}_{|Y^\circ}\big)$. We will need the following easy observation.

Lemma 2.17. *Let $f\colon X \to Y$ be a finite étale cover of smooth projective varieties, and let $\mathscr{F}$ be a regular foliation on Y. Then the following holds.*

(1) *The foliation $f^{-1}\mathscr{F}$ is regular. Moreover $f^{-1}\mathscr{F} \cong f^*\mathscr{F}$, and $\mathscr{N}_{f^{-1}\mathscr{F}} \cong f^*\mathscr{N}_{\mathscr{F}}$.*

(2) *We have $\kappa(K_{f^{-1}\mathscr{F}}) = \kappa(K_{\mathscr{F}})$, and $\nu(K_{f^{-1}\mathscr{F}}) = \nu(K_{\mathscr{F}})$.*

Proof. The first part of the statement is obvious. For the second part, note that $K_{f^{-1}\mathscr{F}} = f^*K_{\mathscr{F}}$ by (1). Then we have $\kappa(K_{f^{-1}\mathscr{F}}) = \kappa(K_{\mathscr{F}})$ by [58, Theorem 5.13] and $\nu(K_{f^{-1}\mathscr{F}}) = \nu(K_{\mathscr{F}})$ by [52, Proposition V.2.22]. □

2.7. Ehresmann connections

Let $f\colon X \to Y$ be a surjective proper morphism between smooth varieties. An *Ehresmann connection* on f is a distribution $\mathscr{D} \subseteq T_X$ such that the restriction of the tangent map $Tf\colon T_X \to f^*T_Y$ to $\mathscr{D}$ induces an isomorphism $\mathscr{D} \cong f^*T_Y$. The Ehresmann connection $\mathscr{D}$ is said to be *flat* if $\mathscr{D}$ is a foliation.

If f has an Ehresmann connection, then f is a locally trivial fibration for the analytic topology since complex flows of vector fields on analytic spaces exist.

Suppose that f has a flat Ehresmann connection $\mathscr{D}$. Let F be any fiber of f, and set $y := f(F)$. Let also $\rho\colon \pi_1(Y,y) \to \mathrm{Aut}(F)$ be the monodromy representation of $\mathscr{D}$. Then

$$X \cong (\widetilde{Y} \times F)/\pi_1(Y,y),$$

where $\widetilde{Y}$ denotes the universal cover of Y based at y and $\pi_1(Y,y)$ acts diagonally on the product.

Example 2.18. Let $n \geqslant 2$ be an integer. Let $A = \mathbb{C}^{n-1}/\Lambda$ be a complex abelian variety, and let $\rho\colon \pi_1(A) \to \mathrm{PGL}(2,\mathbb{C})$ be a representation of the fundamental group $\pi_1(A) \cong \Lambda$ of A. Then the group $\pi_1(A)$ acts diagonally on $\mathbb{C}^{n-1} \times \mathbb{P}^1$ by $\gamma \cdot (z,p) = \big(\gamma(z),\rho(\gamma)(p)\big)$. Set $X := (\mathbb{C}^{n-1} \times \mathbb{P}^1)/\pi_1(A)$, and denote by $\psi\colon X \to A \cong \mathbb{C}^{n-1}/\pi_1(A)$ the projection morphism, which is $\mathbb{P}^1$-bundle. The foliation on $\mathbb{C}^{n-1} \times \mathbb{P}^1$ induced by the projection $\mathbb{C}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^1$ is invariant under the action of $\pi_1(A)$ and gives a flat Ehresmann connection $\mathscr{F}$ on ψ. Then ρ identifies with the monodromy representation of $\mathscr{F}$.

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2.8. The standard setting

Throughout the present paper, we will be working in the following setup and use the notation below.

Setup 2.19. Let X be a complex projective manifold of dimension $n \geqslant 3$, and let $\mathscr{F} \subset T_X$ be a regular codimension 1 foliation with normal bundle $\mathscr{N}$. Suppose that $\mathscr{F}$ is numerically projectively flat. Let $\rho\colon \pi_1(X) \to \mathrm{PGL}(n-1,\mathbb{C})$ denote a representation that defines the projectively flat structure on $\mathscr{F}$.

Remark 2.20. Setting and notation as in Setup 2.19. Let $f\colon Y \to X$ be a finite étale cover. Then the pull-back $f^{-1}\mathscr{F}$ of $\mathscr{F}$ on Y satisfies all the conditions listed in Setup 2.19 (see Lemma 2.17).

We will need the following observation.

Lemma 2.21. *Setting and notation as in 2.19. Then $\mathscr{F}^*$ is a nef vector bundle.*

Proof. In order to prove the lemma, it suffices to show that $K_{\mathscr{F}}$ is nef since the vector bundle $\mathscr{F}^*|_C$ is semistable for any morphism $C \to X$ from a smooth complete curve C by Theorem 2.11. We argue by contradiction and assume that $K_{\mathscr{F}}$ is not nef. Then there exists a smooth complete curve $C \to X$ such that $\deg_C K_{\mathscr{F}} < 0$. This implies that the vector bundle $\mathscr{F}|_C$ is ample since it is semistable with positive slope by our current assumption. By [9, Main Theorem] (see also [50, Theorem 1]), the leaf F of $\mathscr{F}$ through a general point in C is algebraic and rationally connected. It follows that $\mathscr{F}$ is induced by a morphism with connected fibers $f\colon X \to B$ onto a smooth complete curve B. By [49, Theorem IV.2.10] applied to a general fiber F of f, there exists a minimal free rational curve $\mathbb{P}^1 \to F$ which means that there exists an integer $0 \leqslant p \leqslant n-2$ such that

$$\mathscr{F}|_{\mathbb{P}^1} \cong T_F|_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus n-2-p}.$$

Since $\mathscr{F}|_{\mathbb{P}^1}$ is projectively flat by assumption, we must have $p = 0$ and $n = 2$, yielding a contradiction. This finishes the proof of the lemma. □

3. Preparations for the proof of Theorem 1.1

In this section, we provide technical tools for the proof of our main results.

3.1. Bott (partial) connection and applications

We briefly recall the relevant notions first. Let X be a complex manifold, and let $\mathscr{F} \subset T_X$ be a regular foliation with normal bundle $\mathscr{N}$. Let $p\colon T_X \to \mathscr{N}$ denotes the natural projection. For sections U of $\mathscr{N}$, T of T_X and V of $\mathscr{F}$ over some open subset of X with $U = p(T)$, set $D_V U = p([V,T])$. This expression is well defined, $\mathscr{O}_X$-linear in V and satisfies the Leibnitz rule $D_V(fU) = fD_V U + (Vf)U$ so that D is an $\mathscr{F}$-connection on $\mathscr{N}$ (see [3]). We refer to it as the *Bott connection* on $\mathscr{N}$.

Lemma 3.1. *Let X be a complex manifold of dimension $n \geqslant 2$, and let $\mathscr{F} \subset T_X$ be a regular foliation of codimension $1 \leqslant q \leqslant n-1$ with normal bundle $\mathscr{N}$.*

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(1) *The Bott connection is flat when restricted to the leaves of* $\mathscr{F}$.

(2) *The cohomology class* $c_1(\mathscr{N}) \in H^1(X,\Omega^1_X)$ *lies in the image of the natural map*

$$H^1(X,\mathscr{N}^*) \to H^1(X,\Omega^1_X).$$

In particular, we have $c_1(\mathscr{N})^{q+1} = 0 \in H^{q+1}(X,\Omega^{q+1}_X)$.

Proof. Item (1) follows from an easy (local) computation. The arguments of [3, Proof of Corollary 3.4] give a proof of Item (2). □

The following easy consequence of Lemma 3.1 above will prove to be crucial.

Proposition 3.2. *Let* X *be a complex manifold of dimension* $n \geqslant 3$, *and let* $Y \subseteq X$ *be a compact Kähler submanifold. Let* $\mathscr{F} \subset T_X$ *be a regular codimension* 1 *foliation with normal bundle* $\mathscr{N}$. *Suppose that* $\mathscr{F} \cong \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_{n-1}$ *for some line bundles* $\mathscr{L}_i$ *with* $c_1(\mathscr{L}_1) = \cdots = c_1(\mathscr{L}_{n-1}) \in H^1(X,\Omega^1_X)$. *Then*

$$c_1(\mathscr{L}_i|_Y)^2 = c_1(\mathscr{N}|_Y)^2 = c_1(\mathscr{L}_i|_Y) \cdot c_1(\mathscr{N}|_Y) \equiv 0.$$

Moreover, the linear subspace

$$L := \langle c_1(\mathscr{L}_1|_Y), \ldots, c_1(\mathscr{L}_{n-1}|_Y), c_1(\mathscr{N}|_Y)\rangle \subseteq H^1(Y,\Omega^1_Y)$$

has dimension at most one.

Proof. The second assertion follows immediately from the first using the Hodge index theorem.

Set $c := c_1(\mathscr{L}_1) = \cdots = c_1(\mathscr{L}_{n-1}) \in H^1(X,\Omega^1_X)$. Notice that $\mathscr{L}_i \subset T_X$ defines a regular foliation for any index $i \in I$. Denote by $\mathscr{N}_i$ its normal bundle. Then Lemma 3.1 shows that $c_1(\mathscr{N}_i) = c_1(\mathscr{N}) + (n-2)c \in H^1(X,\Omega^1_X)$ lies in the image of the natural map

$$H^1(X,\mathscr{N}_i^*) \to H^1(X,\Omega^1_X).$$

On the other hand, we have

$$\begin{aligned}
\bigcap_{1\leqslant i\leqslant n-1} \operatorname{Im}\left(H^1(X,\mathscr{N}_i^*) \to H^1(X,\Omega^1_X)\right) &= \bigcap_{1\leqslant i\leqslant n-1} \operatorname{Ker}\left(H^1(X,\Omega^1_X) \to H^1(X,\mathscr{L}_i^*)\right) \\
&= \operatorname{Ker}\left(H^1(X,\Omega^1_X) \to \bigoplus_{1\leqslant i\leqslant n-1} H^1(X,\mathscr{L}_i^*)\right) \\
&= \operatorname{Ker}\left(H^1(X,\Omega^1_X) \to H^1(X,\mathscr{F}^*)\right) \\
&= \operatorname{Im}\left(H^1(X,\mathscr{N}^*) \to H^1(X,\Omega^1_X)\right),
\end{aligned}$$

and hence $c_1(\mathscr{N}) + (n-2)c \in H^1(X,\Omega^1_X)$ lies in the image of the natural map $H^1(X,\mathscr{N}^*) \to H^1(X,\Omega^1_X)$. By Lemma 3.1 again, $c_1(\mathscr{N}) \in H^1(X,\Omega^1_X)$ lies in the image of the natural map

$$H^1(X,\mathscr{N}^*) \to H^1(X,\Omega^1_X).$$

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Now, we have a commutative diagram

$$\begin{array}{ccc} H^1(X,\mathscr{N}^*) & \longrightarrow & H^1(X,\Omega^1_X) \\ \downarrow & & \downarrow \\ H^1(Y,\mathscr{N}^*|_Y) & \longrightarrow & H^1(Y,\Omega^1_Y). \end{array}$$

The first assertion follows easily since $\mathscr{N}$ has rank 1 by assumption. □

The proof of Proposition 3.4 below makes use of the following result.

Lemma 3.3. *Let X be a complex projective variety, and let $\nu\colon \widetilde{X} \to X$ be the normalization morphism. Suppose that ω_X is invertible. Then the following holds.*

(1) *There exists an effective Weil divisor on $\widetilde{X}$ such that $\omega_{\widetilde{X}} \cong \nu^*\omega_X(-E)$.*
(2) *We have $E = 0$ if and only if X is normal.*

Proof. By [45, Proposition 5.67] and [45, Proposition 5.68] together, we have $\nu_*\omega_{\widetilde{X}} \cong \mathscr{C} \otimes \omega_X$, where $\mathscr{C} := \mathscr{H}om_{\mathscr{O}_X}(\nu_*\mathscr{O}_{\widetilde{X}}, \mathscr{O}_X) \subseteq \mathscr{O}_X$ is the conductor ideal of $\widetilde{X}$ in X. This gives $\omega_{\widetilde{X}} \cong \widetilde{\mathscr{C}} \otimes \nu^*\omega_X$, where $\widetilde{\mathscr{C}} := \mathscr{C} \cdot \mathscr{O}_{\widetilde{X}} \subseteq \mathscr{O}_{\widetilde{X}}$. Recall that $\omega_{\widetilde{X}}$ is reflexive since $\widetilde{X}$ is normal. It follows that $\widetilde{\mathscr{C}}$ is reflexive as well. Hence, there exists an effective Weil divisor on $\widetilde{X}$ such that $\widetilde{\mathscr{C}} \cong \mathscr{O}_{\widetilde{X}}(-E)$, proving (1).

If $E = 0$, then $\mathscr{C} \cong \mathscr{O}_X$. This immediately implies that ν is an isomorphism, finishing the proof of the lemma. □

Proposition 3.4. *Let X be a complex manifold of dimension $n \geqslant 3$, and let $Y \subseteq X$ be a compact Kähler submanifold of dimension $m \geqslant 2$. Let $\mathscr{F} \subset T_X$ be a regular codimension 1 foliation with normal bundle $\mathscr{N}$. Suppose that $\mathscr{F} \cong \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_{n-1}$ for some line bundles $\mathscr{L}_i$ with $c_1(\mathscr{L}_1) = \cdots = c_1(\mathscr{L}_{n-1}) \in H^1(X,\Omega^1_X)$. Let $\mathscr{M}$ be a line bundle on Y such that $c_1(\mathscr{M}) \in \langle c_1(\mathscr{L}_1|_Y), \ldots, c_1(\mathscr{L}_{n-1}|_Y), c_1(\mathscr{N}|_Y)\rangle \subseteq H^1(Y,\Omega^1_Y)$. Suppose that $\mathscr{M}$ is nef and $\mathscr{M} \equiv \sum_{i\in I} b_i B_i$, where $b_i \in \mathbb{Q}_{>0}$ and the B_i are distinct prime divisors. Suppose in addition that $\mathscr{N}^*_{Y/X}|_{B_i}$ is nef if $m < n$. Then the following holds.*

(1) *For every index $i \in I$, B_i is a finite étale quotient of a complex torus.*
(2) *We have $B_i \cap B_j = \emptyset$ if $i \neq j$.*
(3) *The vector bundles $\mathscr{N}_{B_i/Y}$ and $\mathscr{N}_{B_i/X}$ are numerically flat. In particular, $B_i \subset Y$ is a nef divisor.*

Proof. Suppose that $m < n$. If $m = n$, the proof of the proposition is very similar and therefore omitted.

We may assume without loss of generality that $c_1(\mathscr{M}) \not\equiv 0$. In other words, assume that I is not empty. By Proposition 3.2, we have $c_1(\mathscr{M})^2 \equiv 0$. It follows that $\mathscr{M}|_{B_i} \equiv 0$ for any $i \in I$ since $\mathscr{M}$ is nef by assumption. By Proposition 3.2 again,

$$L := \langle c_1(\mathscr{L}_1|_Y), \ldots, c_1(\mathscr{L}_{n-1}|_Y), c_1(\mathscr{N}|_Y)\rangle = \langle c_1(\mathscr{M}|_Y)\rangle \subseteq H^1(Y,\Omega^1_Y).$$

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Thus, $\mathscr{L}_j|_{B_i} \equiv 0$ and $\mathscr{N}|_{B_i} \equiv 0$. As a consequence, the vector bundle $\Omega^1_X|_{B_i}$ is numerically flat. Consider the exact sequence

$$0 \to \mathscr{N}^*_{Y/X}|_{B_i} \to \Omega^1_X|_{B_i} \to \Omega^1_Y|_{B_i} \to 0.$$

Since $\mathscr{N}^*_{Y/X}|_{B_i}$ is nef by assumption, we infer that $\mathscr{N}^*_{Y/X}|_{B_i}$ and $\Omega^1_Y|_{B_i}$ are numerically flat as well.

Next, consider the exact sequence of sheaves

$$0 \to \mathscr{O}_{B_i}(-B_i) \to \Omega^1_Y|_{B_i} \to \Omega^1_{B_i} \to 0.$$

Since $-B_i|_{B_i} \equiv \frac{1}{b_i}\sum_{j\neq i} b_j B_j|_{B_i}$, we conclude that $-B_i|_{B_i} \equiv 0$ and that $\mathscr{O}_{B_i}(-B_i)$ is a subbundle of $\Omega^1_Y|_{B_i}$ (see [22, Proposition 1.16]). As a consequence, the sheaf $\Omega^1_{B_i}$ is locally free and numerically flat. Moreover, $B_i \cap B_j = \emptyset$ if $i \neq j$.

Let $\nu_i \colon \widetilde{B}_i \to B_i$ be the normalization morphism. By the adjunction formula,

$$\omega_{B_i} \cong \mathscr{O}_Y(K_Y + B_i)|_{B_i} \equiv 0.$$

In particular, ω_{B_i} is locally free. By Lemma 3.3 above, there exists an effective Weil divisor E_i on $\widetilde{B}_i$ such that $\omega_{\widetilde{B}_i} \cong \nu_i^*\omega_{B_i}(-E_i)$.

Suppose that $E_i \neq 0$. Then $K_{\widetilde{B}_i}$ is not pseudo-effective, and hence B_i is uniruled by [5, Corollary 0.3] applied to a resolution of $\widetilde{B}_i$. Let $C \subseteq B_i$ be a rational curve. Then $\Omega^1_Y|_C$ is the trivial vector bundle since $\Omega^1_Y|_C$ is numerically flat, yielding a contradiction. This shows that $E_i = 0$, and hence B_i is normal by Lemma 3.3. Then [5, Corollary 0.3] applies to show that B_i has canonical singularities since $K_{B_i} \equiv 0$. By the solution of the Zariski–Lipman conjecture for canonical spaces (see [29, Theorem 6.1] or [23, Theorem 1.1]), we conclude that B_i is smooth. In addition, we have $c_1(B_i) \equiv 0$ and $c_2(B_i) \equiv 0$ by [22, Corollary 1.19]. As a classical consequence of Yau's theorem on the existence of a Kähler–Einstein metric, B_i is then covered by a complex torus (see [47, Chapter IV Corollary 4.15]). This finishes the proof of the proposition. □

Proposition 3.5. *Let X be a complex manifold of dimension $n \geqslant 3$, and let $Y \subseteq X$ be a compact Kähler submanifold of dimension $m \geqslant 2$. Let $\mathscr{F} \subset T_X$ be a regular codimension 1 foliation. Suppose that $\mathscr{F} \cong \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_{n-1}$ for some line bundles $\mathscr{L}_i$ with $c_1(\mathscr{L}_1) = \cdots = c_1(\mathscr{L}_{n-1}) \in H^1(X, \Omega^1_X)$. Suppose furthermore that $c_k(\mathscr{N}_{Y/X}) = 0$ for any $k \in \{1, \ldots, m\}$ if $m < n$. Then $c_1(Y)^2 = 0$ and $c_k(Y) = 0$ for any $k \in \{2, \ldots, m\}$. Moreover, we have $\chi(Y, \mathscr{O}_Y) = 0$.*

Proof. Suppose that $m < n$. If $m = n$, the proof of the proposition is very similar and therefore omitted.

The Chern polynomial $c_t(T_X|_Y)$ of $T_X|_Y$ satisfies

$$c_t(T_X|_Y) = c_t(\mathscr{N}|_Y) \cdot \prod_{1 \leqslant i \leqslant n-1} c_t(\mathscr{L}_i|_Y),$$

where $\mathscr{N}$ denotes the normal bundle of $\mathscr{F}$. Therefore, by Proposition 3.2, we have $c_1(T_X|_Y)^2 = 0$ and $c_k(T_X|_Y) = 0$ for any $k \in \{2, \ldots, n\}$. On the other hand,

$$c_t(T_X|_Y) = c_t(\mathscr{N}_{Y/X}) \cdot c_t(T_Y) = c_t(T_Y)$$

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since $c_t(\mathscr{N}_{Y/X}) = 1$ by assumption. This easily implies $c_1(Y)^2 = 0$ and $c_k(Y) = 0$ for any $k \in \{2, \dots, m\}$. Then $\chi(Y, \mathscr{O}_Y) = 0$ by the Hirzebruch–Riemann–Roch theorem. This finishes the proof of the proposition. □

3.2. Abelian schemes

The proof of Theorem 1.1 uses Kollár's characterisation of étale quotients of abelian schemes [48, Theorem 6.3]. We recall the relevant notion first. Let X be a normal projective variety, and let $Y \subseteq X$ be a closed subvariety. We say that X has *generically large fundamental group on* Y if for any very general point $y \in Y$ and for every closed and positive-dimensional subvariety $y \in Z \subseteq Y$ with normalization $\overline{Z}$, the image of the natural morphism $\pi_1(\overline{Z}) \to \pi_1(X)$ is infinite. The following observation will prove to be crucial.

Lemma 3.6. *Let $f\colon X \to Y$ be a projective morphism with connected fibers between smooth algebraic varieties, and let F be a general fiber of f. Suppose that every fiber of f (with its reduced structure) is a finite étale quotient of an abelian variety. Then the homomorphism $\pi_1(F) \to \pi(X)$ is injective. In particular, X has generically large fundamental group on F.*

Proof. By [39, Proof of Lemma 2.2] and [51, Lemma 7.3], there exists an open subset $Y^\circ \subseteq Y$ with complement of codimension at least 2 such that the restriction f° of f to $X^\circ := f^{-1}(Y^\circ)$ is locally homotopically Q-smooth (we refer the reader to [51, Definition 7.2] for this notion). Then [51, Theorem 7.8] applies to show that f° is birational to a Q-smooth Q-torus fibration $g^\circ\colon Z^\circ \to Y^\circ$ over Y°. In other words, there exists a finite morphism $Y_1^\circ \to Y^\circ$ such that the normalization Z_1° of $Y_1^\circ \times_{Y^\circ} Z^\circ$ is étale over Z° and such that the morphism $Z_1^\circ \to Y_1^\circ$ is smooth with fibers being finite étale quotients of abelian varieties. This immediately implies that any fiber of g° (with its reduced structure) is a finite étale quotient of an abelian variety. It then follows that $X^\circ/Y^\circ \cong Z^\circ/Y^\circ$. Let $X_1^\circ \to X^\circ$ be the induced finite étale cover, and denote by $f_1^\circ\colon X_1^\circ \to Y_1^\circ$ the corresponding smooth fibration. By [51, Theorem 3.14], there exists a finite étale cover $X_2^\circ \to X_1^\circ$ such that any fiber of the composition $f_2^\circ\colon X_2^\circ \to X_1^\circ \to Y_1^\circ =: Y_2^\circ$ is an abelian variety. Let F_2 be a general fiber of f_2°. By [16, Proof of Lemma 2.3], the homomorphism $\pi_1(F_2) \to \pi_1(X_2^\circ)$ is injective. Moreover, there is a commutative diagram

$$\begin{array}{ccc} \pi_1(F_2) & \hookrightarrow & \pi_1(X_2^\circ) \\ \downarrow & & \downarrow \\ \pi_1(F) & \longrightarrow & \pi_1(X^\circ), \end{array}$$

where F denotes the image of F_2 in X°. By [51, Corollary 3.7], $\pi_1(F)$ is torsion-free. This easily implies that the homomorphism $\pi_1(F) \to \pi(X^\circ)$ is also injective since $\pi_1(F_2)$ has finite index in $\pi(F)$. Finally, the inclusion induces an isomorphism $\pi_1(X^\circ) \cong \pi_1(X)$ of fundamental groups since $X \setminus X^\circ$ is a closed subset of codimension at least 2. This finishes the proof of the lemma. □

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We will also need the following auxiliary result.

