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ARTICLE

Rough permutations with a fixed set of a given size

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Abstract

Let r, k, n be integers satisfying $1 \leqslant r \leqslant k \leqslant n/2$. Let $\mathcal{R}_r(n,k)$ denote the proportion of permutations $\pi \in \mathcal{S}_n$ that fix a set of size k and have no cycle of length less than r. In this note, we determine the order of magnitude of $\mathcal{R}_r(n,k)$ uniformly for all $2 \leqslant r \leqslant k \leqslant n/2$. This result generalises the corresponding estimate of Eberhard, Ford, and Green for the case $r = 1$.

Keywords: Rough permutation; fixed set; divisor

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1. Introduction

Let $\mathcal{S}_n$ denote the symmetric group on a set of n elements. Recall that each permutation can be written uniquely as a product of disjoint cycles (see, for example, [1] or [24]). For a permutation $\pi \in \mathcal{S}_n$, a *fixed set* of π is a subset of $\{1, \dots, n\}$ that is fixed by π. A fixed set corresponds to a *divisor* of π, that is, a product of some subset of the disjoint cycles in π (we include both the empty set and the whole set $\{1, \dots, n\}$ as fixed sets). Fixed sets play the same role for permutations as divisors do for integers. The existence of fixed sets of a given size has applications to various questions in combinatorial group theory, such as the generation of $\mathcal{S}_n$ by random permutations and the distribution of transitive subgroups of $\mathcal{S}_n$. See, for example, [2–7, 15, 18, 20, 26]. For more information and references, one can refer to [14].

Let k, n be integers with $1 \leqslant k \leqslant n/2$. We consider a random permutation $\pi \in \mathcal{S}_n$, each permutation occurring with probability $1/n!$. A basic problem is to estimate $i(n,k)$, the probability that a random permutation $\pi \in \mathcal{S}_n$ has a fixed set of size k. Here and throughout, we write $f \ll g$ or $f = O(g)$ to mean that there is a positive constant C such that $|f| \leqslant Cg$ holds on the domain of f. The notation $f \asymp g$ means that both $f \ll g$ and $g \ll f$ hold. Unless indicated by subscripts, the implied constant C is absolute – independent of all parameters. For example, $\log x \ll_\epsilon x^\epsilon$ for each fixed $\epsilon > 0$ and all $x \geqslant 1$. The upper bound $i(n,k) \ll k^{-1/100}$ was first established by Łuczak and Pyber [18]. Fifteen years later, Diaconis, Fulman, and Guralnick [3] obtained the lower bound $\lim_{n\to\infty} i(n,k) \gg \log k/k$. A sharper estimate was subsequently proved by Pemantle, Peres, and Rivin [20], who showed that $\lim_{n\to\infty} i(n,k) = k^{-\mathcal{E}+o(1)}$, where

$$\mathcal{E} = 1 - \frac{1 + \log\log 2}{\log 2} \approx 0.086071332 \tag{1.1}$$

is referred to as the Erdős-Tenenbaum-Ford constant by some authors [17, 22]. The authors of [20] also pointed out a surprising connection between the problem of estimating $i(n,k)$ and the

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problem of estimating the proportion of integers $n \leqslant x$ with a divisor in a given dyadic interval $(y, 2y]$ which was solved up to a constant factor by Ford in [10, 11]. This in turn relates to the classical Erdős multiplication table problem ([8, 9, 25]), which asks for an estimate of the number $A(N)$ of distinct products of the form ab with $a \leqslant N$, $b \leqslant N$. We remark that the same connection was independently noted by Diaconis and Soundararajan [23, p. 14]. Building on the remarkable connection, Eberhard, Ford, and Green [5] formulated a parallel theory of permutations, and ultimately proved that

$$i(n, k) \asymp k^{-\mathcal{E}}(1 + \log k)^{-3/2} \tag{1.2}$$

holds uniformly for $1 \leqslant k \leqslant n/2$.