Lemma 3.7. *Let $f\colon X \to Y$ be a smooth projective morphism with connected fibers between smooth algebraic varieties. Suppose that there exists $y_0 \in Y$ such that the fiber $f^{-1}(y_0)$ of f over y_0 is a finite étale quotient of an abelian variety. Then every fiber of f is a finite étale quotient of an abelian variety. Moreover, there exist a dense open set $Y^\circ \subseteq Y$ and a finite étale cover $X_1^\circ \to X^\circ := f^{-1}(Y^\circ)$ such that the fibration $X_1^\circ \to Y_1^\circ$ obtained as the Stein factorization of the composition $X_1^\circ \to X^\circ \to Y^\circ$ is an abelian scheme equipped with a level three structure.*

Proof. By [51, Theorem 3.14], there exists a finite étale cover $X_1 \to X$ such that the fiber $f_1^{-1}(y_0)$ of the composition $f_1\colon X_1 \to X \to Y =: Y_1$ is an abelian variety. Notice that f_1 is a smooth morphism. By Ehresmann's theorem, any fiber F_1 of f_1 is homeomorphic to an abelian variety. Then [13, Theorem 4.8] applies to show that F_1 is an abelian variety. Finally, shrinking Y_1 if necessary, we can find a suitable finite étale cover $Y_2 \to Y_1$ such that the projection morphism $Y_2 \times_{Y_1} X_1 \to Y_2$ has a section. Then $Y_2 \times_{Y_1} X_1$ is an abelian scheme over Y_2. Replacing Y_2 by a further finite étale cover, we may assume that $Y_2 \times_{Y_1} X_1/Y_2$ admits a level three structure, finishing the proof of the lemma. □

The proof of Theorem 1.1 makes use of the following result, which might be of independent interest.

Lemma 3.8. *Let $f\colon X \to Y$ be an abelian scheme over a projective base equipped with a level three structure, and let $\beta\colon X \to M$ be a birational morphism onto a smooth projective variety. Then M is an abelian scheme over a smooth projective base N and there exists a birational morphism $Y \to N$ such that $X/Y \cong Y \times_N M/Y$.*

Proof. Let $\Sigma \subseteq X$ be the neutral section of f, and let N be the normalization of $\Lambda := \beta(\Sigma)$. Let also $\mathscr{A}(3)$ be the fine moduli space of polarized abelian varieties with a level three structure.

Recall that there exists a composition $Z \to M$ of a finite number of blow-ups with smooth centers such that the induced rational map $Z \dashrightarrow X$ is a morphism. This easily implies that every positive-dimensional fiber of β is rationally chain connected.

Let $C \subseteq X$ be a rational curve such that $\dim \beta(C) = 0$, and set $B := f(C)$. By [48, Lemma 5.9.3], $\left(f^{-1}(B)/B, \Sigma|_{f^{-1}(B)}\right) \cong (B \times A/B, B \times \{0_A\})$, where $(A, 0_A)$ is an abelian variety. Notice that $C = B \times \{a_0\}$ for some point $a_0 \in A$. In particular, either C is disjoint from Σ or C is contained in Σ. This immediately implies that $\Sigma = \beta^{-1}(\Lambda)$ since every positive-dimensional fiber of β is rationally chain connected. By the rigidity lemma (see [19, Lemma 1.15]), the morphism $Y \to \mathscr{A}(3)$ corresponding to f factorizes through the map $\eta\colon Y \cong \Sigma \to N$ induced by $\beta|_\Sigma$. In other words, the abelian scheme f is the pull-back of an abelian scheme M_1 over N via η. Let $\beta_1\colon X \to M_1$ be the induced morphism.

To prove the statement, it suffices to show that there exists an isomorphism $\tau\colon M \to M_1$ such that $\beta_1 = \tau \circ \beta$ or equivalently that β (resp. β_1) factorizes through β_1 (resp. β).

Let F be a positive-dimensional fiber of β. Recall that F is rationally chain connected. Set $G := f(F)$. By [48, Lemma 5.9.3] again, $\left(f^{-1}(G)/G, \Sigma|_{f^{-1}(G)}\right) \cong (G \times A_2/G, G \times \{0_{A_2}\})$, where $(A_2, 0_{A_2})$ is an abelian variety. Moreover, $F = G \times \{a_2\}$ for some point

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$a_2 \in A_2$. Since G is a fiber of η, F is contracted by β_1. By the rigidity lemma (see [19, Lemma 1.15]), β_1 factorizes through β. Let now F_1 be a fiber of β_1. Set $G_1 := f(F_1)$. Then G_1 is a fiber of η. Moreover, $\big(f^{-1}(G_1)/G_1, \Sigma|_{f^{-1}(G_1)}\big) \cong (G_1 \times A_1/G_1, G_1 \times \{0_{A_1}\})$ for some abelian variety $(A_1, 0_{A_1})$, and $F_1 \cong G_1 \times \{a_1\}$ for some point $a_1 \in A_1$. Since $G_1 \times \{0_{A_1}\} \cong \Sigma \cap f^{-1}(G_1)$ is contracted by β, F_1 is contracted by β as well. By the rigidity lemma (see [19, Lemma 1.15]), β factorizes through β_1. This finishes the proof of the lemma. □

3.3. Descent of line bundles

Let $f\colon X \to Y$ be a morphism of algebraic varieties, and let $\mathscr{L}$ be a line bundle on X. We give sufficient conditions that guarantee that there exist a line bundle $\mathscr{M}$ on Y and a positive integer m such that $\mathscr{L}^{\otimes m} \cong f^*\mathscr{M}$. The following observation is rather standard. We include a proof here for the reader's convenience.

Lemma 3.9. *Let $f\colon X \to Y$ be a flat projective morphism with integral fibers between smooth complex algebraic varieties. Let $\mathscr{L}$ be a line bundle on X, and let F be a general fiber of f. Suppose that $\mathscr{L}|_F$ is a torsion line bundle. Then there exists a line bundle $\mathscr{M}$ on Y and a positive integer m such that $\mathscr{L}^{\otimes m} \cong f^*\mathscr{M}$.*

Proof. Let m be a positive integer such that the set of points $y \in Y$ with $\mathscr{L}^{\otimes m}|_{X_y} \cong \mathscr{O}_{X_y}$ is dense. Because the functions $y \mapsto h^0\big(X_y, \mathscr{L}^{\otimes \pm m}|_{X_y}\big)$ are upper semicontinuous in the Zariski topology on Y (see [36, Theorem 12.8]), we have $h^0\big(X_y, \mathscr{L}^{\otimes \pm m}|_{X_y}\big) = 1$ for every $y \in Y$. This implies that $\mathscr{L}^{\otimes m}|_{X_y} \cong \mathscr{O}_{X_y}$ since X_y is integral by assumption. By [36, Corollary 12.9] together with the base change theorem [36, Theorem 12.11], $\mathscr{M} := f_*\mathscr{L}^{\otimes m}$ is a line bundle and the formation of $f_*\mathscr{L}^{\otimes m}$ commutes with arbitrary base change. This easily implies that $\mathscr{L}^{\otimes m} \cong f^*\mathscr{M}$. □

Remark 3.10. Setting and notation as in Lemma 3.9. Suppose in addition that $\mathscr{L}|_F \cong \mathscr{O}_F$. Then the proof of Lemma 3.9 shows that there exists a line bundle $\mathscr{M}$ on Y such that $\mathscr{L} \cong f^*\mathscr{M}$.

Proposition 3.11. *Let X be a complex projective manifold, and let $f\colon X \to B$ be a surjective morphism with connected fibers onto a smooth complete curve. Let $\mathscr{L}$ be a nef line bundle on X such that $\mathscr{L}|_F$ is torsion, where F is a general fiber of f. Then there exist a line bundle $\mathscr{M}$ on B and a positive integer m such that $\mathscr{L}^{\otimes m} \cong f^*\mathscr{M}$. In particular, $\mathscr{L} \equiv_B 0$.*

Proof. Let $B^\circ \subseteq B$ be a dense open set such that $f^\circ := f|_{X^\circ}$ is a smooth morphism, where $X^\circ := f^{-1}(B^\circ)$. Let m be a positive integer. Because the functions $b \mapsto h^0\big(X_b, \mathscr{L}^{\otimes \pm m}|_{X_b}\big)$ are upper semicontinuous in the Zariski topology on B (see [36, Theorem 12.8]), the set of points b on B° such that $\mathscr{L}^{\otimes m}|_{X_b} \cong \mathscr{O}_{X_b}$ is closed. As a consequence, there exists a positive integer m_1 such that $\mathscr{L}^{\otimes m_1}|_{X_b} \cong \mathscr{O}_{X_b}$ for all $b \in B^\circ$. From [36, Corollary 12.9], we see that $f^\circ_*(\mathscr{L}^{\otimes m_1}|_{X^\circ})$ is a line bundle. Thus, there exists a divisor D on X with $f(\operatorname{Supp} D) \subsetneq B$ such that $\mathscr{L}^{\otimes m_1} \cong \mathscr{O}_X(D)$.

Let $S \subset X$ be a two-dimensional complete intersection of general elements of a very ample linear system on X. This is a smooth surface in X and the restriction $f|_S$ of f to

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S is a surjective morphism with connected fibers onto B. By Zariski's lemma (see [10, Lemma III.8.2]), we have $D|_S^2 \leqslant 0$. Since D is nef by assumption, we must have $D|_S^2 = 0$. By Zariski's lemma again, there exist a positive integer m_2 and a divisor G on B such that $m_2 D|_S = f|_S^* G$. The claim now follows from the Grothendieck–Lefchetz hyperplane theorem. □

The following result is an easy consequence of Proposition 3.11 above.

Corollary 3.12. *Let $f\colon X \to Y$ be an equidimensional morphism with connected fibers between normal projective varieties, and let $\mathscr{L}$ be a nef line bundle on X such that $\mathscr{L}|_F$ is torsion, where F is a general fiber of f. Then $\mathscr{L} \equiv_Y 0$.*

Proof. Notice that a general fiber of f is irreducible by [19, 5.5]. By a theorem of Chevalley (see [34, Corollaire 14.4.4], the morphism f is universally open.

Let $y \in Y$ be a point, and let $B \to Y$ be a morphism from a smooth complete curve B whose image contains y and a general point in Y. Observe that product $B \times_Y X$ is irreducible since the projection $B \times_Y X \to B$ is open with irreducible general fibers. Let Z be a resolution of $(B \times_Y X)_{\mathrm{red}}$. Our claim follows from Proposition 3.11 applied to the pull-back of $\mathscr{L}$ to Z. □

We will also need the following elementary observation.

Lemma 3.13. *Let $f\colon X \to Y$ be a surjective morphism of normal projective varieties with $n := \dim X \geqslant 2$ and $m := \dim Y \geqslant 1$. Let D be a prime $\mathbb{Q}$-Cartier divisor on X such that $\dim f(D) = 0$. Suppose in addition that $D^2 \cdot H^{n-2} \geqslant 0$ for some very ample divisor H on X. Then $\dim Y = \kappa(D) = 1$, and f is induced by the linear system $|mD|$ for m sufficiently large.*

Proof. Let us first show that $\dim Y = 1$. We argue by contradiction and assume that $\dim Y \geqslant 2$. Let $S \subseteq X$ be a two-dimensional complete intersection of general elements of $|H|$, and set $C := S \cap D$. By Bertini's theorem (see [41, Théorème I.6.10]), we may assume without loss of generality that S and C are reduced and irreducible. Then $D^2 \cdot S = (D|_S)^2 < 0$ since C is contracted by the generically finite morphism $f|_S$, yielding a contradiction since $D^2 \cdot S = D^2 \cdot H^{n-2} \geqslant 0$ by assumption. This shows $\dim Y = 1$.

Zariski's lemma then applies to show that some nonzero multiple of $D|_S = C$ is a fiber of $f|_S$. This easily implies that some nonzero multiple of D is a fiber of f, and hence $\kappa(D) = 1$. This finishes the proof of the lemma. □

3.4. Abundance

We will later use the following special case of the abundance conjecture, whose proof relies on an extension theorem proved in [21].

Proposition 3.14. *Let X be a smooth complex projective manifold of dimension $n \geqslant 2$, and let B be a nonzero divisor on X. Let $B := \sum_{i \in I} B_i$ be its decomposition into irreducible components. Suppose that B_i is smooth with $K_{B_i} \equiv 0$ and $B_i^2 \equiv 0$. Suppose moreover that $B_i \cap B_j = \emptyset$ if $i \neq j$. Suppose finally $\kappa(X) \geqslant 0$ and K_X nef. Then there exists a morphism with connected fibers $f\colon X \to C$ onto a smooth complete curve such that B_i is a fiber of f*

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for every $i \in I$. In addition, K_X is semiample, $K_X|_F \sim_{\mathbb{Q}} 0$ for a general fiber F of f and $\kappa(X) = \nu(K_X) \leqslant 1$. In particular, $K_X + B$ is semiample with $\kappa(K_X + B) = \nu(K_X + B) \leqslant 1$.

Proof. Let D be an effective $\mathbb{Q}$-divisor such that $K_X \sim_{\mathbb{Q}} D$. Notice that $D \cdot B_i = K_X \cdot B_i \equiv 0$ since $B_i^2 \equiv 0$ and $K_{B_i} \equiv 0$ by assumption. Thus, there is a decomposition $D = D_1 + D_2$, where D_1 and D_2 are effective $\mathbb{Q}$-divisor with $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2 = \emptyset$ and $\operatorname{Supp} D_2 \subseteq \operatorname{Supp} B$.

Let us show that D_1 is nef. We argue by contradiction and assume that there exists a curve $C \subset X$ such that $D_1 \cdot C < 0$. Then $C \subseteq \operatorname{Supp} D_1$ and hence $C \cap \operatorname{Supp} D_2 = \emptyset$. Thus, $K_X \cdot D = D \cdot C = D_1 \cdot C < 0$, yielding a contradiction. Notice also that B_i is nef since $B_i^2 \equiv 0$ by assumption.

Pick $i_0 \in I$. Then $K_X + B_{i_0} + \varepsilon\big(D_1 + \sum_{i \in I \setminus \{i_0\}} B_i\big)$ is nef as well for every $\varepsilon > 0$. Let $0 < \varepsilon \ll 1$ be a rational number such that the pair $\big(X, B_{i_0} + \varepsilon\big(D_1 + \sum_{i \in I \setminus \{i_0\}} B_i\big)\big)$ is plt. Then [21, Corollary 1.8] applies to show that there exists an effective $\mathbb{Q}$-divisor G_1 such that $K_X + B_{i_0} + \varepsilon\big(D_1 + \sum_{i \in I \setminus \{i_0\}} B_i\big) \sim_{\mathbb{Q}} G_1$ and $B_{i_0} \cap G_1 = \emptyset$ since $\big(K_X + B_{i_0} + \varepsilon\big(D_1 + \sum_{i \in I \setminus \{i_0\}} B_i\big)\big)|_{B_{i_0}} \sim_{\mathbb{Q}} K_{B_{i_0}}$ is torsion. Set $G_2 := D + B_{i_0} + \varepsilon\big(D_1 + \sum_{i \in I \setminus \{i_0\}} B_i\big)$. Then $G_2 \sim_{\mathbb{Q}} G_1$ and $\operatorname{mult}_{B_{i_0}} G_2 \geqslant 1$.

Let m be a positive integer such that mG_1 and mG_2 are $\mathbb{Z}$-divisors, and let $X \dashrightarrow \mathbb{P}^1$ be the corresponding rational map. Let $p\colon Y \to X$ and $q\colon Y \to \mathbb{P}^1$ be a common resolution of singularities. We may assume without loss of generality that p restricts to an isomorphism over some Zariski open neighborhood of B_{i_0} since $B_{i_0} \cap G_1 = \emptyset$ by construction. Let $g\colon Y \to C$ be the Stein factorization of q. By Zariski's lemma, some positive multiple of the strict transform of B_{i_0} in Y is a fiber of g since $B_{i_0}^2 \equiv 0$. This immediately implies that the rational map $f\colon X \dashrightarrow C$ is a morphism. By Zariski's lemma again, for every $i \in I$, some positive multiple of B_i is a fiber of f. Let now F be a general fiber of f. Then, by construction, $\big(K_X + B_{i_0} + \varepsilon\big(D_1 + \sum_{i \in I \setminus \{i_0\}} B_i\big)\big)|_F = K_X|_F$ is torsion. But $K_X|_F \sim_{\mathbb{Z}} K_F$ by the adjunction formula. The statement easily follows from Proposition 3.11. □

3.5. Foliations and rational curves

In order to study foliations on nonminimal varieties it is useful to understand their behavior with respect to rational curves. The following result is a minor generalization of [28, Lemma 2.18].

Lemma 3.15. *Let X be a complex projective manifold of dimension $n \geqslant 2$, and let $\mathscr{F} \subset T_X$ be a regular codimension 1 foliation. Let S be a rational ruled surface, and let $\gamma\colon S \to X$ be a generically finite morphism. Then there exists a rational curve $C \subset S$ which is not contracted by γ such that $\gamma(C)$ is tangent to $\mathscr{F}$.*

Proof. Let $\mathscr{N}$ denote the normal bundle of $\mathscr{F}$, and let $f\colon S \to \mathbb{P}^1$ denote the natural morphism. Recall that $\mathrm{N}^1(S) \cong \mathbb{Z}[C_0] \oplus \mathbb{Z}[F]$, where F is any fiber of f and C_0 is a section of f with $C_0^2 = -e \leqslant 0$.

Set $\mathscr{N}_S := \mathscr{N}|_S$. By Lemma 3.1, we have $c_1(\mathscr{N}_S)^2 = 0$. Write $c_1(\mathscr{N}_S) \equiv aC_0 + bF$. Then

$$0 = c_1(\mathscr{N}_S)^2 = (aC_0 + bF)^2 = a(-ae + 2b),$$

$$c_1(\mathscr{N}_S) \cdot F = a$$

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and

$$c_1(\mathscr{N}_S)\cdot C_0 = -ae + b.$$

We may obviously assume that F is not tangent to $\mathscr{F}$. Then the composition $T_{\mathbb{P}^1} \cong T_F \to T_X|_F \to \mathscr{N}|_F = \mathscr{N}_S|_F$ is nonzero, and hence $a = c_1(\mathscr{N}_S)\cdot F \geqslant 2 > 0$. It follows that $2b = ae$. In particular, we have $b \geqslant 0$.

If C_0 is contracted by γ, then $c_1(\mathscr{N}_S)\cdot C_0 = 0$. This immediately implies $b = e = 0$. Then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and f identifies with the projection onto the first factor. Moreover, the image of a general fiber of the projection onto the second factor is tangent to $\mathscr{F}$.

Suppose from now on that C_0 is not contracted by γ and that $\gamma(C_0)$ is not tangent to $\mathscr{F}$. Then, as above, we must have $c_1(\mathscr{N}_S)\cdot C_0 \geqslant 2$. It follows that $b \geqslant ae + 2$, and hence $b + 2 \leqslant 0$, yielding a contradiction. This finishes the proof of the lemma. □

We will also need the following auxiliary results.

Lemma 3.16. *Let X be a complex projective manifold of dimension $n \geqslant 2$, and let $\mathscr{F} \subset T_X$ be a regular codimension 1 foliation. Let $\varphi\colon X \to Y$ be an elementary Fano–Mori contraction. Suppose that there exists an irreducible component F_1 of a positive-dimensional fiber F of φ which is not tangent to $\mathscr{F}$. Then Y is smooth, and either φ is the blow-up of a smooth codimension 2 submanifold or φ is a conic bundle.*

Proof. Let $\mathscr{N}$ denote the normal bundle of $\mathscr{F}$, and let $C \subseteq F$ be a rational curve. Suppose that $\mathscr{N}\cdot C = 0$. Then C is tangent to $\mathscr{F}$ and moreover $\mathscr{N}|_F \equiv 0$ since φ is elementary by assumption. This in turn implies that any irreducible component of F is tangent to $\mathscr{F}$ since F is rationally chain connected by [38, Corollary 1.4], yielding a contradiction. This shows that $\mathscr{N}\cdot C \neq 0$. On the other hand, $c_1(\mathscr{N})^2 \equiv 0$ by Lemma 3.1. This immediately implies that any positive-dimensional fiber of φ is one-dimensional. The claim now follows from [62, Theorem 1.2]. □

Lemma 3.17. *Let X be a complex projective manifold of dimension $n \geqslant 2$, and let $\mathscr{F}$ be a regular codimension 1 foliation. Suppose that $\mathscr{F}$ is projectively flat, and let $C \subset X$ be a rational curve such that $K_{\mathscr{F}}\cdot C \leqslant 0$. Then $T_X|_C$ is nef. In particular, X is uniruled.*

Proof. Let $\mathscr{N}$ denote the normal bundle of $\mathscr{F}$, and let $\mathbb{P}^1 \to C \subset X$ be the normalization morphism. Since $\mathscr{F}$ is projectively flat, there exists $a \in \mathbb{Z}$ such that $\mathscr{F}|_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus n-1}$. Notice that $a \geqslant 0$ since $K_{\mathscr{F}}\cdot C \leqslant 0$ by assumption.