In the web seminar on analytic and probabilistic number theory held on December 3, 2020, Ford [12] proposed a more general problem aimed at estimating the proportion of permutations $\pi \in \mathcal{S}_n$ for which there exist two divisors π_1 and π_2 with prescribed properties such that $\pi = \pi_1\pi_2$. Let r be a positive integer. Motivated by the work on rough integers of Ford [13], a natural set of permutations to be considered is

$$\mathcal{S}_n^{(r)} = \{\pi \in \mathcal{S}_n : \pi \text{ has no cycle of length less than } r\}. \tag{1.3}$$

For analogy with rough integers in [13], the permutations in $\mathcal{S}_n^{(r)}$ are called $(r-1)$-*rough permutations*. It is evident that $\mathcal{S}_n^{(1)}$ is precisely the symmetric group $\mathcal{S}_n$ of degree n. Also note that $\mathcal{S}_n^{(2)}$ coincides with the set of all derangements, i.e., permutations in $\mathcal{S}_n$ without fixed points. The asymptotic behaviour of the size of $\mathcal{S}_n^{(r)}$ has received considerable attention (see, e.g., [14, 16, 21]). In [13], Ford determined the number of integers $n \leqslant x$ that have a divisor in $(y, 2y]$ and no prime factor $\leqslant w$ uniformly in x, y, w with $4 \leqslant y \leqslant \sqrt{x}$ and $4 \leqslant w \leqslant y/8$. Making analogy with Ford's work in [13], we estimate the proportion $\mathcal{R}_r(n, k)$ of those permutations $\pi \in \mathcal{S}_n$ having a fixed set of size k and having no cycle factors of length less than r. In this note, we determine the exact order of magnitude of $\mathcal{R}_r(n, k)$ uniformly for all n, k, r with $2 \leqslant r \leqslant k \leqslant n/2$.

Theorem 1.1. *Let n, k, r be integers with $2 \leqslant r \leqslant k \leqslant n/2$ and let $\delta = \frac{\log r}{\log k}$. Then we have*

$$\mathcal{R}_r(n, k) \asymp \begin{cases} \frac{1}{r^2}, & \text{if } 1 - 1/\log 4 \leqslant \delta \leqslant 1, \\ \delta B(r, k) k^{-\mathcal{E} + \frac{\log(1-\delta)}{\log 2}}, & \text{if } 0 < \delta < 1 - 1/\log 4, \end{cases}$$

where $\mathcal{E}$ is defined as in (1.1) and

$$B(r, k) = \min\left(1, (\log k)^{-1/2}((1-\delta)\log 4 - 1)^{-1}\right). \tag{1.4}$$

Note that one can check that $\delta B(r, k) k^{-\mathcal{E} + (\log(1-\delta))/\log 2}$ is of the same order of magnitude as $\frac{1}{r^2}$ if δ approaches $1 - 1/\log 4$ from the left, and $\delta B(r, k) k^{-\mathcal{E} + (\log(1-\delta))/\log 2}$ is of the same order of magnitude as $k^{-\mathcal{E}}(1 + \log k)^{-3/2}(\log r)(r^{-1/\log 2})$ if $\log r \leqslant \sqrt{\log k}$. In particular, if $r = O(1)$, then $\mathcal{R}_r(n, k) \asymp_r i(n, k)$.

For $r = 1$, $\mathcal{R}_r(n, k)$ reduces to the probability $i(n, k)$ that a random permutation $\pi \in \mathcal{S}_n$ has a fixed set of size k, which was determined by Eberhard, Ford, and Green [5]. Our Theorem 1.1 generalizes their result. Notably, for $r = 2$, we have $\mathcal{R}_2(n, k) \asymp i(n, k)$; this corresponds to the probability that a random permutation is a derangement with a fixed set of size k ($k \geqslant 2$).

Given the strength of the analogy with [13, Theorem 1.1], one might hope to be able to quickly deduce Theorem 1.1 by directly using transference ideas. Unfortunately, such a direct approach does not seem feasible. Instead, our proof of Theorem 1.1 combines the ideas of Ford [13] with those of Eberhard, Ford, and Green [5].