If C is tangent to $\mathscr{F}$, then $\mathscr{N}|_{\mathbb{P}^1}$ is a flat line bundle by Lemma 3.1, and hence $\mathscr{N}|_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}$. If C is not tangent to $\mathscr{F}$, then the composition $T_{\mathbb{P}^1} \to T_X|_{\mathbb{P}^1} \to \mathscr{N}|_{\mathbb{P}^1}$ is nonzero, and hence $\mathscr{N}\cdot C \geqslant 2$. In either case, the exact sequence

$$0 \to \mathscr{F}|_{\mathbb{P}^1} \to T_X|_{\mathbb{P}^1} \to \mathscr{N}|_{\mathbb{P}^1} \to 0$$

shows that $T_X|_{\mathbb{P}^1}$ is nef. But then X is uniruled by [49, Theorem IV.1.9]. □

3.6. Special cases

In this subsection, we treat a few easy special cases of Theorem 1.1.

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Proposition 3.18. *Setting and notation as in 2.19. Suppose in addition that $K_{\mathscr{F}} \equiv 0$. Then one of the following holds.*

(1) *There exists a $\mathbb{P}^1$-bundle structure $\varphi\colon X \to Y$ onto a finite étale quotient of an abelian variety, and $\mathscr{F}$ induces a flat Ehresmann connection on φ.*

(2) *There exists an abelian variety A and a finite étale cover $\gamma\colon A \to X$ such that $\gamma^{-1}\mathscr{F}$ is a linear foliation on A.*

(3) *There exists a smooth complete curve B of genus at least 2 as well as an abelian variety A, and a finite étale cover $\gamma\colon B \times A \to X$ such that $\gamma^{-1}\mathscr{F}$ is induced by the projection morphism $B \times A \to B$.*

Proof. This follows easily from [56, Théorème 1.2] together with [24, Lemma 5.9]. □

Remark 3.19. In the setup of Proposition 3.18, there exists a finite étale cover $\eta\colon Z \to X$ such that $\eta^{-1}\mathscr{F} \cong \mathscr{O}_Z^{\oplus n-1}$.

Proposition 3.20. *Setting and notation as in 2.19. Suppose in addition that $\mathscr{F}$ is algebraically integrable. Then one of the following holds.*

(1) *There exists an abelian variety A as well as a smooth complete curve C of genus at most 1 and a finite étale cover $\gamma\colon A \times C \to X$ such that $\gamma^{-1}\mathscr{F}$ is induced by the projection $A \times C \to C$.*

(2) *There exists a smooth abelian scheme $f\colon B \to C$ over a smooth complete curve C of genus at least 2 as well as a finite étale cover $\gamma\colon B \to X$ such that $\gamma^{-1}\mathscr{F}$ is induced by f.*

In particular, $K_{\mathscr{F}}$ is semiample with $\kappa(\mathscr{F}) = \nu(\mathscr{F}) \in \{0,1\}$.

Proof. By assumption, $\mathscr{F}$ is induced by a morphism with irreducible fibers $f\colon X \to C$ onto a smooth complete curve C. By [42, Theorem 0.1], any leaf of $\mathscr{F}$ is a finite étale quotient of an abelian variety. By Lemma 3.6 and [48, Theorem 6.3] together, after replacing X by a finite étale cover, the fibration f is birational to an abelian scheme over C. This easily implies that f is an abelian scheme. Replacing X by a further finite étale cover, we may also assume that f is polarized with a level three structure. Let $\mathscr{A}(3)$ be the corresponding fine moduli space of polarized abelian varieties with a level three structure, and let $C \to \mathscr{A}(3)$ be the moduli map. By [49, Lemma 5.9.3], either the moduli map is the constant morphism or C has genus at least 2. The proposition now follows easily. □

Remark 3.21. In the setup of Proposition 3.20 (2), we have $\Omega^1_{B/C} \cong f^*\mathscr{E}$ with $\mathscr{E} := f_*\Omega^1_{B/C}$. Then $\mathscr{F}$ is numerically projectively flat if and only if $\mathscr{E}$ is semistable by Lemma 2.14.

Lemma 3.22. *Setting and notation as in 2.19. Suppose in addition that X is a finite étale quotient of an abelian variety. Then $K_{\mathscr{F}} \equiv 0$.*

Proof. Replacing X by a finite étale cover, if necessary, we may assume without loss of generality that X is an abelian variety. Then $\mathscr{N}$ is generated by its global section. On the

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other hand, we have $\mathscr{N}^2 \equiv 0$ by Lemma 3.1. If $\mathscr{N} \cong \mathscr{O}_X$, then $K_{\mathscr{F}} \equiv 0$ by the adjunction formula. Suppose that $\mathscr{N} \not\equiv 0$. Then, after a further finite étale cover, we have $X \cong A \times E$, where A is an abelian variety of dimension $n-1$ and E is an elliptic curve by Poincaré's complete reducibility theorem. Moreover, $\mathscr{N} \cong f^*\mathscr{M}$ for some line bundle $\mathscr{M}$ on E, where $f\colon X \to E$ denotes the morphism induced projection onto E. Then Lemma 3.23 below applies to show that $K_{\mathscr{F}} \equiv 0$ since $n \geqslant 3$ by assumption. This implies $\mathscr{N} \equiv 0$, yielding a contradiction. This finishes the proof of the lemma. □

Lemma 3.23. *Let C be a smooth complete curve, and let A be an abelian variety of dimension $n-1 \geqslant 1$. Set $X := C \times A$ and $f := \mathrm{pr}_C$. Let $\mathscr{F}$ be a codimension 1 foliation on X with normal bundle $\mathscr{N} \cong f^*\mathscr{L}$ for some line bundle $\mathscr{L}$ on C. Then $\mathscr{F}$ is regular and one of the following conditions holds.*

(1) *We have $\mathscr{F} \cong \mathscr{O}_X^{\oplus n-1}$.*
(2) *We have $\mathscr{F} \cong \mathscr{O}_X^{\oplus n-2} \oplus f^*(T_C \otimes \mathscr{L}^*)$.*
(3) *We have $\mathscr{L} \cong \mathscr{O}_C$, and there is an exact sequence*

$$0 \to \mathscr{O}^{\oplus n-2} \to \mathscr{F} \to f^*T_C \to 0.$$

Proof. Let $\omega \in H^0(X, \Omega^1_X \otimes \mathscr{N})$ be a twisted 1-form defining $\mathscr{F}$. Let $F \cong A$ be a fiber of f. Then $T_X|_F \cong \mathscr{O}_F^{\oplus n}$ and $\mathscr{N}|_F \cong \mathscr{O}_F$. This immediately implies that $\mathscr{F}|_F \cong \mathscr{O}_F^{\oplus n-1}$ and that $\mathscr{F}$ is regular since the zero set of ω has codimension at least 2 in X. Moreover, there exists a vector bundle $\mathscr{E}$ on C such that $\mathscr{F} \cong f^*\mathscr{E}$.

We may obviously assume that $\mathscr{F} \neq T_{X/C}$. If F is a general fiber of f, then $\mathscr{F} \cap T_{X/C}|_F \subset T_F$ is a linear codimension 1 foliation on $F \cong A$. Hence, there exists a subbundle $\mathscr{K} \subset \mathscr{E}$ of rank $n-2$ such that $\mathscr{F} \cap T_{X/C} = f^*\mathscr{K} \subset \mathscr{F}$. Set $g := \mathrm{pr}_A$. Write $\omega = \alpha + \beta \in H^0(X, \Omega^1_X \otimes \mathscr{N}) \cong H^0(X, f^*\omega_C \otimes \mathscr{N}) \oplus H^0(X, g^*\Omega^1_A \otimes \mathscr{N})$ with $\alpha \in H^0(X, f^*\omega_C \otimes \mathscr{N})$ and $\beta \in H^0(X, g^*\Omega^1_A \otimes \mathscr{N}) \cong H^0(A, \Omega^1_A) \otimes H^0(C, \mathscr{L})$. By construction, there is an effective divisor D_1 on C and an exact sequence

$$0 \to \mathscr{K} \to H^0(A, \Omega^1_A)^* \otimes \mathscr{O}_C \to \mathscr{L}(-D_1) \to 0,$$

where the map $H^0(A, \Omega^1_A)^* \otimes \mathscr{O}_C \to \mathscr{L}(-D_1)$ is induced by β. Since $\mathscr{F} \neq T_{X/C}$, the natural map $\mathscr{O}_X^{\oplus n-1} \cong T_{X/C} \to \mathscr{N}$ is nonzero, and hence $h^0(X, \mathscr{L}) \geqslant 1$. Let D_2 be any effective divisor on C such that $\mathscr{L} \cong \mathscr{O}_C(D_2)$. We view ω, α and β as rational 1-forms on X with poles along the divisor f^*D_2. We have $\mathrm{d}\alpha = 0$ and $\mathrm{d}\beta \wedge \alpha = 0$. But then $\mathrm{d}\beta \wedge \beta = \mathrm{d}\omega \wedge \omega = 0$. Let $\gamma_1, \ldots, \gamma_{n-1}$ be a basis of $H^0(A, \Omega^1_A)$, and let $U \subset C \setminus \operatorname{Supp} D_2$ be an open affine subset. Then

$$\beta|_{U \times A} = \sum_{1 \leqslant i \leqslant n-1} t_i \gamma_i$$

for some regular functions t_i on U. Moreover, $\mathrm{d}\beta \wedge \beta = 0$ if and only if $\frac{\mathrm{d}t_i}{t_i} = \frac{\mathrm{d}t_j}{t_j}$ for all indices i and j. Suppose that $t_j \neq 0$. Then $t_i = c_{ij} t_j$ for some $c_{ij} \in \mathbb{C}$. This implies that there exists $s \in H^0(Y, \mathscr{L})$ such that $\beta = \gamma \otimes s$, where $\gamma := \sum_{1 \leqslant i \leqslant n-1} c_{ij} \gamma_i \in H^0(A, \Omega^1_A)$. Then $\operatorname{div} s = D_1$ and $\mathscr{K} \cong \mathscr{O}_C^{\oplus n-2}$.

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Set $\mathscr{Q} := \mathscr{E}/\mathscr{K}$. Then $\mathscr{Q} \cong T_C \otimes \mathscr{L}^*$. If $h^0(C,\mathscr{L}^*) = 0$, then $h^1(C,\mathscr{Q}^*) = h^1(C,\omega_C \otimes \mathscr{L}) = h^0(C,\mathscr{L}^*) = 0$ by Serre duality, and hence the exact sequence

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$$

splits. Then $\mathscr{F} \cong \mathscr{O}_X^{\oplus n-2} \oplus f^*(T_C \otimes \mathscr{L}^*)$.

Suppose finally that $h^0(C,\mathscr{L}^*) \geqslant 1$. Then $\mathscr{L} \cong \mathscr{O}_C$ since we also have $h^0(C,\mathscr{L}) \geqslant 1$. Then $\mathscr{Q} \cong T_C$, finishing the proof of the lemma. □

4. Normal bundles of regular foliations

In this section, we prove Theorem 1.2 and give a few applications to codimension 1 foliations on varieties of general type. If C is a smooth complete curve, then $\pm K_C$ is pseudo-effective. Theorem 1.2 can be seen as the analogous statement for the canonical divisor of the space of leaves of a codimension 1 regular foliation.

We will need the following Bertini-type result.

Lemma 4.1. *Let X be a projective variety of dimension $n \geqslant 2$, and let $Y \subset X$ be a subvariety of codimension $c \geqslant 2$. Let $1 \leqslant k \leqslant c-1$ be an integer, and let $H_1,\dots,H_k$ be very ample divisors on X. Suppose that X is Cohen–Macaulay. Then there exist large enough positive integers $m_1,\dots,m_k$ and dense open subsets $V_i \subseteq |\mathscr{I}_Y(m_iH_i)|$ such that for any tuple $(A_1,\dots,A_k) \in V_1 \times \cdots \times V_k$, the associated complete intersection $S := A_1 \cap \cdots \cap A_k$ is reduced and irreducible. In addition, S is Cohen–Macaulay.*

Proof. To prove the statement, we argue by induction on k.

Suppose $k = 1$. Let $\beta\colon Z \to X$ be the blow-up of X along Y, and let E be the effective divisor on Z corresponding to the invertible sheaf $\beta^{-1}\mathscr{I}_Y \cdot \mathscr{O}_Z$. Notice that Z is reduced and irreducible. Let m_1 be a positive integer such that $\mathscr{I}_Y(m_1H_1)$ is generated by its global sections. Then the linear system $|m_1\beta^*H_1 - E|$ induces a morphism $\varphi_1\colon Z \to \mathbb{P}^{N_1}$, where $N_1 := \dim|\mathscr{I}_Y(m_1H_1)|$. We may also assume without loss of generality that $\dim\varphi_1(Z) \geqslant 2$. By Bertini's theorem (see [41, Théorème I.6.10]), a general member of the linear system $|m_1\beta^*H_1 - E|$ is reduced and irreducible. This immediately implies that a general member A_1 of $|\mathscr{I}_Y(m_1H_1)|$ is generically reduced and irreducible. Since A_1 is Cohen–Macaulay (see [27, Proposition 18.13]), we see that A_1 is reduced by Serre's criterion.

Suppose $k \geqslant 2$. Then the induction hypothesis applied to A_1 and $H_2|_{A_1},\dots,H_k|_{A_1}$ show that there exist positive integers $m_2,\dots,m_k$ and dense open subsets $V_i \subseteq |\mathscr{I}_Y(m_iH_i|_{A_1})|$ such that for any tuple $(B_2,\dots,B_k) \in V_2 \times \cdots \times V_k$, the associated complete intersection $S := B_2 \cap \cdots \cap B_k$ is reduced and irreducible. In addition, S is Cohen–Macaulay. We may also assume without loss of generality that $h^1(X,\mathscr{O}_X(m_iH_i - m_1H_1)) = 0$ for any $i \geqslant 2$. This implies that $|\mathscr{I}_Y(m_iH_i)|_{A_1} = |\mathscr{I}_Y(m_iH_i|_{A_1})|$. The claim now follows easily. □

The proof of Theorem 4.4 below makes use of the following elementary results.

Lemma 4.2. *Let S be a smooth projective surface, and let D be a divisor on S such that $D^2 = 0$. Then $\pm D$ is pseudo-effective.*

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Proof. Let H be an ample divisor on S. Replacing D by $-D$ if necessary, we may assume without loss of generality that $D \cdot H \geqslant 0$. Let $\varepsilon > 0$ be a rational number, and set $D_\varepsilon := D + \epsilon H$. Then

$$D_\varepsilon \cdot H = D \cdot H + \varepsilon H^2 > 0 \quad \text{and} \quad D_\varepsilon^2 = \varepsilon(2D \cdot H + \varepsilon H^2) > 0.$$

Applying [36, Corollary V.1.8], we see that D_ε is $\mathbb{Q}$-effective. This shows that D is pseudo-effective, proving the lemma. □

Lemma 4.3. *Let S be a smooth projective surface, and let D be a pseudo-effective divisor on S with $D^2 = 0$. Suppose in addition that $\kappa(D) \leqslant 1$. Then D is nef.*

Proof. Let $D = P + N$ be the Zariski decomposition of D. Then P is a nef $\mathbb{Q}$-divisor while N is an effective $\mathbb{Q}$-divisor. In addition, $P \cdot N = 0$ and, either $N = 0$ or the intersection matrix $(N_i \cdot N_j)_{i,j \in I}$ is negative definite, where the N_i with $i \in I$ are the irreducible components of N. Now, $\kappa(P) \leqslant \kappa(D) \leqslant 1$ by assumption, and hence $P^2 = 0$. It follows that $0 = D^2 = N^2$, and hence $N = 0$, proving the lemma. □

Theorem 4.4. *Let X be a complex projective manifold of dimension $n \geqslant 1$, and let D be a divisor on X such that $D^2 \equiv 0$. Suppose in addition that D is not pseudo-effective and that $h^0(X, \Omega^1_X(D)) \geqslant 1$. Then $-D$ is nef.*

Proof. If $n = 1$, then $\deg D \leqslant 0$ since D is not pseudo-effective by assumption, and hence $-D$ is $\mathbb{Q}$-effective. Suppose from now on that $n \geqslant 2$. By [5, Theorem 1.5], there exists a smooth projective variety Z, as well as a birational morphism $\beta\colon Z \to X$, and very ample divisors $H_1, \ldots, H_{n-1}$ on Z such that

$$\beta^* D \cdot H_1 \cdots H_{n-1} < 0.$$

Let $C \subset Z$ be a curve. By Lemma 4.1 there exist positive integers $m_1, \ldots, m_{n-2}$ and a tuple $(A_1, \ldots, A_{n-2}) \in |\mathscr{I}_C(m_1 H_1)| \times \cdots \times |\mathscr{I}_C(m_{n-2} H_{n-2})|$ such that the surface $S := A_1 \cap \cdots \cap A_{n-2}$ is reduced and irreducible.

We have $\beta^* D|_S \cdot H_{n-1}|_S < 0$ and $\beta^* D|_S^2 = 0$. Let $S_1 \to S$ be a resolution of singularities. Applying Lemma 4.2 to the pull-back $\beta^* D|_{S_1}$ of $\beta^* D|_S$ on S_1, we conclude that $-\beta^* D|_{S_1}$ is pseudo-effective.

Let $\omega \in H^0(X, \Omega^1_X(D)) \setminus \{0\}$. The twisted 1-form ω yields an injective map $\mathscr{O}_X(-D) \to \Omega^1_X$ of sheaves. We may assume without loss of generality that the composition $\mathscr{O}_X(-D)|_{S_1} \to \Omega^1_X|_{S_1} \to \Omega^1_{S_1}$ is nonzero. By the Bogomolov–Sommese vanishing theorem, we then have $\kappa(\mathscr{O}_X(-D)|_{S_1}) \leqslant 1$. Applying Lemma 4.3, we conclude that $-\beta^* D|_{S_1}$ is nef. In particular, $-\beta^* D \cdot C \geqslant 0$. This shows that $-D$ is nef, completing the proof of the proposition. □

Proof of Theorem 1.2. This is an immediate consequence of Theorem 4.4 since $\mathscr{N}^2 \equiv 0$ by Lemma 3.1. □

The same argument used in the proof of Theorem 4.4 above shows that the following holds.

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Theorem 4.5. *Let X be a complex projective manifold of dimension $n \geqslant 2$, and let $\mathscr{L} \subseteq \Omega^1_X$ be a locally free subsheaf of rank 1. Suppose $\mathscr{L} \cdot H^{n-1} > 0$ and $\mathscr{L}^2 \cdot H^{n-2} = 0$ for some ample divisor H on X. Then $\mathscr{L}$ is nef.*

The following is an immediate consequence of Theorem 4.4. Note that Touzet described the structure of codimension 1 foliations with pseudo-effective conormal bundle in [57].

Proposition 4.6. *Let X be a complex projective manifold, and let $\mathscr{F}$ be a codimension 1 foliation on X with normal bundle $\mathscr{N}$. Suppose $\mathscr{N}^2 \equiv 0$ and $\nu(K_{\mathscr{F}}) \leqslant \nu(K_X) - 1$. Then $\mathscr{N}^*$ is nef.*

Proof. If $\mathscr{N}$ is pseudo-effective, then $\nu(K_{\mathscr{F}}) \geqslant \nu(K_X)$ by [52, Proposition V.2.7] since $K_{\mathscr{F}} = K_X + c_1(\mathscr{N})$ by the adjunction formula, yielding a contradiction. The claim now follows from Theorem 4.4. □

The following result extends [11, Lemme 9] to arbitrary regular codimension 1 foliations on complex projective varieties of general type.

Corollary 4.7. *Let X be a smooth projective variety of general type, and let $\mathscr{F}$ be a regular codimension 1 foliation on X with normal bundle $\mathscr{N}$. Suppose $\nu(K_{\mathscr{F}}) \leqslant \dim X - 1$. Then $\mathscr{N}^*$ is nef.*

Proof. By Lemma 3.1, we have $\mathscr{N}^2 \equiv 0$. The statement follows from Proposition 4.6 above. □

5. Projectively flat foliations on nonminimal varieties

In this section, we address regular codimension 1 foliations with numerically projectively flat tangent bundle on nonminimal complex projective manifolds. The following is the main result of this section.

Theorem 5.1. *Setting and notation as in 2.19. Suppose in addition that $n \geqslant 4$ and that K_X is not nef. Then there exists a $\mathbb{P}^1$-bundle structure $\varphi\colon X \to Y$ onto a finite étale quotient of an abelian variety, and $\mathscr{F}$ induces a flat Ehresmann connection on φ.*

Proof. For the reader's convenience, the proof is subdivided into a number of steps.

Step 1. We will need the following easy observation.

Claim 5.2. Any rational curve on X is generically transverse to $\mathscr{F}$.

Proof. Let C be a rational curve on X, and let $\mathbb{P}^1 \to C$ be the normalization morphism. Suppose that $T_{\mathbb{P}^1} \subset \mathscr{F}|_{\mathbb{P}^1}$. Then $\mathscr{F}|_{\mathbb{P}^1}$ is an ample vector bundle since $\mathscr{F}$ is numerically projectively flat by assumption. But this contradicts Lemma 2.21, proving the claim. □

Let $\varphi\colon X \to Y$ be an elementary Fano–Mori contraction. By Lemma 3.16 and Claim 5.2 together, Y is a smooth variety, and either φ is the blow-up of a smooth codimension 2 submanifold or φ is a conic bundle. The proof of Theorem 5.1 uses the following notation.

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Setup 5.3. Suppose that φ is a singular conic bundle, and let $\Delta := \{y \in Y \,|\, \varphi^{-1}(y) \text{ is not smooth}\}$ be the discriminant locus of φ. The same argument used in the proof of [6, Proposition 1.2] shows that Δ has pure codimension 1. Moreover, Δ has normal crossings in codimension 1. Let $Z_{-2} \subseteq \Delta$ be an irreducible component, and let E_{-1} be the normalization of $E_{-2} := \varphi^{-1}(Z_{-2})$. Then there exists a normal projective variety Z_{-1} as well as morphisms $E_{-1} \to Z_{-1}$ and $Z_{-1} \to Z_{-2}$, and a commutative diagram

$$\begin{array}{ccc} E_{-1} & \xrightarrow{\nu} & E_{-2} \\ {\scriptstyle p_{-1}}\downarrow & & \downarrow{\scriptstyle p_{-2}} \\ Z_{-1} & \xrightarrow{\iota} & Z_{-2}. \end{array}$$

In addition, p_{-1} is a $\mathbb{P}^1$-bundle and ι is a finite morphism of degree 2. By [6, Lemme 1.5.2], there exists an open set $Z^\circ_{-2} \subseteq Z_{-2}$ with complement of codimension at least 2 such that $Z^\circ_{-1} := \iota^{-1}(Z^\circ_{-2})$ is smooth and such that the morphisms $E^\circ_{-1} \to X$ and $Z^\circ_{-1} \to Y$ induced by $\nu|_{E^\circ_{-1}}$ and $\iota|_{Z^\circ_{-1}}$ are unramified, where $E^\circ_{-1} := p_{-1}^{-1}(Z^\circ_{-1})$. Let $Z \to Z_{-1}$ be a quasi-étale cover with Z normal and projective, and let $E := Z \times_{Z_{-1}} E_{-1}$ with natural morphism $p\colon E \to Z$. Let Z° be the inverse image of Z°_{-1} and set $E^\circ := p^{-1}(Z^\circ)$ and $p^\circ := p|_{E^\circ}$. By the Nagata–Zariski purity theorem, the morphism $Z^\circ \to Z^\circ_{-1}$ is étale. As a consequence, the natural morphisms $E^\circ \to X$ and $Z^\circ \to Y$ are unramified.