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Suppose $n = 2k$. We say that a permutation $\sigma \in \mathcal{S}_n$ is *perfectly balanced* if it has a divisor of size k (see [12]). In this case, σ is the product of two divisors, each of size k. The enumeration of perfectly balanced rough permutations is the direct analogue, in the context of permutations, of the restricted multiplication table problem for rough integers studied by Ford (see [13, Corolary 1.4]). By applying Theorem 1.1, one can readily determine the probability that a uniformly random permutation in $\mathcal{S}_{2k}$ has no cycle of length less than r and is perfectly balanced.

Corollary 1.2. *Let k, r be integers with $2 \leqslant r \leqslant k$. If $\log r \geqslant (1 - 1/\log 4)\log k$, then we have*

$$\frac{1}{(2k)!}\#\{\pi \in \mathcal{S}_{2k}^{(r)} : \pi = \sigma_1\sigma_2, |\sigma_1| = |\sigma_2| = k\} \asymp \frac{1}{r^2}.$$

Otherwise, we have

$$\frac{1}{(2k)!}\#\{\pi \in \mathcal{S}_{2k}^{(r)} : \pi = \sigma_1\sigma_2, |\sigma_1| = |\sigma_2| = k\} \asymp \frac{B(r,k)\log r}{k^{\mathcal{E}-(\log(1-\delta))/\log 2}\log k},$$

where $\mathcal{S}_{2k}^{(r)}$ and $B(r,k)$ are defined as in (1.3) and (1.4) respectively.

2. Preliminaries

2.1 Notation

For convenience, we first give some notation:

- Let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$. For any positive integers i, j with $i \leqslant j$, let $[i,j] = \{m \in \mathbb{N} : i \leqslant m \leqslant j\}$ and $[j] = \{m \in \mathbb{N} : 1 \leqslant m \leqslant j\}$.
- $C_j(\pi)$ is the number of cycles of length j in the permutation π, and $C_I(\pi)$ is the number of cycles whose lengths lie in the set I.
- $\beta|\pi$ means that β is a fixed set or divisor of the permutation π, i.e., a product of some subset of the cycles of π.
- $|\beta|$ is the size of β.
- For any positive integer t with $t \geqslant r$, let

$$\mathcal{C}_{r,t} = \{(c_1, \dots, c_r, \dots, c_t) \in \mathbb{N}_0^t : c_1 = \cdots = c_{r-1} = 0\}.$$

- $H_m = 1 + 1/2 + \cdots + 1/m$ is the m-th harmonic sum.
- $\mathbb{1}(S)$ is the indicator function of statement S; $\mathbb{1}(S) = 1$ if S is true, and $\mathbb{1}(S) = 0$ if S is false.
- $\mathbb{P}(A)$ is the probability of the event A.
- $\mathbb{E}(X)$ is the expectation of the random variable X.
- $\mathcal{C}_i^{(n)}$ is the set of cycles of length i in $\mathcal{S}_n$.
- For a vector $\mathbf{c} = (c_1, \dots, c_k)$ of nonnegative integers, let

$$\mathscr{L}(\mathbf{c}) = \{m_1 + 2m_2 + \cdots + km_k : 0 \leqslant m_j \leqslant c_j \text{ for } j = 1, 2, \dots, k\} \text{ and } S(\mathbf{c}) = \max \mathscr{L}(\mathbf{c}).$$

- Let $\mathbf{X}_{r,k}$ denote the random vector $(X_r, \dots, X_k)$, where $X_r, \dots, X_k$ are independent Poisson random variables with parameters $1/r, \dots, 1/k$, respectively. Furthermore, we write

$$S(\mathbf{X}_{r,t}) = \max \mathscr{L}(\mathbf{X}_{r,t}) = rX_r + \cdots + tX_t.$$

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2.2 Preliminary lemmas

We first record the inequalities

$$\log(m+1) \leqslant H_m \leqslant 1 + \log m, \ (m \geqslant 1). \tag{2.1}$$

Moreover, we state Stirling's formula as follows.

Stirling's formula. For all sufficiently large positive integers n,

$$n! = \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right),$$

or equivalently,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \quad (n \to \infty).$$

The following result is an analogue for permutations of a basic lemma from sieve theory. The readers can refer to [5] or [16].