Suppose now that φ is a $\mathbb{P}^1$-bundle. Set $Z_{-1} := Y$. Let $Z \to Z_{-1}$ be a finite étale cover, and set $E := Z \times_{Z_{-1}} E_{-1}$ with natural morphism $p\colon E \to Z$. Set also $E^\circ := E$ and $Z^\circ := Z$.

Suppose finally that φ is the blow-up of a smooth subvariety Z_{-1} of codimension 2 in Y with exceptional locus E_{-1}. Let $Z \to Z_{-1}$ be a finite étale cover, and set $E := Z \times_{Z_{-1}} E_{-1}$ with natural morphism $p\colon E \to Z$. Set also $E^\circ := E$ and $Z^\circ := Z$.

We obtain a diagram

$$\begin{array}{ccccc} & & \overset{\text{unramified}}{\frown} & & \\ E^\circ & \hookrightarrow & E & \to & X \\ {\scriptstyle p^\circ}\downarrow & & \downarrow{\scriptstyle p,\ \mathbb{P}^1\text{-bundle}} & & \downarrow{\scriptstyle \varphi} \\ Z^\circ & \hookrightarrow & Z & \to & Y. \end{array}$$

By choice of E°, T_{E° is a subbundle of $T_X|_{E^\circ}$.

Step 2. By Claim 5.2, the composition $T_{E/Z} \to T_X|_E \to \mathscr{N}|_E$ is nonzero.

Claim 5.4. To prove Theorem 5.1, it suffices to show that the map $T_{E/Z} \to \mathscr{N}|_E$ is an isomorphism.

Proof. Suppose that the map $T_{E/Z} \to \mathscr{N}|_E$ is an isomorphism, and let F be a fiber of p. Since $\mathscr{F}$ is projectively flat, there exists $a \in \mathbb{Z}$ such that $\mathscr{F}|_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(a)^{\oplus n-1}$. In particular, we have $K_{\mathscr{F}} \cdot F = (n-1)a$. If φ is a singular conic bundle or a blow-up, then $K_X \cdot F = -1$ by choice of F. The adjunction formula $K_{\mathscr{F}} = K_X + c_1(\mathscr{N})$ then gives

$$(n-1)a = K_{\mathscr{F}} \cdot F = K_X \cdot F + c_1(\mathscr{N}) \cdot F = -1 + 2 = 1.$$

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This is impossible and hence φ is a $\mathbb{P}^1$-bundle. Moreover, $\mathscr{F}$ is a flat Ehresmann connection on φ. It follows that T_Y is numerically projectively flat. But then [42, Theorem 0.1] applies to show that Y is a finite étale quotient of an abelian variety. □

We argue by contradiction and assume the following.

Assumption 5.5. The map $T_{E/Z} \to \mathscr{N}|_E$ is not an isomorphism.

There are finitely many prime divisors $(\Sigma_i)_{i\in I}$ on E and positive integers m_i for $i \in I$ such that

$$\mathscr{N}|_E \cong T_{E/Z}(\sum_{i\in I} m_i\Sigma_i).$$

By Claim 5.2 again, the restriction of p to Σ_i is a finite morphism $\Sigma_i \to Z$.

Claim 5.6. Let $i \in I$. Then Σ_i is smooth in codimension 1, and the finite morphism $\Sigma_i \to Z$ is quasi-étale. Moreover, the vector bundles $T_{E/Z}|_{\Sigma_i}$, $T_X|_{\Sigma_i}$, $\mathscr{N}_{\Sigma_i/E}$, $\mathscr{F}|_{\Sigma_i}$ and $\mathscr{N}|_{\Sigma_i}$ are numerically flat. In addition, $\mathscr{O}_E(\Sigma_j)|_{\Sigma_i} \equiv 0$ for all indices $j \in I$.

Proof. Set $\Sigma_i^\circ := \Sigma_i \cap E^\circ$. Let $C \subset Z$ be a complete intersection curve of general elements of very ample linear systems on Z. By general choice of C, we may assume without loss of generality that $C \subset Z^\circ$. Set $S := p^{-1}(C) \subset E^\circ$ and $f := p|_S$. Write $T_f := T_{S/C} = T_{E/Z}|_S$. By Bertini's theorem (see [41, Théorème I.6.10]), we may assume that S is smooth and that the complete curve $B_i := \Sigma_i \cap S$ is reduced and irreducible.

We show that $B_i \to C$ is étale and that

$$T_f \cdot B_i = \mathscr{N} \cdot B_i = B_j \cdot B_i = 0$$

for all indices $j \in I$. Let $\mathscr{E}_C$ be a normalized vector bundle on C such that $S \cong \mathbb{P}_C(\mathscr{E}_C)$. Recall that $\mathrm{N}^1(S) \cong \mathbb{Z}[C_0] \oplus \mathbb{Z}[F]$, where $F \cong \mathbb{P}^1$ is any fiber of f and C_0 is a section of f with $C_0^2 = -e = \deg \mathscr{E}_C$. In addition, $T_f \equiv 2C_0 + eF$, and hence $T_f^2 = 0$. By Lemma 3.1, we have $(\mathscr{N}|_S)^2 = 0$. This easily implies that

$$\mathscr{N}|_S \equiv \frac{\mathscr{N}|_S \cdot F}{2} T_f$$

so that $(\sum_{i\in I} m_i B_i)^2 = 0$.

Let a_i and b_i be integers such that $B_i \equiv a_i C_0 + b_i F$. Then $a_i = B_i \cdot F > 0$. By construction, we have $T_f|_{B_i} \subset \mathscr{F}|_{B_i}$. By Theorem 2.11 and Lemma 2.21 together, we have

$$T_f \cdot B_i = (2C_0 + eF) \cdot (a_i C_0 + b_i F) = -ea_i + 2b_i \leqslant 0.$$

Suppose first that $e \geqslant 0$.

If $I = \{i\}$ and $B_i = C_0$, then $e = 0$ since $0 = (\sum_{i\in I} m_i B_i)^2 = -m_i^2 e$. Moreover, $T_f \cdot B_i = \mathscr{N} \cdot B_i = B_i \cdot B_i = 0$. Otherwise, there exists $j \in I$ such that $B_j \neq C_0$. Then $b_j \geqslant ea_j$ by [36, Proposition 2.20], and hence $e = 0$, $b_j = 0$ and $B_j \equiv a_j C_0$. Hence, we have $T_f \cdot B_i = \mathscr{N} \cdot B_i = B_j \cdot B_i = 0$ for all indices $j \in I$. Moreover, the morphism $B_i \to C$ is étale by [36, Proof of Proposition 2.21].

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Suppose now that $e<0$.

Then $2b_i \leqslant ea_i < 0$. Moreover, $B_i \neq C_0$ since $T_f \cdot C_0 = -e > 0$. Therefore, we have $a_i \geqslant 2$ and $2b_i \geqslant ea_i$ by [36, Proposition 2.21], and hence $2b_i = ea_i$. This implies $B_i \cdot B_j = 0$ for all $j \in I$ and $T_f \cdot B_i = \mathscr{N} \cdot B_i = 0$. In addition, the morphism $B_i \to C$ is étale by [36, Proof of Proposition 2.21].

Since the morphism $B_i \to C$ is étale, we see that Σ_i is smooth in codimension 1 and that $\Sigma_i \to Z$ is quasi-étale. Let $\widetilde{\Sigma}_i$ denote the normalization of Σ_i. Replacing E by $\widetilde{\Sigma}_i \times_Z E$, if necessary, we may assume without loss of generality that Σ_i is a section of p. Then Kleiman's criterion for numerical triviality (see [30, Lemma 4.1]) applies to show that the vector bundles $T_{E/Z}|_{\Sigma_i}$, $\mathscr{N}_{\Sigma_i/E}$ and $\mathscr{N}|_{\Sigma_i}$ are numerically flat. In addition, $\mathscr{O}_E(\Sigma_j)|_{\Sigma_i} \equiv 0$ for all indices $j \in I$. Recall that $T_f|_{B_i} \subset \mathscr{F}|_{B_i}$. By Theorem 2.11 and Lemma 2.21 together, we see that $\mathscr{F}|_{B_i}$ is numerically flat. As a consequence, $K_{\mathscr{F}} \cdot B_i = 0$. By Kleiman's criterion for numerical triviality again, we conclude that $K_{\mathscr{F}}|_{\Sigma_i} \equiv 0$. This in turn implies that $\mathscr{F}|_{\Sigma_i}$ is numerically flat by Theorem 2.11. Since $\mathscr{N}|_{\Sigma_i} \equiv 0$, the vector bundle $T_X|_{\Sigma_i}$ is numerically flat as well, finishing the proof of Claim 5.6. □

Step 3. Let us show Z is a finite quasi-étale quotient of an abelian variety.

By Claim 5.6, the morphisms $\Sigma_i \to Z$ are quasi-étale covers. Let $i \in I$. To lighten notation, set $\Sigma := \Sigma_i$ and $\Sigma^\circ := \Sigma_i^\circ$. Let $\widetilde{\Sigma}$ denote the normalization of Σ. Replacing E by $\widetilde{\Sigma} \times_Z E$, if necessary, we may assume without loss of generality that Σ is a section of p. Set $\mathscr{E} := p_*\mathscr{O}_E(\Sigma)$. Then $E \cong \mathbb{P}_Z(\mathscr{E})$ and there is an exact sequence

$$0 \to \mathscr{O}_Z \to \mathscr{E} \to \mathscr{L} \to 0,$$

where $\mathscr{L} \cong \mathscr{N}_{\Sigma/E}$ is a flat line bundle on Z by Claim 5.6. Notice that $T_{E/Z} \cong \mathscr{O}_E(2\Sigma) \otimes p^*\mathscr{L}^*$.

Let F be any fiber of p. Notice that $\mathscr{N}|_E \equiv cT_{E/Z} \equiv 2c\Sigma$ with $c := \frac{\mathscr{N}\cdot F}{2}$ since $\mathscr{N}|_E \cdot F = cT_{E/Z} \cdot F$ and $\mathscr{N}|_\Sigma \equiv T_{E/Z}|_\Sigma \equiv 0$ by Claim 5.6.

Let now k be an integer such that $\mathscr{F}(k\Sigma)|_F \cong \mathscr{O}_F^{\oplus n-1}$. Then $\mathscr{F}|_E \cong p^*\mathscr{K} \otimes \mathscr{O}_E(-k\Sigma)$ for some vector bundle $\mathscr{K}$ on Z. Since $\mathscr{F}|_\Sigma$ and $\mathscr{N}_{\Sigma/E}$ are numerically flat by Claim 5.6, the locally free sheaf $\mathscr{K}$ is numerically flat as well. This immediately implies that

$$\begin{aligned} c_1(\mathscr{F}|_E) &= p^*c_1(\mathscr{K}) - k(n-1)\Sigma \\ &\equiv -k(n-1)\Sigma \end{aligned}$$

and

$$\begin{aligned} c_2(\mathscr{F}|_E) &= p^*c_2(\mathscr{K}) - (n-2)k\Sigma \cdot p^*c_1(\mathscr{K}) + \frac{(n-1)(n-2)}{2}k^2\Sigma^2 \\ &\equiv 0 \end{aligned}$$

since $\Sigma^2 \equiv 0$. The exact sequence

$$0 \to \mathscr{F}|_E \to T_X|_E \to \mathscr{N}|_E \to 0$$

then gives

$$\begin{aligned} c_1(T_X|_E) &= c_1(\mathscr{F}|_E) + c_1(\mathscr{N}|_E) \\ &\equiv \Sigma \end{aligned}$$

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using $K_X \cdot F = -1$, and

$$\begin{aligned} c_2(T_X|_E) &= c_2(\mathscr{F}|_E) + c_1(\mathscr{F}|_E) \cdot c_1(\mathscr{N}|_E) \\ &\equiv 0 \end{aligned}$$

using $c_1(\mathscr{N}) \cdot \Sigma \equiv 0$. In particular, there exists a flat line bundle $\mathscr{M}$ on Z such that

$$\mathscr{N}|_E \cong p^*\mathscr{M} \otimes \mathscr{O}_E((k(n-1)+1)\Sigma).$$

By the projection formula, there exist rational numbers a_1, a_2 and a_3 such that

$$p_* c_2(T_X|_E) = a_1 c_1(\mathscr{K}) + a_2 c_1(\mathscr{M}) + a_3 c_1(\mathscr{N}_{\Sigma/E}) \tag{5.1}$$

as cycle classes in $\mathrm{CH}^1(Z)$. The exact sequence

$$0 \to T_{E^\circ/Z^\circ} \to T_{E^\circ} \to (p^\circ)^* T_{Z^\circ} \to 0$$

gives

$$\begin{aligned} c_1(T_{E^\circ}) &= 2\Sigma^\circ - (p^\circ)^* c_1(\mathscr{L}|_{Z^\circ}) + (p^\circ)^* c_1(T_{Z^\circ}) \\ &\equiv 2\Sigma^\circ + (p^\circ)^* c_1(T_{Z^\circ}) \end{aligned}$$

and

$$\begin{aligned} c_2(T_{E^\circ}) &= c_1(T_{E^\circ/Z^\circ}) \cdot (p^\circ)^* c_1(T_{Z^\circ}) + (p^\circ)^* c_2(T_{Z^\circ}) \\ &= (2\Sigma^\circ - (p^\circ)^* c_1(\mathscr{L}|_{Z^\circ})) \cdot (p^\circ)^* c_1(T_{Z^\circ}) + (p^\circ)^* c_2(T_{Z^\circ}). \end{aligned}$$

Notice that the cokernel $\mathscr{N}_{E^\circ/X^\circ}$ of the tangent map $T_{E^\circ} \to T_X|_{E^\circ}$ is locally free since the morphism $E^\circ \to X$ is unramified by choice of E°. Then we have

$$\begin{aligned} c_2(T_X|_{E^\circ}) &= c_2(T_{E^\circ}) + c_1(T_{E^\circ}) \cdot c_1(\mathscr{N}_{E^\circ/X^\circ}) \\ &= c_2(T_{E^\circ}) + c_1(T_{E^\circ}) \cdot (c_1(T_X|_{E^\circ}) - c_1(T_{E^\circ})) \\ &\equiv 2\Sigma^\circ \cdot (p^\circ)^* c_1(T_{Z^\circ}) + (p^\circ)^* c_2(T_{Z^\circ}) + (2\Sigma^\circ + (p^\circ)^* c_1(T_{Z^\circ})) \\ &\quad \cdot (-\Sigma^\circ - (p^\circ)^* c_1(T_{Z^\circ})) \\ &\equiv -\Sigma^\circ \cdot (p^\circ)^* c_1(T_{Z^\circ}) + (p^\circ)^* c_2(T_{Z^\circ}) - (p^\circ)^* c_1(T_{Z^\circ})^2. \end{aligned}$$

By the projection formula again, there exist rational numbers b_1, b_2, b_3 et b_4 such that

$$(p^\circ)_* c_2(T_X|_{E^\circ}) = -c_1(T_{Z^\circ}) + b_1 c_1(\mathscr{K}|_{Z^\circ}) + b_2 c_1(\mathscr{M}|_{Z^\circ}) + b_3 c_1(\mathscr{L}|_{Z^\circ}) + b_4 c_1(\mathscr{N}_{\Sigma^\circ/E^\circ}) \tag{5.2}$$

as cycle classes in $\mathrm{CH}^1(Z^\circ)$. Equations (5.1) and (5.2) together then show that K_Z is $\mathbb{Q}$-Cartier using the fact that $Z \setminus Z^\circ$ has codimension at least 2 in Z. Moreover, $K_Z \cdot C = 0$ for any complete intersection curve C of general elements of very ample linear systems on Z so that Kleiman's criterion for numerical triviality (see [30, Lemma 4.1]) applies to show that $K_Z \equiv 0$.

Suppose that φ is a $\mathbb{P}^1$-bundle. Then T_Σ is numerically projectively flat since both $T_X|_\Sigma$ and $\mathscr{N}_{\Sigma/X}$ are numerically projectively flat by Claim 5.6 above. This implies that T_Y is numerically projectively flat as well. Then [42, Theorem 0.1] applies to show that $Y = Z$ is a finite étale quotient of an abelian variety.

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Suppose now that φ is a singular conic bundle or a blow-up. By Lemma 3.15 and Claim 5.2 together, Z contains no rational curve. Then [5, Corollary 0.3] applied to a resolution of Z shows that Z has canonical singularities. By [22, Proposition 1.16], the map $\det T_\Sigma \to \wedge^{n-2}(T_X|_\Sigma)$ does not vanish anywhere since $T_X|_\Sigma$ is numerically flat by Claim 5.6. This immediately implies that T_Σ is locally free and numerically flat. By the solution of the Zariski–Lipman conjecture for klt spaces (see [29, Theorem 6.1] or [23, Theorem 1.1]), we see that $Z \cong \Sigma$ is smooth. Moreover, by [22, Corollary 1.19], we have $c_2(Z) = 0$. As a classical consequence of Yau's theorem on the existence of a Kähler–Einstein metric, Z is then covered by a complex torus (see [47, Chapter IV Corollary 4.15]).

This shows that Z is a quasi-étale quotient of an abelian variety.

Step 4. Finally, let us obtain a contradiction.

Claim 5.7. The vector bundle $\mathscr{F}|_E$ is not numerically flat.

Proof. Suppose first that φ is a $\mathbb{P}^1$-bundle. Using Claim 5.2, we then conclude that $\mathscr{F}$ induces a flat Ehresmann connection on p. This immediately implies that the map $T_{E/Z} \to \mathscr{N}|_E$ is an isomorphism, which contradicts Assumption 5.5.

Suppose now that φ is a singular conic bundle or a blow-up. Then $T_X|_F \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}^{n-2} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. But this is impossible by Lemma 3.17, proving our claim. □

By Step 3, replacing Z by a finite quasi-étale cover, if necessary, we may also assume that Z is an abelian variety. Recall also that $E \cong \mathbb{P}_Z(\mathscr{E})$, where $\mathscr{E}$ is numerically flat. In particular, $\mathscr{E}$ is flat by the Corlette—Simpson correspondence (see Theorem 2.3). As a consequence, there is a flat Ehresmann connection $\mathscr{H} \cong \mathscr{O}_E^{\oplus n-2}$ on p.

Let $\omega \in H^0(X, \Omega_X^1 \otimes \mathscr{N})$ be a twisted 1-form defining $\mathscr{F}$, and let $\mathscr{G}$ be the foliation on E induced by $\mathscr{F}$. Suppose that $\omega|_E \in H^0(E, \Omega_E^1 \otimes \mathscr{N}|_E)$ vanishes at some point $x \in E$. Then $x \in \Sigma_i$ for some $i \in I$. In addition, $\omega|_E$ vanishes along Σ_i since $\Omega_E^1 \otimes \mathscr{N}|_{\Sigma_i}$ is numerically flat (see [22, Proposition 1.16]). Repeating the argument, we see that there exist integers $0 \leqslant k_i \leqslant m_i$ such that $\mathscr{N}_\mathscr{G} \cong T_{E/Z}(\sum_{i \in I} k_i \Sigma_i)$ and that $\mathscr{G}$ is regular.

Suppose first that $\mathscr{O}_E^{\oplus n-2} \cong \mathscr{H} = \mathscr{G} \subset T_E$. Then $\mathscr{F}|_E$ is numerically flat by Theorem 2.11 and Lemma 2.21 together. But this contradicts Claim 5.7.

Suppose from now on that $\mathscr{H} \neq \mathscr{G} \subset T_E$, and let $x \in X$ be a general point. Then there exists $v \in H^0(E, \mathscr{H})$ with $v(x) \in \mathscr{G}_x \setminus \{0\}$ since $n \geqslant 3$. If $h^0(E, \mathscr{F}|_E) \geqslant 1$, then $\mathscr{F}|_E$ is numerically flat by Theorem 2.11 and Lemma 2.21 together, which is impossible by Claim 5.7 again. Thus, we have $h^0(E, \mathscr{F}|_E) = 0$. But then v induces a nonzero section $s \in H^0(E, \mathscr{N}|_E)$ vanishing at x. Let $\sum_{j \in J} d_j \Lambda_j$ be the zero divisor of s. Since $\mathscr{N}|_E \equiv 2c\Sigma$ is nef and $\Sigma^2 \equiv 0$ we see that $\Lambda_j \cdot \Sigma \equiv 0$. This easily implies $\Lambda_j^2 \equiv 0$ since $\mathscr{E}$ is numerically flat. Arguing as in the proof of Claim 5.6, we conclude that Λ_j is smooth in codimension 1 and that $\Lambda_j \to Z$ is quasi-étale. By Serre's criterion for normality, Λ_j is normal and $\Lambda_j \to Z$ is étale by Nagata–Zariski purity theorem. Let $j \in J$ such that $x \in \Lambda_j$. Then $\Lambda_j \cap \Sigma = \emptyset$ since $x \notin \Sigma$. Replacing Z by a finite étale cover, if necessary, we may therefore assume that p has 3 pairwise disjoint sections. Then $E \cong \mathbb{P}^1 \times Z$. In addition, $\mathscr{N}_\mathscr{G} \cong q^*\mathscr{R}$ for some line bundle $\mathscr{R}$ on $\mathbb{P}^1$, where $q := \mathrm{pr}_{\mathbb{P}^1}$. By Lemma 3.23, we have $h^0(E, \mathscr{F}|_E) \geqslant$

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$h^0(E,\mathscr{G}) \geqslant 1$ since $n \geqslant 4$ by assumption, yielding a contradiction. This finishes the proof of Theorem 5.1. □

6. Abundance

In this section, we prove a special case of the abundance conjecture (see Proposition 6.3 below). More precisely, we prove that if the pair $(X,\mathscr{F})$ satisfies all the conditions listed in Setup 2.19, then $\kappa(X) = \nu(X)$. First, we consider foliations with projectively trivial tangent bundle. The proof of Proposition 6.1 below follows an argument that goes back at least as far as [12, Chapter 9].

Proposition 6.1. *Let X be a complex projective manifold of dimension $n \geqslant 3$, and let $\mathscr{F}$ be a regular codimension 1 foliation. Suppose that $\mathscr{F} \cong \mathscr{L}^{\oplus n-1}$ for some line bundle $\mathscr{L}$. Then $\mathscr{L}^*$ is semiample with $\nu(\mathscr{L}^*) \leqslant 1$.*

Proof. For the reader's convenience, the proof is subdivided into a number of steps. Notice that $\mathscr{L}^*$ is nef by Lemma 2.21 and that $\nu(\mathscr{L}^*) \leqslant 1$ by Proposition 3.2. Therefore, in order to prove the proposition, it suffices to show that $\kappa(\mathscr{L}^*) = \nu(\mathscr{L}^*)$ (see [43, Proposition 2.1]). Notice also that $\kappa(\mathscr{L}^*) \leqslant \nu(\mathscr{L}^*)$ by [43, Proposition 2.2].

Step 1. We show that $\kappa(\mathscr{L}^*) \geqslant 0$.

Suppose first that $q(X) > 0$.

If $h^0(X,\mathscr{L}^*) \geqslant 1$, then there is nothing to show. Suppose now that $h^0(X,\mathscr{L}^*) = 0$. Let $a\colon X \to A$ be the Albanese morphism, and let $f\colon X \to Y$ be its Stein factorization. Notice that $\dim Y \geqslant 1$ since $q(X) > 0$ by assumption. Now, the standard pull-back map of Kähler differentials

$$H^0(X,\Omega^1_X) \otimes \mathscr{O}_X \cong a^*\Omega^1_A \to \Omega^1_X$$

identifies with the evaluation map. It follows that $\mathscr{F}$ is contained in the kernel of the tangent map

$$Ta\colon T_X \to a^*T_A$$

since $h^0(X,\mathscr{L}^*) = 0$ by our current assumption. Thus, $\dim Y = 1$, and $\mathscr{F}$ is induced by the fibration f. Let F be a general fiber of f. By [42, Theorem 0.1], F is a finite étale quotient of an abelian variety. In particular, $\mathscr{L}|_F$ is a torsion line bundle. This easily implies that there exists a line bundle $\mathscr{M}$ on Y and a positive integer m such that $\mathscr{L}^{\otimes m} \cong f^*\mathscr{M}$ (see also Proposition 3.11). Notice that $\mathscr{M}^*$ is nef since $\mathscr{L}^*$ is nef. But then $\kappa(\mathscr{M}^*) \geqslant 0$ and hence $\kappa(\mathscr{L}^*) \geqslant 0$.

Suppose now that $q(X) = 0$.

If $\mathscr{L} \equiv 0$, then $\mathscr{L}$ is a torsion line bundle, and hence $\kappa(\mathscr{L}^*) = 0$. Suppose now that $\mathscr{L} \not\equiv 0$. From Proposition 3.5, we get $\chi(X,\mathscr{O}_X) = 0$ and hence $h^p(X,\mathscr{O}_X) > 0$ for some $p > 0$. By Hodge symmetry, we then have $h^0(X,\Omega^p_X) > 0$. Using the exact sequence

$$0 \to \mathscr{N}^* \to \Omega^1_X \to \mathscr{F}^* \to 0$$

we see that $h^0\big(X,(\mathscr{N}^*)^{\otimes u} \otimes (\mathscr{L}^*)^{\otimes v}\big) > 0$ for some nonnegative integers u and v such that $u + v = p$. By Proposition 3.2, there exist integers $r > 0$ and s such that $\mathscr{N}^{\otimes r} \cong$

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$\mathscr{L}^{\otimes s}$. It follows that $h^0\big(X,(\mathscr{L}^*)^{\otimes su+rv}\big)>0$. Since $\mathscr{L}^*$ is nef and $\mathscr{L}^*\not\equiv 0$, we must have $su+rv\geqslant 0$. This shows that $\kappa(\mathscr{L}^*)\geqslant 0$ in the case where $q(X)=0$.

Step 2. Write

$$-c_1(\mathscr{L})\sim_{\mathbb{Q}}\sum_{i\in I}b_iB_i \tag{6.1}$$

with $b_i\in\mathbb{Q}_{>0}$. If $\mathscr{L}\equiv 0$, then $\mathscr{L}$ is a torsion line bundle since $\kappa(\mathscr{L}^*)\geqslant 0$ by Step 1, proving the proposition in this case. Suppose from now on that $\mathscr{L}\not\equiv 0$. In other words, assume that I is not empty. Then Proposition 3.4 applies to show that for any index $i\in I$, B_i is a finite étale quotient of an abelian variety. Moreover, $B_i\cap B_j=\emptyset$ if $i\neq j$ and $\mathscr{N}_{B_i/X}$ is a flat line bundle. Notice that $\mathscr{L}|_{B_i}$ is a flat line bundle as well by Equation (6.1).

Step 3. Suppose now that the composition $\mathscr{L}^{\oplus n-1}|_{B_i}\cong\mathscr{F}\to T_X|_{B_i}\to\mathscr{N}_{B_i/X}$ is nonzero. In other words, assume that B_i is not a leaf of $\mathscr{F}$. Then there exists a foliation by curves $\mathscr{G}\cong\mathscr{L}\subset T_X$ such that B_i is everywhere transverse to $\mathscr{L}$ since both $\mathscr{L}|_{B_i}$ and $\mathscr{N}_{B_i/X}$ are flat line bundles. In particular, we have $\mathscr{N}_{B_i/X}\cong\mathscr{L}|_{B_i}$. Moreover, there is a section $s\in H^0(X,\mathscr{O}_X(mK_{\mathscr{G}}))$ for some integer $m\geqslant 1$ that vanishes on B_i and nowhere else in some analytic open neighborhood of B_i by Equation 6.1 again. Arguing as in the proof of [12, Theorem 9.2] on page 112, we conclude that there exists a fibration $f\colon X\to C$ onto a smooth complete curve C such that B_i is a (set-theoretic) fiber of f. This easily implies that $\kappa(\mathscr{L}^*)=\nu(\mathscr{L}^*)=1$.

Step 4. Suppose finally that B_i is a leaf of $\mathscr{F}$ for all indices $i\in I$. Then $T_{B_i}\cong\mathscr{F}|_{B_i}\cong(\mathscr{L}|_{B_i})^{\oplus n-1}$ so that Lemma 6.2 below applies to show that B_i is an abelian variety and $\mathscr{L}|_{B_i}\cong\mathscr{O}_{B_i}$. Moreover, $\mathscr{N}_{B_1/X}$ is a flat vector bundle by Lemma 3.1.

Let m be a positive integer such that $-mc_1(\mathscr{L})\sim_{\mathbb{Z}}\sum_{i\in I}mb_iB_i=:B$. In particular, mb_i is an integer. Let $\gamma\colon X_1\to X$ be the associated cyclic cover (see [45, Definition 2.52]) which is étale away from B. Notice that X_1 is smooth. Set $B_1:=\gamma^{-1}(B)$. By [25, Lemma 3.4(1)], the foliation $\mathscr{F}_1$ induced by $\mathscr{F}$ on X_1 is regular and $\mathscr{F}_1\cong\gamma^*\mathscr{F}$. Using [58, Lemma 5.12], one readily checks that $\mathscr{L}^*$ is semiample if and only if so is $\gamma^*\mathscr{L}^*$. Moreover, $\nu(\mathscr{L}^*)=\nu(\gamma^*\mathscr{L}^*)$. Therefore, replacing X by X_1, we may assume without loss of generality that $-c_1(\mathscr{L})\sim_{\mathbb{Z}}\sum_{i\in I}b_iB_i=:B$ with $b_i\in\mathbb{Z}$.

Suppose first that $q(X)>0$.

Let $a\colon X\to A$ be the Albanese morphism, and let $f\colon X\to Y$ be its Stein factorization. By assumption, we have $\dim Y\geqslant 1$.

Suppose that $\dim f(B_i)\geqslant 1$ for some $i\in I$. Then there exists a global 1-form on X whose restriction to $\mathscr{F}|_{B_i}$ is nonzero at a general point in B_i. This implies $\kappa(\mathscr{L}^*)=\nu(\mathscr{L}^*)=1$.

Suppose now that $\dim f(B_i)=0$ for any $i\in I$. Then Lemma 3.13 applies to show that $\dim Y=1$. Now, by Zariski's lemma, some nonzero multiple of B_i is a fiber of f since $B_i^2\equiv 0$. This again implies $\kappa(\mathscr{L}^*)=\nu(\mathscr{L}^*)=1$.

Suppose finally that $q(X)=0$.

Then $\mathscr{O}_X(b_iB_i)|_{B_i}\cong\mathscr{L}^*|_{B_i}\cong\mathscr{O}_{B_i}$. In other words, $\mathscr{O}_X(B_i)|_{B_i}$ is torsion, say of order m_i. In particular, for any $1\leqslant k\leqslant m_i-1$, $h^1(B_i,\mathscr{O}_X(kB_i)|_{B_i})=0$. The short exact sequence

$$0\to\mathscr{O}_X((k-1)B_i)\to\mathscr{O}_X(kB_i)\to\mathscr{O}_X(kB_i)|_{B_i}\to 0$$

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then gives $h^1(X,\mathscr{O}_X(kB_i))=0$ for all $1\leqslant k\leqslant m_i-1$. It follows that the restriction map

$$H^0(X,\mathscr{O}_X(m_iB_i))\to H^0(X,\mathscr{O}_X(m_iB_i)|_{B_i})\cong H^0(X,\mathscr{O}_{B_i})$$

is surjective. This implies $\kappa(\mathscr{L}^*)\geqslant\kappa(\mathscr{O}_X(B_i))\geqslant 1$, and hence $\kappa(\mathscr{L}^*)=\nu(\mathscr{L}^*)=1$. □

Lemma 6.2. *Let X be a compact Kähler manifold of dimension $n\geqslant 2$. Suppose that $T_X\cong\mathscr{L}^{\oplus n}$ for some line bundle $\mathscr{L}$ on X. Then X is a torus.*

Proof. By [31, Proposition 4.3], $c_1(\mathscr{L})=0$. If $q(X)=0$, then $\mathscr{L}$ is torsion. Otherwise, we have $\mathscr{L}\cong\mathscr{O}_X$. In either case, $\mathscr{L}$ is torsion, say of order m. Let $\gamma\colon Y\to X$ be the associated cyclic cover (see [45, Definition 2.52]) which is étale with Galois group $G=\langle\tau\rangle$. By construction, $T_Y\cong\gamma^*T_X\cong\mathscr{O}_Y^{\oplus n}$. By a result proved by Wang ([61, Corollary 2]), Y is a complex torus, $Y=V/\Lambda$, where V is complex vector space of dimension n and $\Lambda\subset V$ is a lattice. Set $y_0:=\tau(0)$. There exists a $\mathbb{C}$-linear map $L\colon V\to V$ satisfying $L(\Lambda)\subseteq\Lambda$ such that $\tau=t_{y_0}\circ\overline{L}$, where $\overline{L}$ is the automorphism of Y induced by L. Notice that the linear map L can be identified with τ_* acting on $H^0(Y,T_Y)\cong V$. Since $T_Y\cong\gamma^*T_X\cong(\gamma^*\mathscr{L})^{\oplus n}$, we see that $L=\lambda\mathrm{Id}$ for some $\lambda\in\mathbb{C}$ with $\lambda^m=1$. By [8, Lemma 13.1.1], the set of fixed points of $\overline{L}$ is a positive-dimensional analytic subvariety since τ has no fixed point. But this immediately implies that $\lambda=1$. As a consequence, τ is a translation by a point of finite order. This in turn implies that X is torus and that $\mathscr{L}\cong\mathscr{O}_X$. □

The following is the main result of this section.

Proposition 6.3. *Setting and notation as in 2.19. Suppose in addition that K_X is pseudo-effective. Then K_X is abundant.*

Proof. Let $\mathrm{H}\subseteq\mathrm{PGL}(n-1,\mathbb{C})$ be the Zariski closure of $\rho(\pi_1(X))$. This is a linear algebraic group which has finitely many connected components. Applying Selberg's lemma and passing to an appropriate finite étale cover of X, we may assume without loss of generality that H is connected and that the image of the induced representation

$$\rho_1\colon\pi_1(X)\to\mathrm{H}\to\mathrm{H}/\mathrm{Rad}(\mathrm{H})$$

is torsion-free, where Rad(H) denotes the radical of H. Let

$$\mathrm{sh}_{\rho_1}\colon X\dashrightarrow Y$$

be the ρ_1-Shafarevich map, where sh_{ρ_1} is dominant and Y is a smooth projective variety. The rational map sh_{ρ_1} is almost proper with connected fibers. By [14, Théorème 1], we may assume without loss of generality that Y is of general type and that the representation ρ_1 factorizes through sh_{ρ_1}. If $\dim Y=\dim X$, then X is of general type and there is nothing to show. Suppose from now on that $\dim Y<\dim X$.

Let F be a general (smooth) fiber of sh_{ρ_1}. By [26, Proposition 2.23], in order to prove that K_X is abundant, it suffices to prove that K_F is abundant.

Suppose first $\dim Y=0$ and $q(X)=0$.

Since $\rho(\pi_1(X))\subseteq\mathrm{Rad}(\mathrm{H})$, the restriction of ρ to a finite index subgroup of $\pi_1(X)$ is the trivial representation. Thus, there exists a finite étale cover $\gamma\colon Y\to X$, as well as a line

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bundle $\mathscr{L}$ on Y such that $\gamma^{-1}\mathscr{F} \cong \mathscr{L}^{\oplus n-1}$. Then Proposition 6.1 applies to show that $\mathscr{L}^*$ is semiample. This in turn implies that $\mathscr{O}_X(K_{\mathscr{F}})$ is semiample ([58, Lemma 5.12]).

If $K_{\mathscr{F}} \equiv 0$, then our claim follows from Proposition 3.18.

Suppose now that $K_{\mathscr{F}} \not\equiv 0$. By Proposition 3.2 and using the fact that $q(X) = 0$, there exist integers $r > 0$ and s such that $\mathscr{N}^{\otimes r} \cong \mathscr{O}_X(K_{\mathscr{F}})^{\otimes s}$. The claim now follows easily from the adjunction formula $\mathscr{O}_X(K_X) \cong \mathscr{O}_X(K_{\mathscr{F}}) \otimes \mathscr{N}^*$.

Suppose from now on that either $\dim Y > 0$ or $q(X) > 0$.

Let $a\colon F \to \mathrm{A}$ be the Albanese morphism. By construction, we have $\dim \mathrm{A} = q(F)$. Let G be a general (smooth) fiber of the Stein factorization of $F \to a(F)$. By [40, Theorem 1.1], in order to prove that K_F is abundant, it suffices to prove that K_G is abundant. Thus, we may assume without loss of generality that $\dim G > 0$. Repeating the process finitely many times, we may also assume that $q(G) = 0$. Since $\rho(\pi_1(G)) \subseteq \mathrm{Rad}(\mathrm{H})$, the restriction of ρ to a finite index subgroup of $\pi_1(G)$ is the trivial representation.

Let $U \subseteq X$ be a neighborhood of G in X for the analytic topology. Shrinking U, if necessary, we may assume that there exists a smooth projective morphism with connected fibers $f\colon U \to T$ onto a simply connected complex manifold T such that G is a fiber of f. Then $\mathscr{N}_{G/U} \cong \mathscr{O}_G^{\oplus \dim X - \dim G}$. Moreover, we have $\dim G < \dim X$ because $\dim Y > 0$ or $q(X) > 0$ by our current assumption. Since $\rho(\pi_1(G))$ is finite, there exists a finite étale cover $g\colon U_1 \to U$ such that the inverse image G_1 of G in U_1 is connected and such that $\mathscr{F}_{U_1} := g^{-1}(\mathscr{F}|_U) \cong \mathscr{L}_{U_1}^{\oplus n-1}$ for some line bundle $\mathscr{L}_{U_1}$ on U_1.

If G_1 is a leaf of $\mathscr{F}_{U_1}$, then G_1 is an abelian variety by Lemma 6.2. This in turn implies that K_G is torsion.

Suppose from now on that either $\dim G_1 \leqslant n-2$ or that $\dim G_1 = n-1$ and G_1 is not a leaf of $\mathscr{F}_{U_1}$. Then the composition

$$\mathscr{F}_{U_1}|_{G_1} \cong \mathscr{L}_{U_1}|_{G_1}^{\oplus n-1} \to T_{U_1}|_{G_1} \to \mathscr{N}_{G_1/U_1} \cong \mathscr{O}_{G_1}^{\oplus \dim X - \dim G}$$

is nonzero, and hence $h^0(G_1, \mathscr{L}_{G_1}^*) \geqslant 1$, where $\mathscr{L}_{G_1} := \mathscr{L}_{U_1}|_{G_1}$.

Suppose that $\mathscr{L}_{G_1} \cong \mathscr{O}_{G_1}$. If $T_{G_1} \subset \mathscr{F}_{U_1}|_{G_1} \cong \mathscr{O}_{G_1}^{\oplus n-1}$, then $\Omega^1_{G_1}$ is generated by its global sections and hence K_{G_1} is abundant. It follows from [58, Theorem 5.13] that K_G is abundant as well. Suppose now that $T_{G_1} \not\subset \mathscr{F}_{U_1}|_{G_1}$. Then $T_{G_1} \cap (\mathscr{F}_{U_1}|_{G_1}) \subset T_{G_1}$ is a codimension 1 foliation with trivial tangent bundle. Then K_{G_1} is abundant by Proposition 3.18, using the fact that $K_{G_1} = K_X|_{G_1}$ is pseudo-effective. Again, by [58, Theorem 5.13], we conclude that K_G is abundant.

Suppose finally that $\mathscr{L}_{G_1} \not\equiv 0$. By Lemma 2.21, $\mathscr{L}_{G_1}^*$ is nef. Moreover, by Proposition 3.2, there is a number $\lambda \in \mathbb{Q}$ such that $K_{G_1} \equiv \lambda c_1(\mathscr{L}_{G_1})$. Observe that $\lambda \leqslant 0$ since $K_G = K_X|_G$ is pseudo-effective. Since $h^0(G_1, \mathscr{L}_{G_1}^*) \geqslant 1$, we see that $\kappa(G_1) \geqslant 0$ by [15, Corollary 3.2]. Notice also that K_{G_1} and hence K_G are nef. Then [58, Theorem 5.13] implies that $\kappa(K_G) \geqslant 0$. Write $K_G \sim_{\mathbb{Q}} \sum_{i\in I} b_i B_i$ with $b_i \in \mathbb{Q}_{>0}$, B_i irreducible and $B_i \neq B_j$ if $i \neq j$. By Proposition 3.4 applied to $K_{G_1} = (g|_{G_1})^* K_G \sim_{\mathbb{Q}} \sum_{i\in I} b_i (g|_{G_1})^* B_i$, B_i is a finite étale quotient of an abelian variety and $B_i \cap B_j = \emptyset$ if $i \neq j$. Moreover, we have $B_i^2 \equiv 0$. Then Proposition 3.14 applies to show that K_G is abundant. This finishes the proof of the proposition. □

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7. Projectively flat foliations with nef conormal bundle

In this section, we address codimension 1 regular foliations with numerically projectively flat tangent sheaf and nef conormal bundle. The proof of Theorem 7.2 below makes use of the following auxiliary result.

Lemma 7.1. *Setting and notation as in 2.19. Suppose that X is an abelian scheme over a smooth projective base Y of dimension $m := \dim Y \leqslant n-1$, and denote by $f\colon X \to Y$ the natural morphism. Suppose in addition that $\mathscr{N} \cong f^*\mathscr{M}$ for some line bundle $\mathscr{M}$ on Y and that $T_{X/Y} \not\subseteq \mathscr{F}$. Then f is isotrivial.*

Proof. Let $\omega \in H^0(X,\Omega^1_X \otimes \mathscr{N})$ be a twisted 1-form defining $\mathscr{F}$, and let $\Sigma \subset X$ be the neutral section of f. By assumption, the composition $\mathscr{F} \to T_X \to f^*T_Y$ is generically surjective. Notice that the sheaf $f_*T_{X/Y}$ is locally free of rank $n-m$. Let F be a general fiber of f. The morphism $\mathscr{O}_F^{\oplus n-m} \cong T_{X/Y}|_F \to \mathscr{N}|_F \cong \mathscr{O}_F$ induced by ω is generically surjective by assumption, and hence surjective. This implies that the saturation $\mathscr{E}$ of $f_*(\mathscr{F} \cap T_{X/Y})$ in $f_*T_{X/Y}$ has rank $n-m-1$. Let $Y^\circ \subseteq Y$ be an open subset with complement of codimension at least 2 such that $\mathscr{E}^\circ := \mathscr{E}|_{Y^\circ}$ is a subbundle of $f_*T_{X/Y}$. Set $X^\circ := f^{-1}(Y^\circ)$ and $f^\circ := f|_{X^\circ}$. Then $(f^\circ)^*\mathscr{E}^\circ = \mathscr{F}^\circ \cap T_{X^\circ/Y^\circ} \subset T_{X^\circ/Y^\circ}$ since $\mathscr{F}^\circ \cap T_{X^\circ/Y^\circ}$ is saturated in T_{X°/Y°. In particular, $\mathscr{F}^\circ \cap T_{X^\circ/Y^\circ}$ is a subbundle of $\mathscr{F}^\circ$. Set $\mathscr{Q}^\circ := \mathscr{F}^\circ/(\mathscr{F}^\circ \cap T_{X^\circ/Y^\circ})$. Notice also that

$$\mathscr{Q}^\circ|_F \cong (f^*T_Y)|_F \cong \mathscr{O}_F^{\oplus m},$$

where F is a general fiber F of f°. In particular, $K_{\mathscr{F}}|_F \sim_{\mathbb{Z}} 0$. Then Lemma 3.9 applies to show that there exists a line bundle $\mathscr{K}$ on Y such that $\mathscr{O}_X(K_{\mathscr{F}}) \cong f^*\mathscr{K}$.