Lemma 2.1. *Suppose that m, n are integers with $1 \leqslant m \leqslant n$. Let $\pi \in \mathcal{S}_n$ be chosen uniformly at random. Then*

$$\frac{1}{2m} \leqslant \mathbb{P}(\pi \text{ has no cycle of length } < m) \leqslant \frac{1}{m}.$$

We also need standard bounds on the Poisson distribution, see, for example, [19, Section 4]. To facilitate use, we employ the following simplified version proposed by Ford [13, Lemma 2.4].

Lemma 2.2. *Uniformly for $h \leqslant m \leqslant x$,*

$$\sum_{h \leqslant k \leqslant m} \frac{x^k}{k!} \asymp \min\left(\sqrt{x}, \frac{x}{x-m}, m-h+1\right)\frac{x^m}{m!}.$$

The following lemma is Cauchy's classical formula. One may refer to [14, Theorem 1.2] or [24, Proposition 1.3.2].

Lemma 2.3. *Suppose that π is chosen uniformly at random. If $m_1 + 2m_2 + \cdots + nm_n = n$, then*

$$\mathbb{P}\big(C_1(\pi) = m_1, \ldots, C_n(\pi) = m_n\big) = \prod_{j=1}^{n} \frac{(1/j)^{m_j}}{m_j!}.$$

We will also use the following estimate of Eberhard, Ford, and Green [5], which applies to permutations whose cycle structure is constrained only at small lengths.

Lemma 2.4. *Let $1 \leqslant m < n$ and $c_1, \ldots, c_m$ be nonnegative integers satisfying*

$$c_1 + 2c_2 + \cdots + mc_m \leqslant n - m - 1.$$

Suppose that $\pi \in \mathcal{S}_n$ is chosen uniformly at random. Then

$$\frac{1}{2m+2}\prod_{i=1}^{m}\frac{(1/i)^{c_i}}{c_i!} \leqslant \mathbb{P}\big(C_1(\pi) = c_1, \ldots, C_m(\pi) = c_m\big) \leqslant \frac{1}{m+1}\prod_{i=1}^{m}\frac{(1/i)^{c_i}}{c_i!}.$$

Applying Lemmas 2.3 and 2.4, we obtain a crude lower bound for $\mathcal{R}_r(n,k)$.

Lemma 2.5. *For any positive integers r, k, n with $1 \leqslant r \leqslant k \leqslant n/2$, we have*

$$\mathcal{R}_r(n,k) \geqslant \frac{1}{2k(k+1)}.$$

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Proof. If $n = 2k$, then we obtain by Lemma 2.3 that

$$\mathcal{R}_r(n,k) \geqslant \mathbb{P}_{\pi\in\mathcal{S}_n}\big(C_1(\pi) = \cdots = C_{k-1}(\pi) = 0, C_k(\pi) = 2\big) = \frac{1}{2k^2} \geqslant \frac{1}{2k(k+1)}.$$

If $n > 2k$, we then have by Lemma 2.4 that

$$\mathcal{R}_r(n,k) \geqslant \mathbb{P}_{\pi\in\mathcal{S}_n}\big(C_1(\pi) = \cdots = C_{k-1}(\pi) = 0, C_k(\pi) = 1\big) \geqslant \frac{1}{2k(k+1)}$$

as desired. □

By using Lemma 2.1, we derive a crude upper bound for $\mathcal{R}_r(n,k)$ as follows.

Lemma 2.6. *For any positive integers r, k, n with $2 \leqslant r \leqslant k \leqslant n/2$, we have*

$$\mathcal{R}_r(n,k) \leqslant \frac{1}{r^2}.$$

Proof. For convenience, let

$$\mathcal{S}_n^{(r)}(k) := \{\pi \in \mathcal{S}_n^{(r)} : \exists\, \sigma \,|\pi \text{ with } |\sigma| = k\}, \tag{2.2}$$

where $\mathcal{S}_n^{(r)}$ is defined as in (1.3). We then obtain by Lemma 2.1 that