Let now F be any fiber of f°. Then the vector bundle $\mathscr{F}|_F$ is numerically flat since $K_{\mathscr{F}}|_F \sim_{\mathbb{Z}} 0$. It follows that $\mathscr{Q}^\circ|_F$ is numerically flat as well. Then Lemma 2.15 applies to show that there exists a vector bundle $\mathscr{G}^\circ$ on Y° such that $\mathscr{Q}^\circ \cong (f^\circ)^*\mathscr{G}^\circ$. By construction, there is an exact sequence

$$0 \to (f^\circ)^*\mathscr{E}^\circ \to \mathscr{F}^\circ \to (f^\circ)^*\mathscr{G}^\circ \to 0.$$

Let again F be a general fiber of f°. To prove the statement, it suffices to show that the exact sequence

$$0 \to T_{X/Y}|_F \to T_X|_F \to (f^*T_Y)|_F \to 0 \tag{7.1}$$

splits. This is because complex flows of vector fields on analytic spaces exist. If the short exact sequence

$$0 \to (f^\circ)^*\mathscr{E}^\circ|_F \to \mathscr{F}^\circ|_F \to (f^\circ)^*\mathscr{G}^\circ|_F \to 0 \tag{7.2}$$

splits, then the exact sequence (7.1) splits as well, proving our claim. Suppose from now on that the exact sequence (7.2) does not split. Then the image of $\mathrm{Id}_{\mathscr{G}^\circ} \in H^0(Y^\circ,\mathscr{E}nd(\mathscr{G}^\circ))$ under the connecting homomorphism

$$H^0(Y^\circ,\mathscr{E}nd(\mathscr{G}^\circ)) \to R^1f^\circ_*\big((f^\circ)^*\mathscr{E}^\circ \otimes ((f^\circ)^*\mathscr{G}^\circ)^*\big) \cong R^1f^\circ_*\mathscr{O}_{X^\circ} \otimes \mathscr{E}^\circ \otimes (\mathscr{G}^\circ)^*$$

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is nonzero. Let C be a complete intersection curve of general elements of a very ample linear system on Y. By general choice of C, we have $C \subseteq Y^\circ$ since $Y \setminus Y^\circ$ has codimension at least 2. Moreover,

$$\mu^{\max}\big((R^1 f^\circ_* \mathscr{O}_{X^\circ} \otimes \mathscr{E}^\circ \otimes (\mathscr{G}^\circ)^*)|_C\big) \geqslant 0.$$

Then

$$\begin{aligned}\mu^{\max}\big((R^1 f^\circ_* \mathscr{O}_{X^\circ} \otimes \mathscr{E}^\circ \otimes (\mathscr{G}^\circ)^*)|_C\big) &= \mu^{\max}\big(R^1 f^\circ_* \mathscr{O}_{X^\circ}|_C\big) + \mu^{\max}\big(\mathscr{E}^\circ|_C\big) + \mu^{\max}\big((\mathscr{G}^\circ)^*|_C\big) \\ &= \mu^{\max}\big(R^1 f^\circ_* \mathscr{O}_{X^\circ}|_C\big) + \mu^{\max}\big(\mathscr{E}^\circ|_C\big) - \mu^{\min}\big(\mathscr{G}^\circ|_C\big),\end{aligned}$$

and hence

$$\mu^{\min}\big(\mathscr{G}^\circ|_C\big) \leqslant \mu^{\max}\big(R^1 f^\circ_* \mathscr{O}_{X^\circ}|_C\big) + \mu^{\max}\big(\mathscr{E}^\circ|_C\big).$$

Since $(R^1 f_* \mathscr{O}_X)^* \cong f_* \Omega^1_{X/Y}$ is nef (see [33, Theorem 5.2]), we have

$$\mu^{\max}\big(R^1 f^\circ_* \mathscr{O}_{X^\circ}|_C\big) \leqslant 0.$$

On the other hand, viewing C as a curve in Σ, there is an exact sequence

$$0 \to \mathscr{E}^\circ|_C \to \mathscr{F}^\circ|_C \to \mathscr{G}^\circ|_C \to 0.$$

It follows that

$$\mu^{\max}\big(\mathscr{E}^\circ|_C\big) \leqslant \mu(\mathscr{F}|_C) \leqslant \mu^{\min}\big(\mathscr{G}^\circ|_C\big)$$

since $\mathscr{F}|_C$ is semistable by assumption. This gives

$$\mu(\mathscr{F}|_C) \leqslant \mu^{\min}\big(\mathscr{G}^\circ|_C\big) \leqslant \mu^{\max}\big(R^1 f^\circ_* \mathscr{O}_{X^\circ}|_C\big) + \mu^{\max}\big(\mathscr{E}^\circ|_C\big) \leqslant \mu^{\max}\big(\mathscr{E}^\circ|_C\big) \leqslant \mu(\mathscr{F}|_C),$$

and hence

$$\mu^{\max}\big(R^1 f^\circ_* \mathscr{O}_{X^\circ}|_C\big) = 0 \quad \text{and} \quad \mu^{\max}\big(\mathscr{E}^\circ|_C\big) = \mu(\mathscr{F}|_C) = \mu^{\min}\big(\mathscr{G}^\circ|_C\big).$$

This implies that the vector bundle $f_* \Omega^1_{X/Y}|_C$ is numerically flat. In particular, we have $\deg(f_* \mathscr{O}(K_{X/Y})|_C) = 0$. Then [59, Corollary 2.3] applies to show that f is isotrivial. This finishes the proof of the lemma. □

The following is the main result of this section.

Theorem 7.2. *Setting and notation as in 2.19. Suppose in addition that the conormal bundle $\mathscr{N}^*$ of $\mathscr{F}$ is nef. Then one of the following holds.*

(1) *There exists an abelian variety A as well as a finite étale cover $\gamma\colon A \to X$ such that $\gamma^{-1}\mathscr{F}$ is a linear foliation.*
(2) *There exists an abelian scheme $f\colon B \to C$ onto a smooth complete curve of genus at least 2 as well as a finite étale cover $\gamma\colon B \to X$ such that $\gamma^{-1}\mathscr{F}$ is induced by f.*
(3) *The variety X is of general type, and $\nu(K_{\mathscr{F}}) = \dim X$.*

Proof. By Lemma 2.21, $\mathscr{F}^*$ is a nef vector bundle. This in turn implies that Ω^1_X is nef since $\mathscr{N}^*$ is nef by assumption. In particular, K_X is nef. On the other hand, K_X is abundant by Proposition 6.3. Then [43, Theorem 1.1] applies to show that K_X

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is semiample. Moreover, by [39, Theorem 1.2], replacing X by a finite étale cover, if necessary, we may assume that X is an abelian scheme over a smooth projective variety Y with ample canonical divisor (see Lemma 2.17). Let us denote by $f\colon X \to Y$ the natural morphism and by Σ any section of f. Set $m := \dim Y$.

Suppose that $m = 0$. Then X is an abelian variety. This easily implies $\mathscr{N}^* \cong \mathscr{O}_X$. As a consequence, $\mathscr{F}$ is a linear foliation on X, and we are in case (1) of Theorem 7.2.

Suppose from now on that $m \geqslant 1$.

Set $\mathscr{E} := f_*\Omega^1_{X/Y}$ so that $\Omega^1_{X/Y} \cong f^*\mathscr{E}$. By [60, Theorem 1 and Remark 2], we have

$$2\mu(\mathscr{E}) \leqslant \mu(\Omega^1_Y), \tag{7.3}$$

where μ denotes the slope with respect to K_Y, which is ample by construction. A celebrated theorem proved by Aubin and Yau moreover asserts that Y is Kähler–Einstein. As a consequence, the tangent bundle T_Y is polystable with respect to K_Y by a result proved by Kobayashi and Lübke independently.

The short exact sequences

$$0 \to \mathscr{N}^* \to \Omega^1_X \to \mathscr{F}^* \to 0 \quad \text{and} \quad 0 \to f^*\Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0$$

give

$$\frac{\mu(\mathscr{N}^*|_\Sigma) + (n-1)\mu(\mathscr{F}^*|_\Sigma)}{n} = \mu(\Omega^1_X|_\Sigma) = \frac{m\mu(\Omega^1_Y) + (n-m)\mu(\mathscr{E})}{n}. \tag{7.4}$$

Suppose first that the composition $\mathscr{N}^*|_\Sigma \to \Omega^1_X|_\Sigma \to \Omega^1_{X/Y}|_\Sigma \cong \mathscr{E}$ vanishes. Then

$$\mathscr{N}^*|_\Sigma \subseteq f^*\Omega^1_Y|_\Sigma \cong \Omega^1_Y$$

and there exists a surjective morphism

$$\mathscr{F}^*|_\Sigma \twoheadrightarrow f^*\mathscr{E}|_\Sigma \cong \mathscr{E}.$$

Moreover, by Lemma 2.14, the vector bundle $\mathscr{F}|_\Sigma$ is numerically projectively flat. It follows that

$$\mu(\mathscr{N}^*|_\Sigma) \leqslant \mu(\Omega^1_Y) \quad \text{and} \quad \mu(\mathscr{F}^*|_\Sigma) \leqslant \mu(\mathscr{E}).$$

Then Equation (7.4) yields

$$m\mu(\Omega^1_Y) + (n-m)\mu(\mathscr{E}) = \mu(\mathscr{N}^*|_\Sigma) + (n-1)\mu(\mathscr{F}^*|_\Sigma) \leqslant \mu(\Omega^1_Y) + (n-1)\mu(\mathscr{E}),$$

and hence, using Equation (7.3), we obtain

$$(m-1)\mu(\Omega^1_Y) \leqslant (m-1)\mu(\mathscr{E}) \leqslant \frac{m-1}{2}\mu(\Omega^1_Y).$$

This immediately implies $m = 1$ since $\mu(\Omega^1_Y) > 0$. Moreover, we have $\mathscr{N}^* = f^*\Omega^1_Y \subset \Omega^1_X$. As a consequence, $\mathscr{F}$ is induced by f, and we are in case (2) of Theorem 7.2.

Suppose finally that the composition $\mathscr{N}^*|_\Sigma \to \Omega^1_X|_\Sigma \to \Omega^1_{X/Y}|_\Sigma \cong \mathscr{E}$ is nonzero. Then the composition $\mathscr{F}|_\Sigma \to T_X|_\Sigma \to f^*T_Y|_\Sigma \cong T_Y$ is generically surjective. Since the vector bundle $\mathscr{F}|_\Sigma$ is numerically projectively flat, we have

$$\mu(\Omega^1_Y) \leqslant \mu(\mathscr{F}^*|_\Sigma). \tag{7.5}$$

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Together with Equations (7.3) and (7.4), we obtain

$$2\mu(\mathscr{N}^*|_\Sigma)+2(n-1)\mu(\Omega^1_Y)\leqslant 2\mu(\mathscr{N}^*|_\Sigma)+2(n-1)\mu(\mathscr{F}^*|_\Sigma)=$$
$$2m\mu(\Omega^1_Y)+2(n-m)\mu(\mathscr{E})\leqslant 2m\mu(\Omega^1_Y)+(n-m)\mu(\Omega^1_Y)=(n+m)\mu(\Omega^1_Y),$$

and thus

$$(n-2-m)\mu(\Omega^1_Y)+2\mu(\mathscr{N}^*|_\Sigma)\leqslant 0.$$

Notice that $\mu(\mathscr{N}^*|_\Sigma)\geqslant 0$ since $\mathscr{N}^*$ is nef and that $\mu(\Omega^1_Y)>0$ since $m\geqslant 1$. In particular, $m\in\{n-2,n-1,n\}$.

Claim 7.3. There exists a line bundle $\mathscr{M}$ on Y such that $\mathscr{N}^*\cong f^*\mathscr{M}$.

Proof. Let F be a general fiber of f. The composition $\mathscr{N}^*\to\Omega^1_X\to\Omega^1_{X/Y}$ is nonzero by our current assumption and induces an injective map $\mathscr{N}^*|_F\to\Omega^1_{X/Y}|_F\cong\mathscr{O}_F^{\oplus n-m}$. Since $\mathscr{N}^*$ is nef, we have $\mathscr{N}^*|_F\cong\mathscr{O}_F$. The claim now follows from Lemma 3.9. □

Suppose that $m=n$. Then $X=Y$ is of general type and $\nu(K_{\mathscr{F}})=\dim X$ by Lemma 7.4 below. We are in case (3) of Theorem 7.2.

Suppose that $m\in\{n-2,n-1\}$. By Lemma 7.1, after replacing X by a finite étale cover, we may assume that $X\cong Y\times A$ for some abelian variety A of dimension $n-m\in\{1,2\}$ and that f is induced by the projection onto Y. Since the composition $\mathscr{N}^*|_\Sigma\to\Omega^1_X|_\Sigma\to\Omega^1_{X/Y}|_\Sigma\cong\mathscr{O}_\Sigma^{\oplus n-m}$ is nonzero and $\mathscr{N}^*$ is nef, we have $\mathscr{N}\cong\mathscr{O}_X$.

If $m=n-1$, then $\mathscr{F}$ is everywhere transverse to f. This implies that T_Y is numerically projectively flat. But then $K_Y\equiv 0$ by [42, Theorem 0.1], yielding a contradiction.

Suppose finally that $m=n-2$. Then Equations (7.3), (7.4) and (7.5) give $\mu(\mathscr{F}^*|_\Sigma)=\mu(\Omega^1_Y)$ and $2\mu(\mathscr{E})=\mu(\Omega^1_Y)$. Now, this is impossible since $\mu(\mathscr{E})=0$ and $\mu(\Omega^1_Y)>0$. This finishes the proof of the theorem. □

Lemma 7.4. *Setting and notation as in 2.19. Suppose in addition that X is of general type. Then $\nu(K_{\mathscr{F}})=\dim X$.*

Proof. We argue by contradiction and assume that $\nu(K_{\mathscr{F}})\leqslant\dim X-1$. By Corollary 4.7, $\mathscr{N}^*$ is nef. It follows that Ω^1_X is nef since $\mathscr{F}^*$ is also nef by Lemma 2.21. This implies that X contains no rational curve, and hence K_X is an ample divisor. A celebrated theorem proved independently to Aubin ([2]) and Yau ([63]) then asserts that X is Kähler-Einstein. As a consequence, the tangent bundle T_X is polystable with respect to K_X by a result proved by Kobayashi and Lübke independently (see [47, Theorem V.8.3] and the references therein).

Set $n:=\dim X$. Then

$$K^n_{\mathscr{F}}=(K_X+c_1(\mathscr{N}))^n=K^n_X+nK^{n-1}_X\cdot c_1(\mathscr{N})=0$$

since $K_{\mathscr{F}}=K_X+c_1(\mathscr{N})$ by the adjunction formula and $c_1(\mathscr{N})^2=0$ by Lemma 3.1. It follows that $\mu(\mathscr{N})=\mu(T_X)$, where μ denotes the slope with respect to K_X, and hence $T_X\cong\mathscr{F}\oplus\mathscr{N}$ since T_X is polystable.

By Lemma 7.5 (1) below, we then have

$$(2nc_2(\mathscr{F})-(n-1)c_1(\mathscr{F})^2)\cdot K^{n-2}_X\geqslant 0.$$

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Since $c_2(\mathscr{F}) = \frac{n-2}{2(n-1)}c_1(\mathscr{F})$ (see Lemma 2.7), we obtain

$$c_1(\mathscr{F})^2 \cdot K_X^{n-2} = 0.$$

But this contradicts Lemma 7.5 (2) below, finishing the proof of the lemma. □

Lemma 7.5. *Let X be a complex projective manifold with K_X ample, and let $\Omega^1_X := \mathscr{A} \oplus \mathscr{B}$ be a decomposition of Ω^1_X into locally free subsheaves. Suppose that $\mathscr{A}$ has rank $a \geqslant 2$, and set $n := \dim X$. Then*

(1) $(2(a+1)c_2(\mathscr{A}) - ac_1(\mathscr{A})^2) \cdot K_X^{n-2} \geqslant 0$, *and*
(2) $c_1(\mathscr{A})^2 \cdot K_X^{n-2} > 0$.

Proof. Let $\mathscr{E} := \Omega^1_X \oplus \mathscr{O}_X$ equipped with the Higgs field ϑ defined by

$$\Omega^1_X \oplus \mathscr{O}_X \ni \alpha + f \mapsto (0+1) \otimes \alpha \in (\Omega^1_X \oplus \mathscr{O}_X) \otimes \Omega^1_X.$$

Notice that $\vartheta(\mathscr{A} \oplus \mathscr{O}_X) \subset (\mathscr{A} \oplus \mathscr{O}_X) \otimes \Omega^1_X$ so that $(\mathscr{G} := \mathscr{A} \oplus \mathscr{O}_X, \vartheta|_{\mathscr{G}})$ is a Higgs bundle as well.

Arguing as in the proof of Lemma 7.4 above, we see that T_X is polystable with respect to K_X. This in turn implies that $\mathscr{A}$ is polystable with respect to K_X with $\mu(\mathscr{A}) = \mu(\Omega^1_X) > 0$, where μ denotes the slope with respect to K_X.

Next, we show that the Higgs bundle $(\mathscr{G}, \vartheta|_{\mathscr{G}})$ is stable with respect to K_X. Let $\mathscr{H} \subset \mathscr{G}$ be a saturated subsheaf of rank $0 < r < a+1$ preserved by $\vartheta|_{\mathscr{G}}$, and let $p\colon \mathscr{H} \to \mathscr{O}_X$ be the composition $\mathscr{H} \to \mathscr{G} = \mathscr{A} \oplus \mathscr{O}_X \to \mathscr{O}_X$. Observe that p is not the zero map since $\mathscr{H}$ is preserved by $\vartheta|_{\mathscr{G}}$ by assumption.

If p is injective, then $\mu(\mathscr{H}) \leqslant 0 < \mu(\mathscr{G}) = \frac{a}{a+1}\mu(\mathscr{A})$. Suppose that p is not injective. Note that we must have $r \geqslant 2$ and $\mu(\operatorname{Im} p) \leqslant 0$. Moreover, $\mu(\operatorname{Ker} p) \leqslant \mu(\mathscr{A})$ since $\operatorname{Ker} p \subseteq \mathscr{A}$ and $\mathscr{A}$ is polystable with respect to K_X. Then, we have

$$\mu(\mathscr{H}) = \frac{(r-1)\mu(\operatorname{Ker} p) + \mu(\operatorname{Im} p)}{r} \leqslant \frac{r-1}{r}\mu(\mathscr{A}) = \frac{r-1}{r}\frac{a+1}{a}\mu(\mathscr{G}) \leqslant \frac{a^2-1}{a^2}\mu(\mathscr{G}) < \mu(\mathscr{G}),$$

proving our claim.

Item (1) now follows from [53, Theorem 1 and Proposition 3.4].

Let $b \geqslant 0$ denote the rank of $\mathscr{B}$. By [7, Lemma 3.1], we have $c_1(\mathscr{A})^{a+1} \equiv 0$ and $c_1(\mathscr{B})^{b+1} \equiv 0$. Then

$$c_1(\mathscr{A})^2 \cdot K_X^{n-2} = c_1(\mathscr{A})^2 \cdot (c_1(\mathscr{A}) + c_1(\mathscr{B}))^{n-2} = \binom{n-2}{b} c_1(\mathscr{A})^a \cdot c_1(\mathscr{B})^b,$$

and

$$K_X^n = \binom{n}{b} c_1(\mathscr{A})^a \cdot c_1(\mathscr{B})^b.$$

It follows that $c_1(\mathscr{A})^2) \cdot K_X^{n-2} > 0$ since $K_X^n > 0$ by assumption, proving Item (2). □

Kodaira fibrations provide examples of regular foliations by curves $\mathscr{F}$ on surfaces of general type with $\kappa(\mathscr{F}) = 2$. However, no example of foliations satisfying the conclusion

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of Theorem 7.2 (3) is known to the author. Let us show that these foliations have no compact leaves.

Proposition 7.6. *Setting and notation as in 2.19. Suppose in addition that K_X is pseudo-effective and that $\mathscr{F}$ has a compact leaf. Then $\kappa(X) = \nu(X) \in \{0,1\}$.*

Proof. By Proposition 6.3, K_X is abundant. Let B denote an algebraic leaf of $\mathscr{F}$. Then B is a finite étale quotient of an abelian variety by [42, Theorem 0.1]. Moreover, $\mathscr{N}|_B \cong \mathscr{O}_X(B)|_B$ is a flat vector bundle by Lemma 3.1. This immediately implies that $B^2 \equiv 0$. Proposition 7.7 below then applies to show that $\nu(X) \leqslant 1$, completing the proof of the proposition. □

We refer to [45] and [46] for standard references concerning the minimal model program.

Proposition 7.7. *Let X be a complex projective manifold of dimension $n \geqslant 2$ with $\kappa(X) = \nu(X) \geqslant 0$. Suppose that there exists a smooth prime divisor $B \subset X$ with $K_B \equiv 0$ and $B^2 \equiv 0$. Then $\nu(X) \leqslant 1$.*

Proof. By [32, Theorem 4.3], we may run a minimal model program for X and end with a minimal model. Hence, there exists a birational map $\varphi\colon X \dashrightarrow Y$, where Y is a normal $\mathbb{Q}$-factorial projective variety with K_Y nef. Moreover, Y has terminal singularities, and φ^{-1} does not contract any divisor. Let $p\colon Z \to X$ and $q\colon Z \to Y$ be a common resolution of singularities. Let C be the strict transform of B in Z. If B is contracted by φ, then it is uniruled by [44, Theorem 1], yielding a contradiction since $K_B \equiv 0$ by assumption. Set $D := \varphi_* B$.

Let $\varepsilon \in \mathbb{Q}$ be a positive number such that φ is a minimal model program for $K_X + \varepsilon B$. By [45, Lemma 3.38], there exists an effective q-exceptional $\mathbb{Q}$-divisor F such that

$$p^*(K_X + \varepsilon B) = q^*(K_Y + \varepsilon D) + F.$$

Moreover, by the negativity lemma, there exists an effective q-exceptional $\mathbb{Q}$-divisor G such that

$$p^*B + G = q^*D$$

since B is nef. Notice also that $K_X|_B \equiv 0$ by the adjunction formula $K_B = (K_X + B)|_B$. We then have

$$0 \equiv p^*(K_X + \varepsilon B)|_C = q^*(K_Y + \varepsilon D)|_C + F|_C \equiv (q^*K_Y)|_C + \varepsilon G|_C + F|_C.$$

Since $G|_C + F|_C$ is effective and $(q^*K_Y)|_C$ is nef, we have

$$K_Y|_D \equiv 0 \quad \text{and} \quad G \cap C = \emptyset.$$

This immediately implies $D^2 \equiv 0$. Now, we have $\kappa(X) = \kappa(Y)$ and $\nu(X) = \nu(Y)$ by [52, Proposition V.2.7], and hence $\kappa(Y) = \nu(Y)$. Then [43, Theorem 1.1] applies to show that K_Y is semiample. We denote the Iitaka fibration of Y by $f\colon Y \to T$. Since $K_Y|_D \equiv 0$, we have $\dim f(D) = 0$. If $\dim T = 0$, then $\nu(X) = 0$ and there is nothing to show. If $\dim T \geqslant 1$, then Lemma 3.13 applies to show $\nu(X) = 1$. This finishes the proof of the proposition. □

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8. The nef reduction map

In this section, we provide a technical tool for the proof of Theorem 9.1 below. We briefly recall the relevant notions first. Let X be a smooth projective variety, and let $\mathscr{L}$ be a nef line bundle on X. Then there exists an almost proper, dominant rational map $f\colon X \dashrightarrow Y$ such that

(1) $\mathscr{L}|_F \equiv 0$ for a general fiber F of f,
(2) for every general point $x \in X$ and every irreducible curve C passing through x with $\dim f(C) = 1$, we have $\mathscr{L}\cdot C > 0$.

The map f is unique up to birational equivalence of Y. We refer to it as the *nef reduction map* of $\mathscr{L}$. The dimension of Y is an invariant of $\mathscr{L}$ called the *nef dimension*. We write $\mathrm{nd}(\mathscr{L}) := \dim Y$. We have $\kappa(\mathscr{L}) \leqslant \nu(\mathscr{L}) \leqslant \mathrm{nd}(\mathscr{L})$. We refer to [4] for further explanations concerning these notions. Let $\mathscr{F}$ be a regular foliation on a complex projective manifold and suppose that $\mathscr{F}$ satisfies all the conditions listed in Setup 2.19. Recall from Lemma 2.21 that $K_{\mathscr{F}}$ is nef. Then Proposition 8.1 below gives sufficient conditions that guarantee $\mathrm{nd}(K_{\mathscr{F}}) < \dim X$.

Proposition 8.1. *Setting and notation as in 2.19. Suppose in addition that K_X is nef and that $\mathscr{N}$ is pseudo-effective. Set $m := \min(\nu(K_X), n-2)$. Then $\nu(K_X) \leqslant n-1$ and $nd(K_{\mathscr{F}}) \leqslant m+1 \leqslant n-1$.*

Proof. Let us first show $\nu(K_X) \leqslant n-1$. Let H be an ample divisor on X. By [35, Theorem B], we have

$$\Big(c_2(X) - \frac{n}{2(n+1)}c_1(X)^2\Big)\cdot K_X^m \cdot H^{n-2-m} \geqslant 0.$$

We have

$$c_2(X) = c_2(\mathscr{F}) + c_1(\mathscr{F})\cdot c_1(\mathscr{N}) \quad \text{and} \quad c_1(X)^2 \equiv c_1(\mathscr{F})^2 + 2c_1(\mathscr{F})\cdot c_1(\mathscr{N})$$

since $c_1(\mathscr{N})^2 \equiv 0$ by Lemma 3.1. In addition, we have

$$c_2(\mathscr{F}) \equiv \frac{n-2}{2(n-1)}c_1(\mathscr{F})^2$$

by [42, Proposition 1.1]. We obtain

$$\Big(-\frac{1}{n^2-1}c_1(\mathscr{F})^2 + \frac{1}{n+1}c_1(\mathscr{F})\cdot c_1(\mathscr{N})\Big)\cdot K_X^m \cdot H^{n-2-m} \geqslant 0. \tag{8.1}$$

On the other hand, $K_{\mathscr{F}} = -c_1(\mathscr{F})$ is nef by Lemma 2.21. This immediately implies

$$-c_1(\mathscr{F})^2 \cdot K_X^m \cdot H^{n-2-m} \leqslant 0$$

and

$$c_1(\mathscr{F})\cdot c_1(\mathscr{N}) \cdot K_X^m \cdot H^{n-2-m} \leqslant 0$$

since $\mathscr{N}$ is pseudo-effective by assumption. Equation (8.1) then gives

$$c_1(\mathscr{F})^2 \cdot K_X^m \cdot H^{n-2-m} = c_1(\mathscr{F})\cdot c_1(\mathscr{N}) \cdot K_X^m \cdot H^{n-2-m} = 0. \tag{8.2}$$

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Therefore, we have $K_X^{m+2}\cdot H^{n-2-m}=0$, and hence $K_X^n=0$ since $m=\min(\nu(K_X),n-2)$. This shows $\nu(K_X)\leqslant n-1$.

Let us now show $\mathrm{nd}(K_{\mathscr{F}})\leqslant m+1$. If $\mathscr{F}$ is algebraically integrable, then $\mathrm{nd}(K_{\mathscr{F}})\leqslant 1$ by Proposition 3.20. Suppose from now on that $\mathscr{F}$ is not algebraically integrable.

By Proposition 6.3 and [43, Theorem 1.1] together, K_X is semiample. We denote the Iitaka fibration by $f\colon X\to Y$. Let F be a general fiber of f. Then F is smooth. By construction, there exists an ample divisor A on Y such that $K_X\sim_{\mathbb{Q}} f^*A$. In particular, we have $K_F\sim_{\mathbb{Q}}0$ by the adjunction formula.

Suppose first that $m=\dim Y=\nu(K_X)\leqslant n-2$.

Then $c_2(T_X|_F)\cdot H|_F^{n-2-m}=\frac{1}{A^m}c_2(X)\cdot K_X^m\cdot H^{n-2-m}=0$ by Equation (8.2). The exact sequence

$$0\to T_F\to T_X|_F\to \mathscr{N}_{F/X}\cong\mathscr{O}_F^{\oplus m}\to 0$$

then gives $c_2(T_F)\cdot H|_F^{n-2-m}=0$. As a classical consequence of Yau's theorem on the existence of a Kähler–Einstein metric, F is then covered by a complex torus (see [47, Chapter IV Corollary 4.15]). Thus, there exist an abelian variety F_1 of dimension at least 2 and a finite étale cover $\eta\colon F_1\to F$. Then [31, Proposition 3.3] applies to show that there exists a line bundle $\mathscr{L}_1$ on F_1 as well as a flat vector bundle $\mathscr{G}_1$ on F_1 such that $\eta^*(\mathscr{F}|_F)\cong\mathscr{G}_1\otimes\mathscr{L}_1$. In addition, $\mathscr{G}_1$ admits a filtration $0=F_0\mathscr{G}_1\subsetneq F_1\mathscr{G}_1\subsetneq\cdots\subsetneq F_{n-1}\mathscr{G}_1=\mathscr{G}_1$ by flat subbundles $F_i\mathscr{G}_1$ with $\operatorname{rank}F_i\mathscr{G}_1=i$. By Equation (8.2), we have

$$c_1(\eta^*(\mathscr{F}|_F))^2\cdot\eta^*(H|_F)^{n-2-m}=0. \tag{8.3}$$

If $K_{\mathscr{F}}|_F\equiv 0$, then $\mathrm{nd}(K_{\mathscr{F}})\leqslant\dim Y=m\leqslant m+1$.

Suppose that $K_{\mathscr{F}}|_F\not\equiv 0$.

If $m=0$, then $F=X$ and there exists a generically surjective morphism $\Omega^1_{F_1}\to\eta^*(\mathscr{F}|_F)^*$. This immediately implies $h^0(F_1,\mathscr{O}_{F_1}(\eta^*(K_{\mathscr{F}}|_F)))\geqslant 1$. Then [18, Théorème VI.5.1] applies to show that the line bundle $\mathscr{O}_{F_1}(\eta^*(K_{\mathscr{F}}|_F))$ is semiample. Notice that $\nu(\mathscr{O}_{F_1}(\eta^*(K_{\mathscr{F}}|_F)))=1$ by Equation (8.3). This easily implies $\mathrm{nd}(K_{\mathscr{F}})=1=m+1$.

Suppose that $m\geqslant 1$. Since $\mathscr{F}$ is not algebraically integrable by our current assumption, the composition $\mathscr{F}|_F\to T_X|_F\to\mathscr{N}_{F/X}\cong\mathscr{O}_F^{\oplus m}$ is nonzero. As a consequence, there is an index $1\leqslant i\leqslant n-1$ such that the line bundle $\mathscr{Q}_i:=\left((F_{i+1}\mathscr{G}_1/F_i\mathscr{G}_1)\otimes\mathscr{L}_1\right)^*$ has a nonzero global section. Then [18, Théorème VI.5.1] again applies to show that $\mathscr{Q}_i$ is semiample. We have $\nu(\mathscr{Q}_i)=1$ since $c_1(\mathscr{Q}_i)\equiv c_1(\mathscr{L}_1)\equiv\frac{1}{n-1}\eta^*(K_{\mathscr{F}}|_F)$. This implies $\mathrm{nd}(K_{\mathscr{F}})\leqslant\dim Y+1=m+1$.

Suppose finally that $\nu(K_X)=n-1$. In particular, $m=n-2$.

Then F is a smooth curve of genus 1. By Equation (8.2), we have

$$K_{\mathscr{F}}\cdot F=\frac{1}{A^{n-1}}K_{\mathscr{F}}\cdot K_X^{n-1}=\frac{1}{A^{n-1}}c_1(\mathscr{F})\cdot(c_1(\mathscr{F})+c_1(\mathscr{N}))\cdot K_X^{n-2}=0.$$

As before, this implies $\mathrm{nd}(K_{\mathscr{F}})\leqslant n-1=m+1$, completing the proof of Proposition 8.1. □

The following is the main result of this section.

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Proposition 8.2. *Setting and notation as in 2.19. Suppose in addition that K_X is nef and that $\mathscr{N}$ is pseudo-effective. Then, there exist an abelian scheme $f_1\colon X_1 \to Y_1$ over a smooth projective base and a finite étale cover $\gamma\colon X_1 \to X$ such that f_1 is the nef reduction map of $\gamma^* K_{\mathscr{F}}$. Moreover, there exist line bundles $\mathscr{M}_1$ and $\mathscr{K}_1$ on Y_1 such that $\gamma^*\mathscr{N} \cong f_1^*\mathscr{M}_1$ and $\gamma^*\mathscr{O}_X(K_{\mathscr{X}}) \cong f_1^*\mathscr{K}_1$. In addition, if F_1 is a general fiber of f_1, then $(\gamma^{-1}\mathscr{F})|_{F_1} \cap T_{F_1}$ is a linear foliation on F_1 of codimension at most 1.*

Proof. For the reader's convenience, the proof is subdivided into a number of steps. Let

$$f\colon X \dashrightarrow Y$$

be the nef reduction map of $K_{\mathscr{F}}$. By Proposition 8.1 above, we have $m := \dim Y \leqslant n-1$.

Step 1. Fibers of the nef reduction map.

Let us show that a general fiber F of f is a finite étale quotient of an abelian variety and that $K_{\mathscr{F}}|_F \sim_{\mathbb{Q}} 0$. Then $\mathscr{N}|_F$ is a torsion line bundle by the adjunction formula $K_{\mathscr{F}} = K_X + c_1(\mathscr{N})$.

Suppose first that F is a very general fiber. Then $\mathscr{N}|_F$ is pseudo-effective while $K_X|_F$ is nef and $K_{\mathscr{F}}|_F \equiv 0$ by construction. It follows that $K_X|_F \equiv 0$ and $\mathscr{N}|_F \equiv 0$ since $K_{\mathscr{F}} = K_X + c_1(\mathscr{N})$ by the adjunction formula. Let now F be a general fiber of f. By [20, Lemma 2.13], we have $K_X|_F \equiv 0$. By the adjunction formula again, we have $\mathscr{N}|_F \equiv 0$. Now, the vector bundle $\mathscr{F}|_F$ is numerically flat by Theorem 2.11 since $K_{\mathscr{F}}|_F \equiv 0$. It follows that the composition

$$\mathscr{F}|_F \to T_X|_F \to \mathscr{N}_{F/X} \cong \mathscr{O}_F^{\oplus m}$$

has constant rank, and hence $\mathscr{F}|_F \cap T_F$ is a numerically flat subbundle of T_F.

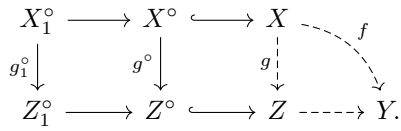
If $T_F \subseteq \mathscr{F}|_F$, then F is a finite étale quotient of an abelian variety by [47, Chapter IV Corollary 4.15]. Moreover, $K_{\mathscr{F}}|_F \sim_{\mathbb{Q}} 0$ since $\mathscr{F}|_F/T_F \subset \mathscr{N}_{F/X} \cong \mathscr{O}_F^{\oplus m}$ is numerically flat.

Suppose that $\mathscr{F}|_F \cap T_F \subsetneq T_F$. Then F is again a finite étale quotient of an abelian variety by Proposition 3.18 since $K_F \equiv K_X|_F \equiv 0$ by the adjunction formula. Moreover, $K_{\mathscr{F}|_F \cap T_F} \sim_{\mathbb{Q}} 0$, and hence $K_{\mathscr{F}}|_F \sim_{\mathbb{Q}} 0$ since $\mathscr{F}|_F/(\mathscr{F}|_F \cap T_F) \cong \mathscr{N}_{F/X} \cong \mathscr{O}_F^{\oplus m}$. This proves our claim.

Step 2. Next, we show that X has generically large fundamental group on F.

We argue by contradiction and assume that X does not have generically large fundamental group on F. There exists a positive-dimensional finite étale quotient of an abelian variety $G \subseteq F$ passing through a general point of F such that the image of the natural morphism $\pi_1(G) \to \pi_1(X)$ is finite. By [48, 3.10], the algebraic varieties G fit together to form an almost proper rational map $g\colon X \dashrightarrow Z$ such that f factorizes through g. Let $X^\circ \subseteq X$ and $Z^\circ \subseteq Z$ be dense open sets such that the restriction $g^\circ\colon X^\circ \to Z^\circ \subseteq Z$ of g to X° is a smooth projective morphism with connected fibers. Shrinking Z°, we may assume that there exists a finite étale cover $X_1^\circ \to X^\circ$ such that the fibration $g_1^\circ\colon X_1^\circ \to Z_1^\circ$ obtained as the Stein factorization of the composition $X_1^\circ \to X^\circ \to Z^\circ$ is an abelian scheme equipped with a level three structure (see Lemma 3.7). We obtain a diagram as follows:

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Let G_1 be a fiber of g_1°. There is a split exact sequence of fundamental groups

$$1 \longrightarrow \pi_1(G_1) \longrightarrow \pi_1(X_1^\circ) \longrightarrow \pi_1(Z_1^\circ) \longrightarrow 1.$$

Let $K \subseteq \pi_1(G_1)$ denotes the kernel of the representation

$$\pi_1(G_1) \to \pi_1(X_1^\circ) \to \pi_1(X) \to \mathrm{PGL}(n-1,\mathbb{C})$$

induced by ρ. Then the subset $K \cdot \pi_1(Z_1^\circ)$ of $\pi_1(X_1^\circ)$ is a (normal) subgroup. Moreover, it has finite index since the image of the composition $\pi_1(G_1) \to \pi_1(X_1^\circ) \to \pi_1(X)$ is finite by our current assumption. Therefore, replacing X_1° by a further finite étale cover, if necessary, we may assume that the representation

$$\pi_1(X_1^\circ) \to \pi_1(X) \to \mathrm{PGL}(n-1,\mathbb{C})$$

induced by ρ factorizes through Z_1°. Shrinking Z_1°, if necessary, we may assume in addition that $\mathscr{F}|_{X_1^\circ} \cong \mathscr{L}_{X_1^\circ}^{\oplus n-1}$ for some line bundle $\mathscr{L}_{X_1^\circ}$ on X_1°. Notice that by construction, $\mathscr{L}_{X_1^\circ}|_{G_1} \equiv 0$. The short exact sequence

$$0 \to T_{G_1} \cong \mathscr{O}_{G_1}^{\oplus n-m} \to T_{X_1^\circ}|_{G_1} \to \mathscr{N}_{G_1/X_1^\circ} \cong \mathscr{O}_{G_1}^{\oplus m} \to 0$$

then shows that $\mathscr{L}_{X_1^\circ}|_{G_1} \cong \mathscr{O}_{G_1}$. Therefore, shrinking Z°, we may assume that $\mathscr{F}|_{X_1^\circ} \cong \mathscr{O}_{X_1^\circ}^{\oplus n-1}$.

Claim 8.3. The fibration g_1° is locally trivial for the analytic topology, and hence $X_1^\circ/Z_1^\circ \cong Z_1^\circ \times A/Z_1^\circ$ for some positive-dimensional abelian variety A.

Proof of Claim 8.3. Suppose first that $T_G \subseteq \mathscr{F}|_G$ for a general fiber G of g.

Then, there exists a regular foliation $\mathscr{G}^\circ$ on Z° such that $\mathscr{F}|_{X^\circ} = (g^\circ)^{-1}\mathscr{G}^\circ$. Let L_1 be a leaf of $\mathscr{G}_1^\circ := \mathscr{G}^\circ|_{Z_1^\circ}$, and set $M_1 := (g_1^\circ)^{-1}(L_1)$. Notice that M_1 is a leaf of $\mathscr{F}|_{X_1^\circ}$. By construction, there is an exact sequence:

$$0 \to T_{M_1/L_1} \to \mathscr{F}|_{M_1} \cong T_{M_1} \to (g_1^\circ|_{M_1})^* T_{L_1} \to 0.$$

Since $\mathscr{F}|_{X_1^\circ} \cong \mathscr{O}_{X_1^\circ}^{\oplus n-1}$, the above exact sequence is locally split. A classical result of complex analysis says that complex flows of vector fields on analytic spaces exist. As a consequence, the fibration $g_1^\circ|_{M_1} \colon M_1 \to L_1$ is locally trivial for the analytic topology. Let $\mathscr{A}(3)$ be the fine moduli space of polarized abelian varieties with a level three structure, and let $Z_1^\circ \to \mathscr{A}(3)$ be the morphism corresponding to g_1°. Then every leaf of $\mathscr{G}_1^\circ$ is contained in a fiber of $Z_1^\circ \to \mathscr{A}(3)$. Thus, either $\mathscr{G}_1^\circ$ has algebraic leaves, or the morphism $Z_1^\circ \to \mathscr{A}(3)$ is constant. In the former case, $\mathscr{F}$ is algebraically integrable. By Proposition 3.20, we must have $K_{\mathscr{F}} \equiv 0$ and $F = X$ is a finite étale quotient of an abelian variety since $\mathscr{N}$ is pseudo-effective by assumption. In particular, X has generically large

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fundamental group on F, yielding a contradiction. In the latter case, $X_1^\circ/Z_1^\circ \cong Z_1^\circ \times A/Z_1^\circ$ for some positive-dimensional abelian variety A, proving the claim in this case.

Suppose now that $T_G \not\subseteq \mathscr{F}|_G$ for a general fiber G of g.

Then the composition

$$\mathscr{F}|_{G_1} \cong \mathscr{O}_{G_1}^{\oplus n-1} \to T_{X_1^\circ}|_{G_1} \to (g_1^\circ)^* T_{Z_1^\circ}|_{G_1} \cong \mathscr{O}_{G_1}^{\oplus \dim Z^\circ}$$

is generically surjective, and hence surjective. This immediately implies that the exact sequence

$$0 \to T_{X_1^\circ/Z_1^\circ} \to T_{X_1^\circ} \to (g_1^\circ)^* T_{Z_1^\circ} \to 0$$

is locally split. As before, it follows that the fibration g_1° is locally trivial for the analytic topology, and hence $X_1^\circ/Z_1^\circ \cong Z_1^\circ \times A/Z_1^\circ$ for some positive-dimensional abelian variety A. This finishes the proof of Claim 8.3. □

Let W be the normalization of the closed subvariety of the Chow variety of X whose general point parametrizes a general fiber of g, and let U be the normalization of the universal cycle. By Claim 8.3, the morphism p is generically isotrivial. Hence, there exists a smooth projective variety W_1 as well as a generically finite morphism $\eta\colon W_1 \to W$ and a rational map $W_1 \times A \dashrightarrow U$ fitting into the following commutative diagram:

$$\begin{array}{ccccc} W_1 \times A & \dashrightarrow & U & \xrightarrow{q} & X \\ {\scriptstyle \mathrm{pr}_{W_1}}\downarrow & & \downarrow{\scriptstyle p} & & \vdots{\scriptstyle g} \\ W_1 & \xrightarrow{\eta} & W & \dashrightarrow & Z. \end{array}$$

Claim 8.4. The map $W_1 \times A \dashrightarrow U$ is a morphism.

Proof. Let $p_1\colon Z \to W_1 \times A$ be a composition of a finite number of blow-ups with smooth centers such that the induced rational map $q_1\colon Z \to U$ is a morphism. Let $Z_{(w_1,a)}$ be a positive-dimensional fiber of p_1 over $(w_1,a) \in W_1 \times A$. Then $Z_{(w_1,a)}$ is rationally chain connected, and the composition

$$q_1(Z_{(w_1,a)}) \subseteq U_{\eta(w_1)} \to X$$

is finite, where $U_{\eta(w_1)}$ denotes the fiber of p over the point $\eta(w_1)$. Thus, if $\dim q_1(Z_{(w_1,a)}) \geqslant 1$, then $U_{\eta(w_1)}$ contains a rational curve C such that $\dim q(C) \geqslant 1$. By Corollary 3.12, we must have $q(C) \cdot K_{\mathscr{F}} = 0$. But this contradicts Lemma 3.17 since K_X is nef by assumption. Therefore, every fiber of p_1 is contracted by q_1. Our claim now follows from the rigidity lemma (see [19, Lemma 1.15]). □

A consequence of Claim 8.4 is that every fiber of p is irreducible. This implies that the exceptional locus E of q satisfies $E = p^{-1}(p(E))$. Set $X^\bullet := X \setminus q(E)$ and $W^\bullet := W \setminus p(E)$. Then the rational map $p \circ q^{-1}$ induces an equidimensional morphism $p^\bullet\colon X^\bullet \to W^\bullet$ with irreducible fibers. By construction, $X \setminus X^\bullet$ has codimension at leat 2. Shrinking $W^\bullet$, if necessary, we may assume without loss of generality that $W^\bullet$ is smooth and that there exists a smooth divisor $D^\bullet$ on $W^\bullet$ such that the restriction of $p^\bullet$ to $(p^\bullet)^{-1}(W^\bullet \setminus D^\bullet)$ is

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a smooth morphism. Notice that every smooth fiber of $p^\bullet$ is a finite étale quotient of an abelian variety by Lemma 3.7.

Claim 8.5. Let $w \in D^\bullet$ be a general point. Then the fiber $(p^\bullet)^{-1}(w)$ (with its reduced structure) is a finite étale quotient of an abelian variety.