$$\begin{aligned}\big|\mathcal{S}_n^{(r)}(k)\big| &\leqslant \sum_{\substack{I\subseteq[n]\\|I|=k}} \big|\{\pi_1 \in \mathcal{S}_I : C_{[r-1]}(\pi_1) = 0\}\big|\big|\{\pi_2 \in \mathcal{S}_{[n]\setminus I} : C_{[r-1]}(\pi_2) = 0\}\big| \\ &\leqslant \sum_{\substack{I\subseteq[n]\\|I|=k}} \frac{k!}{r}\frac{(n-k)!}{r} = \frac{n!}{r^2},\end{aligned}$$

where $\mathcal{S}_I$ denotes the symmetric group consisting of all permutations of I, and for $j = 1, 2$, the symbol $C_{[r-1]}(\pi_j)$ counts the number of cycles of π_j whose lengths belong to the set $[r-1] = \{1, \ldots, r-1\}$. Consequently, we derive that

$$\mathcal{R}_r(n,k) \leqslant \frac{1}{r^2}$$

as required. □

Moreover, we also need the cycle lemma from combinatorics. See [11, Lemma 2.4], for example.

Lemma 2.7. *For positive real numbers $x_1, \ldots, x_t$ with product A, let $x_{t+i} = x_i$ for $i \geqslant 1$. Then*

$$\frac{1}{\max(1,A)} \leqslant \sum_{j=0}^{t-1}\left(\sum_{h=1}^{t} x_{1+j}\cdots x_{h+j}\right)^{-1} \leqslant \frac{1}{\min(1,A)}.$$

Finally, we refer to a result derived by Ford [10, Lemma 4.9] for the volume of a specific complex region in $\mathbb{R}^t$ with $t \in \mathbb{N}$.

Lemma 2.8. *Let M be a sufficiently large integer. Suppose $v \geqslant 1$, $10M \leqslant t \leqslant 100(v-1)$, $s \geqslant M/2 + 1$ and $0 \leqslant t - v \leqslant s - M/3 - 1$. Let $Y_t(s,v)$ be the set of $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_t) \in \mathbb{R}^t$ satisfying the following:*

(i) $0 \leqslant \xi_1 \leqslant \cdots \leqslant \xi_t \leqslant 1$;
(ii) *For $1 \leqslant i \leqslant \sqrt{t-M}$, $\xi_{M+i^2} > i/v$ and $\xi_{t+1-(M+i^2)} < 1 - i/v$;*
(iii) $\sum_{j=1}^{t} 2^{j-v\xi_j} \leqslant 2^s$.

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Then we have

$$Vol(Y_t(s,v)) \gg \frac{t-v+1}{(t+1)!}.$$

3. A global-to-local principle

For a random permutation $\pi \in \mathcal{S}_n$, let A be the event that π has no cycles of length less than r, and let B be the event that π has a fixed set of size k. Let $\mathbf{c}(\pi) = (C_1(\pi), C_2(\pi), \ldots, C_k(\pi))$ denote the vector of cycle type only listing the number of cycles of length $1, 2, \ldots, k$, respectively, in π. The event that a random permutation π has a fixed set of size k is equivalent to the condition that k lies in the set $\mathscr{L}(\mathbf{c}(\pi))$. Note then that $\mathcal{R}_r(n,k) = \mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B \mid A)$ and $\mathbb{P}(B \mid A) = \mathbb{P}\big(k \in \mathscr{L}(\mathbf{c}(\pi)) \mid \pi \in \mathcal{S}_n^{(r)}\big)$, where $\mathbb{P}(B \mid A)$ is the conditional probability of B relative to A. By Lemma 2.1, we derive that $\mathbb{P}(A) = \mathbb{P}(\pi \in \mathcal{S}_n^{(r)})$ is of the same order as $1/r$. So it is sufficient to estimate the probability $\mathbb{P}\big(k \in \mathscr{L}(\mathbf{c}(\pi)) \mid \pi \in \mathcal{S}_n^{(r)}\big)$. Instead of estimating $\mathbb{P}\big(k \in \mathscr{L}(\mathbf{c}(\pi)) \mid \pi \in \mathcal{S}_n^{(r)}\big)$ directly, however, we apply a global-to-local principle used in [5, 10, 13] to reduce the estimate of $\mathcal{R}_r(n,k)$ to studying the average size of $\mathscr{L}(\mathbf{X}_{r,k})$. In fact, we prove the following result.