Proof of Claim 8.5. The statement is local on $D^\bullet$, hence we may shrink $W^\bullet$ and assume that $D^\bullet$ is irreducible. Let m denote the multiplicity of $(p^\bullet)^*D^\bullet$, and let $\gamma^\bullet\colon W_1^\bullet \to W^\circ$ be a quasi-finite morphism with $W_1^\bullet$ smooth such that $(\gamma^\bullet)^*D^\bullet = mD_1^\bullet$ with $D_1^\bullet$ smooth (and $D_1^\bullet \neq \emptyset$). We may also assume that $\gamma^\bullet$ is étale away from $\operatorname{Supp} D_1^\bullet$. Let $X_1^\bullet$ be the normalization of the product $W_1^\bullet \times_{W^\bullet} X^\bullet$. Then an easy local computation shows that the natural morphism $X_1^\bullet \to X^\bullet$ is étale away from a closed subset of codimension at least 2, and hence étale by the Nagata–Zariski purity theorem. In particular, $X_1^\bullet$ is smooth. Shrinking $W_1^\bullet$ further, if necessary, we may assume that the natural morphism $g_1^\bullet\colon X_1^\bullet \to W_1^\bullet$ has generically reduced fibers. Observe also that its fibers are irreducible since p has irreducible fibers and $X_1^\bullet \to X^\bullet$ is étale.

Let $w \in D_1^\bullet$ be a general point. By [27, Proposition 18.13], the scheme theoretic fiber $G_w := (g_1^\bullet)^{-1}(w)$ is Cohen–Macaulay, and hence reduced by Serre's criterion. Moreover, by the adjunction formula, $\omega_{G_w} \cong \omega_{X_1^\bullet}|_{G_w}$. In particular, G_w is Gorenstein. By miracle flatness, the morphism $g_1^\bullet$ is flat (see [55, Tag 00R4]). Because the functions $w \mapsto h^0\bigl(G_w, \omega_{X_1^\bullet}^{\otimes \pm 1}|_{G_w}\bigr)$ are upper semicontinuous in the Zariski topology on $W_1^\bullet$ (see [36, Theorem 12.8]), we have $\omega_{G_w} \cong \mathscr{O}_{G_w}$. Let $\nu_w\colon \widetilde{G}_w \to G_w$ be the normalization morphism. By Lemma 3.3, there exists an effective Weil divisor E_w on $\widetilde{G}_w$ such that $\omega_{\widetilde{G}_w} \cong \nu_w^*\omega_{G_w}(-E_w) \cong \mathscr{O}_{\widetilde{G}_w}(-E_w)$. Suppose that $E_w \neq 0$. Then $K_{\widetilde{G}_w}$ is not pseudo-effective and hence G_w is uniruled by [5, Corollary 0.3] applied to a resolution of $\widetilde{G}_w$. Let $C \subseteq G_w$ be a rational curve. By Corollary 3.12, we have $C \cdot K_{\mathscr{F}} = 0$. But this contradicts Lemma 3.17 since K_X is nef. This shows that $E_w = 0$, and hence G_w is normal by Lemma 3.3. Then [5, Corollary 0.3] applies to show that G_w has canonical singularities since $K_{G_w} = 0$.

Now, replacing $W_1^\bullet$ with a neighborhood of w for the analytic topology, we may assume without loss of generality the $W_1^\bullet$ is biholomorphic to the open unit ball in $\mathbb{C}^{\dim W_1^\bullet}$. The same argument used in the proof of [48, Lemma 5.2.2] shows that there is a surjective homomorphism

$$\pi_1(G_{w_1}) \twoheadrightarrow \pi_1(X_1^\bullet),$$

where $\omega_1 \in W_1^\bullet \setminus D_1^\bullet$ is any point. This implies that the image of the composition

$$\pi_1(X_1^\bullet) \to \pi_1(X^\circ) \to \pi_1(X)$$

is finite by our current assumption. Thus, replacing $X_1^\bullet$ by a finite étale cover, if necessary, we may assume that there exists a line bundle $\mathscr{L}_{X_1^\bullet}$ on $X_1^\bullet$ such that $\mathscr{F}|_{X_1^\bullet} \cong \mathscr{L}_{X_1^\bullet}^{\oplus n-1}$. By Step 1, the restriction of $\mathscr{L}_{X_1^\bullet}$ to a general (smooth) fiber of $g_1^\bullet$ is a torsion line bundle. Moreover, shrinking $(W_1^\bullet, w)$ further, we may assume without loss of generality that $\mathscr{L}_{X_1^\bullet}$ is a torsion line bundle as well (see Lemma 3.9). Thus, replacing $X_1^\bullet$ by a finite étale cover,

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if necessary, we may therefore assume that $\mathscr{L}_{X_1^\bullet} \cong \mathscr{O}_{X_1^\bullet}$. It follows that the composition

$$\mathscr{F}|_{G_w} \to T_{X_1^\bullet}|_{G_w} \to \mathscr{N}_{G_w/X_1^\bullet} \cong \mathscr{O}_{G_w}^{\oplus m}$$

has constant rank. This in turn implies that the reflexive sheaf $\mathscr{F}|_{G_w} \cap T_{G_w}$ is a numerically flat locally free sheaf.

Suppose first that $T_{G_w} \subseteq \mathscr{F}|_{G_w}$ for a general point $w \in D_1^\bullet$. Then T_{G_w} is locally free. By the solution of the Zariski–Lipman conjecture for klt spaces (see [29, Theorem 6.1] or [23, Theorem 1.1]), we see that G_w is smooth. The claim now follows from Lemma 3.7.

Suppose finally that $T_{G_w} \not\subseteq \mathscr{F}|_{G_w}$ for a general point $w \in D_1^\bullet$. Then the composition

$$\mathscr{F}|_{G_w} \to T_{X_1^\bullet}|_{G_w} \to (g_1^\bullet)^* T_{W_1^\bullet}|_{G_w}$$

is generically surjective, and hence surjective. This in turn implies that $g_1^\bullet$ is a locally trivial fibration for the analytic topology in a neighborhood of a general point in $D_1^\bullet$. In particular, G_w is smooth. The claim now follows again from Lemma 3.7. □

By Claim 8.5, replacing $W^\bullet$ by an open subset with complement of codimension at least 2, we may assume that every fiber of $p^\bullet$ (with its reduced structure) is a finite étale quotient of an abelian variety. By Lemma 3.6 applied to $p^\bullet \colon X^\bullet \to W^\bullet$, $X^\bullet$ has generically large fundamental group on G. On the other hand, the inclusion induces an isomorphism $\pi_1(X^\bullet) \cong \pi_1(X)$ of fundamental groups since $X \setminus X^\bullet$ is a closed subset of codimension at least 2 by construction. It follows that X has generically large fundamental group on G, yielding a contradiction. This shows that X has generically large fundamental group on F.

Step 3. End of proof.

By [48, Theorem 6.3] and Step 2 together, there exists a finite étale cover $\gamma \colon X_1 \to X$ such that the fibration $f_1 \colon X_1 \dashrightarrow Y_1$ obtained as the Stein factorization of the almost proper map $X_1 \to X \dashrightarrow Y$ is birational to an abelian group scheme $f_1 \colon X_2 \to Y_2$ over a projective base equipped with a level three structure.

Let W_1 be the normalization of the closed subvariety of the Chow variety of X_1 whose general point parametrizes a general fiber of f_1, and let U_1 be the normalization of the universal cycle. Replacing Y_2 by a birational modification, if necessary, we may assume that there exist a birational morphism $Y_2 \to W_1$ and a commutative diagram

$$\begin{array}{ccccc} X_2 & \dashrightarrow & U_1 & \longrightarrow & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow & & \vdots\,{\scriptstyle f_1} \\ Y_2 & \longrightarrow & W_1 & \overset{\text{birational}}{\dashrightarrow} & Y_1. \end{array}$$

Arguing as in the proof of Claim 8.4, we see that rational map $X_2 \dashrightarrow U_1$ is a morphism. Then Lemma 3.8 applies to show that X_1 is an abelian scheme $f_3 \colon X_1 \to Y_3$ over a smooth projective base Y_3 and that there exists a birational morphism $Y_2 \to Y_3$ such that $X_2/Y_2 \cong Y_2 \times_{Y_3} X_1/Y_2$. This finishes the proof of first statement of Proposition 8.2.

Finally, we show that there exist line bundles $\mathscr{M}_3$ and $\mathscr{K}_3$ on Y_3 such that $\gamma^*\mathscr{N} \cong f_3^*\mathscr{M}_3$ and $\gamma^*\mathscr{O}_X(K_\mathscr{K}) \cong f_3^*\mathscr{K}_3$. By Step 1, $K_\mathscr{F}|_F \sim_\mathbb{Q} 0$ and $\mathscr{N}|_F$ is a torsion line bundle. Then Lemma 3.9 applies to show that there exists a positive integer k such that $\gamma^*\mathscr{N}^{\otimes k} \cong f_3^*\mathscr{M}_3$

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and $\gamma^*\mathscr{O}_X(kK_{\mathscr{K}}) \cong f_3^*\mathscr{K}_3$, where $\mathscr{M}_3$ and $\mathscr{K}_3$ are line bundles on Y_3. Let $\Sigma_3 \subseteq X_1$ be the neutral section of f_3. Then $\mathscr{M}_3 \cong (\gamma^*\mathscr{N}|_{\Sigma_3})^{\otimes k}$ viewing $(\gamma^*\mathscr{N}|_{\Sigma_3})^{\otimes k}$ as a line bundle on Y_3. Hence $\gamma^*\mathscr{N}^{\otimes k} \cong f_3^*(\gamma^*\mathscr{N}|_{\Sigma_3})^{\otimes k}$. Replacing X_1 by the associated cyclic étale cover (see [45, Definition 2.52]), we may assume without loss of generality that $k = 1$. This completes the proof of the proposition. □

Corollary 8.6. *Setting and notation as in 2.19. Suppose in addition that X is of general type and $\dim X \geqslant 4$. Then $\mathscr{N}^*$ is nef.*

Proof. By Theorem 5.1, K_X is nef. Then Proposition 8.1 applies to show that $\mathscr{N}$ is not pseudo-effecive. The claim now follows from Theorem 4.4. □

9. Projectively flat foliations with pseudo-effective normal bundle

Theorem 9.1. *Setting and notation as in 2.19. Suppose in addition that K_X is nef and that $\mathscr{N}$ is pseudo-effective. Then $K_{\mathscr{F}} \equiv 0$.*

Proof. Let $f\colon X \dashrightarrow Y$ be the nef reduction map of $K_{\mathscr{F}}$, and let F be a general fiber of f. Set $m := \dim Y$. Notice that $m \leqslant n-1$ by Proposition 8.1. By Proposition 8.2, replacing X by a finite étale cover, if necessary, we may assume without loss of generality that $f\colon X \to Y$ is an abelian group scheme over a smooth projective base equipped with a level three structure. We may also assume that $\mathscr{F}|_F \cap T_F$ is a linear foliation of codimension at most 1 and that there exist line bundles $\mathscr{M}$ and $\mathscr{K}$ on Y such that $\mathscr{N} \cong f^*\mathscr{M}$ and $\mathscr{O}_X(K_{\mathscr{F}}) \cong f^*\mathscr{K}$. Let $\Sigma \subset X$ be the neutral section of f. Notice that $\mathscr{K} \cong \mathscr{O}_X(K_{\mathscr{F}})|_\Sigma$.

Notice that $m = 0$ if and only if $K_{\mathscr{F}} \equiv 0$.

Let us first show that $T_{X/Y} \not\subseteq \mathscr{F}$. We argue by contradiction and assume that $T_{X/Y} \subseteq \mathscr{F}$. In particular, we have $m \geqslant 1$.

If $m = 1$, then $\mathscr{F} = T_{X/Y}$. In other words, $\mathscr{F}$ is algebraically integrable. By Proposition 3.20, X is a finite étale quotient of an abelian variety and $K_{\mathscr{F}} \equiv 0$ since $\mathscr{N}$ is pseudo-effective by assumption, yielding a contradiction.

Suppose now that $m \geqslant 2$. There is a regular foliation $\mathscr{G}$ on Y such that $\mathscr{F} = f^{-1}\mathscr{G}$ since $T_{X/Y} \subseteq \mathscr{F}$ by assumption (see [1, Lemma 6.7]). Then $\mathscr{M} \cong \mathscr{N}_{\mathscr{G}}$. Moreover, there is an exact sequence

$$0 \to T_{X/Y} \to \mathscr{F} \to f^*\mathscr{G} \to 0.$$

In addition, the composition

$$T_{X/Y}|_\Sigma \subset \mathscr{F}|_\Sigma \subset T_X|_\Sigma \to \mathscr{N}_{\Sigma/X}$$

is an isomorphism, and hence

$$\mathscr{F}|_\Sigma \cong T_{X/Y}|_\Sigma \oplus \mathscr{G}.$$

By Lemma 2.14, the vector bundles $\mathscr{F}|_\Sigma$ and $\mathscr{G}$ are numerically projectively flat. Moreover, we have

$$\frac{1}{n-1}c_1(\mathscr{F}|_\Sigma) \equiv \frac{1}{d}c_1(T_{X/Y}|_\Sigma) \equiv \frac{1}{m-1}c_1(\mathscr{G}).$$

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Since f is the nef reduction map of $K_{\mathscr{F}}$, we must have $\mathrm{nd}(\mathscr{K}) = m$. It follows that the nef reduction map of $K_{\mathscr{G}} \equiv \frac{m-1}{n-1} c_1(\mathscr{K})$ is the identity map. Notice that $\mathscr{N}_{\mathscr{G}}$ is pseudo-effective because $\mathscr{N}$ is pseudo-effective by assumption.

Claim 9.2. The divisor K_Y is nef.

Proof. We argue by contradiction and assume that K_Y is not nef. Then there is a rational curve C on Y such that $K_Y \cdot C < 0$. Let $B \to C$ denotes the normalization of C, and set $S := B \times_Y X$ and $\sigma := B \times_Y \Sigma$. By [48, Lemma 5.9.3], we have $(S/B,\sigma) \cong (B \times A/B, B \times 0_A)$ for some abelian variety A. Moreover, $K_{X/Y}|_S = K_{S/B} \sim_{\mathbb{Z}} 0$. This immediately implies that K_X is not nef, yielding a contradiction. This proves the claim. □

Applying Proposition 8.1 to $\mathscr{G}$, we see that $m = 2$.

If $\kappa(Y) = 2$, then $\mathscr{N}_{\mathscr{G}}^*$ is pseudo-effective by [11, Lemme 9] and [11, Corollaire 1], and hence $\mathscr{N} \equiv 0$ since $\mathscr{N} \cong f^*\mathscr{N}_{\mathscr{G}}$ is pseudo-effective by assumption. By Theorem 7.2 and Proposition 8.1 together, we have $K_{\mathscr{F}} \equiv 0$, yielding a contradiction.

From [11, Section 5], we see that either Y is an elliptic surface or the minimal model of Y is an abelian variety. In the latter case, X is an abelian variety by [48, Lemma 5.9.3]. Then Lemma 3.22 applies to show that $K_{\mathscr{F}} \equiv 0$. This again gives a contradiction. Suppose finally that Y is an elliptic surface over a smooth connected curve B. By [11, Proposition 2], the fibers (with their reduced structure) of $Y \to B$ are smooth curves of genus 1. Then, by [11, Proposition 3], we have $m = \mathrm{nd}(\mathscr{K}) \leqslant 1$. This again yields a contradiction.

This shows that $T_{X/Y} \not\subseteq \mathscr{F}$.

Then Lemma 7.1 applies to show that, after replacing X by a finite étale cover, we may assume that $X \cong Y \times A$ for some abelian variety A of dimension and that f is induced by the projection onto Y.

Let us show that $m \leqslant 1$. The composition

$$\mathscr{O}_X^{\oplus n-m} \cong T_{X/Y} \to T_X \to \mathscr{N}$$

is generically surjective, and hence $h^0(X,\mathscr{N}) \geqslant 1$. Since $\mathscr{N} \cong f^*\mathscr{M}$, there exist prime divisors D_i on Y for $i \in I$ and positive integers $m_i \geqslant 1$ such that $\mathscr{N} \cong \mathscr{O}_X(\sum_{i\in I} m_i B_i)$, where $B_i := f^*D_i$.

Then $\mathscr{O}_{B_i}^{\oplus d} \cong T_{X/Y}|_{B_i} \subseteq \mathscr{F}|_{B_i}$. Since $\mathscr{F}$ is numerically projectively flat and $K_{\mathscr{F}}$ is nef by Lemma 2.21, we see that $\mathscr{F}|_{B_i}$ is numerically flat.

If B_i is a leaf of $\mathscr{F}$, then it is a finite étale quotient of an abelian variety by [42, Theorem 0.1]. Moreover, $\mathscr{N}|_{B_i}$ is a flat line bundle by Lemma 3.1, and hence $B_i^2 \equiv 0$ since $\mathscr{N}|_{B_i} \cong \mathscr{N}_{B_i/X}$.

Suppose now that B_i is not a leaf of $\mathscr{F}$. Then the composition

$$\mathscr{F}|_{B_i} \to T_X|_{B_i} \to \mathscr{N}_{B_i/X}$$

is generically surjective. Since $\mathscr{F}|_{B_i}$ is numerically flat, it follows that B_i^2 is numerically equivalent to an effective cycle.

Now, by Lemma 3.1, we have $\mathscr{N}^2 \equiv 0$. This immediately implies that $B_i^2 \equiv 0$ for every $i \in I$ and that $B_i \cap B_j = \emptyset$ if $i \neq j$.

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If $\mathscr{N} \equiv 0$, then $K_{\mathscr{F}} \equiv 0$ by Theorem 7.2 and Proposition 8.1 together. Then $m = 0$. Suppose from now on that $\mathscr{N} \not\equiv 0$.

Let $i \in I$ such that B_i is not a leaf of $\mathscr{F}$. Let us show that B_i is smooth with $K_{B_i} \equiv 0$. The exact sequence

$$0 \to \mathscr{F}|_{B_i} \to T_X|_{B_i} \to \mathscr{N}|_{B_i} \to 0$$

shows that $T_X|_{B_i}$ is numerically flat. In particular, $K_X|_{B_i} \equiv 0$, and hence $\omega_{B_i} \equiv 0$ by the adjunction formula. Let $\nu_i \colon \widetilde{B}_i \to B_i$ be the normalization morphism. By Lemma 3.3, there exists an effective Weil divisor E_i on $\widetilde{B}_i$ such that $\omega_{\widetilde{B}_i} \cong \nu_i^* \omega_{B_i}(-E_i)$. Suppose that $E_i \neq 0$. Then $K_{\widetilde{B}_i}$ is not pseudo-effective and hence B_i is uniruled by [5, Corollary 0.3] applied to a resolution of $\widetilde{B}_i$. Let $C \subseteq B_i$ be a rational curve. By Corollary 3.12, we have $C \cdot K_{\mathscr{F}} = 0$. But this contradicts Lemma 3.17 since K_X is nef. This shows that $E_i = 0$, and hence B_i is normal by Lemma 3.3. Then [5, Corollary 0.3] applies to show that B_i has canonical singularities since $K_{B_i} \equiv 0$. The natural map $T_X|_{B_i} \to \mathscr{N}_{B_i/X}$ is generically surjective and hence surjective since $T_X|_{B_i}$ and $\mathscr{N}_{B_i/X}$ are numerically flat vector bundles. It follows that the tangent sheaf T_{B_i} is locally free and numerically flat. By the solution of the Zariski–Lipman conjecture for canonical spaces (see [29, Theorem 6.1] or [23, Theorem 1.1]), we conclude that B_i is smooth.

By Proposition 6.3, we have $\kappa(X) = \nu(X) \geqslant 0$. Applying Proposition 3.14 to $(X, \sum_{i \in I} B_i)$, we see that $K_{\mathscr{F}}$ is semiample with $\nu(K_{\mathscr{F}}) \leqslant 1$. This proves that $m = 1$.

By Lemma 3.23 and Lemma 2.21 together, the vector bundle $\mathscr{F}$ is then numerically flat, and hence $K_{\mathscr{F}} = 0$, yielding a contradiction. This finishes the proof of the theorem. □

10. Proof of Theorem 1.1

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose first that K_X is not nef. Then there exists a $\mathbb{P}^1$-bundle structure $\varphi \colon X \to Y$ onto a finite étale quotient of an abelian variety, and $\mathscr{F}$ induces a flat Ehresmann connection on φ by Theorem 5.1.

Suppose from now on that K_X is nef.

If $\mathscr{N}$ is pseudo-effective, then Theorem 9.1 applies to show that $K_{\mathscr{F}} \equiv 0$. By Proposition 3.18, there exists an abelian variety A and a finite étale cover $\gamma \colon A \to X$ such that $\gamma^{-1}\mathscr{F}$ is a linear foliation on A.

Suppose finally that $\mathscr{N}$ is not pseudo-effective. Then $\mathscr{N}^*$ is nef by Theorem 1.2. Thus Theorem 7.2 applies to show that either there exists a smooth complete curve B of genus at least 2 as well as an abelian variety A, and a finite étale cover $\gamma \colon B \times A \to X$ such that $\gamma^{-1}\mathscr{F}$ is induced by the projection morphism $B \times A \to B$, or X is of general type and $\nu(K_{\mathscr{F}}) = \dim X$. In the latter case, Ω^1_X is nef by Lemma 2.21, and hence K_X is ample. In addition, $\kappa(K_{\mathscr{F}}) = \dim X$ by Lemma 7.4. This finishes the proof of theorem. □

Remark 10.1. If we relax the integrability condition, then the conclusion of Theorem 1.1 is false. Indeed, let E be an elliptic curve, and let B be an abelian variety of dimension $\dim B = n - 1 \geqslant 1$. Set $A := E \times B$. Let $\mathscr{M}$ be a line bundle of degree 2 on E, and set $\mathscr{L} := \mathrm{pr}_E^* \mathscr{M}^*$. Let $(s_i, t_i) \in H^0(A, \mathscr{L}^*)^{\times 2}$ for $i \in \{1, \ldots, n-1\}$. Suppose that the zero sets

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of the s_i and the t_i are pairwise disjoint. Write $T_A = \oplus_{1 \leqslant i \leqslant n} \mathscr{O}_A v_i$. Then the sections s_i and t_i give an injective map of vector bundles $\mathscr{L} \subset \mathscr{O}_A v_i \oplus \mathscr{O}_A v_{i+1}$. In addition, the induced map $\mathscr{D} = \mathscr{L}^{\oplus n-1} \to T_A$ is an injective map of vector bundles as well. The vector bundle is obviously numerically projectively flat.

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