Proposition 3.1. *For any integers n, k, r with $1 \leqslant k \leqslant n/2$ and $1 \leqslant r \leqslant 0.04k$, we have*

$$\mathcal{R}_r(n,k) \asymp \frac{1}{rk}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|.$$

To prove Proposition 3.1, we need the following result due to Eberhard, Ford, and Green [5].

Lemma 3.2. *Let $k \in \mathbb{N}$, $\mathbf{c} = (c_1, \ldots, c_k)$, $\mathbf{c}' = (c_1', \ldots, c_k')$ with $\mathbf{c}, \mathbf{c}' \in \mathbb{N}_0^k$. Suppose that $I \subset [k]$, $c_i' = 0$ for $i \in I$ and $c_i' = c_i$ for $i \in [k] \setminus I$. Then*

$$|\mathscr{L}(\mathbf{c}')| \leqslant |\mathscr{L}(\mathbf{c})| \leqslant |\mathscr{L}(\mathbf{c}')| \prod_{i \in I}(c_i + 1).$$

First, it is easy to see that for any positive integer t with $t \geqslant r$, there is a natural bijection from $\mathbb{N}_0^{t-r+1} = \{(c_r, \ldots, c_t) : c_i \in \mathbb{N}_0 \text{ for } r \leqslant i \leqslant t\}$ to $\mathcal{C}_{r,t}$ by mapping $\mathbf{c}' = (c_r, \ldots, c_t) \in \mathbb{N}_0^{t-r+1}$ to $\mathbf{c} = (c_1, \ldots, c_t) \in \mathcal{C}_{r,t}$. Moreover, for $\mathbf{c}' = (c_r, \ldots, c_t) \in \mathbb{N}_0^{t-r+1}$, we always have $\mathscr{L}(\mathbf{c}) = \mathscr{L}(\mathbf{c}')$ with $\mathbf{c} = (0, \ldots, 0, c_r, \ldots, c_t) \in \mathcal{C}_{r,t}$. So we obtain by the independence of $X_r, \ldots, X_k$ that

$$\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,t})| = \sum_{\mathbf{c}=(c_1,\ldots,c_t) \in \mathcal{C}_{r,t}} |\mathscr{L}(\mathbf{c})| \mathbb{P}(X_r = c_r, \ldots, X_t = c_t). \tag{3.1}$$

For $1 \leqslant r \leqslant l' \leqslant l$, we establish the connection between $\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l})|$ and $\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l'})|$ as follows.

Lemma 3.3. *Suppose that $1 \leqslant r \leqslant l' \leqslant l$. Then*

$$\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l'})| \leqslant \mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l})| \leqslant \frac{l+1}{l'+1}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l'})|.$$

Proof. By (3.1) and Lemma 3.2, we have

$$\begin{aligned}
\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l})| &= \sum_{\mathbf{c}=(c_1,\ldots,c_l) \in \mathcal{C}_{r,l}} |\mathscr{L}(\mathbf{c})| \mathbb{P}(X_r = c_r, \ldots, X_l = c_l) \\
&\geqslant \sum_{\mathbf{c}=(c_1,\ldots,c_l) \in \mathcal{C}_{r,l}} |\mathscr{L}((c_1, \ldots, c_{l'}))| \mathbb{P}(X_r = c_r, \ldots, X_l = c_l) \\
&= \sum_{\mathbf{c}=(c_1,\ldots,c_{l'}) \in \mathcal{C}_{r,l'}} |\mathscr{L}(\mathbf{c})| \mathbb{P}(X_r = c_r, \ldots, X_{l'} = c_{l'}) = \mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l'})|.
\end{aligned}$$

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By Lemma 3.2, we have $|\mathscr{L}(\mathbf{X}_{r,l})| \leqslant (1+X_{l'+1})\cdots(1+X_l)|\mathscr{L}(\mathbf{X}_{r,l'})|$. Since $\mathbb{E}X_i = \frac{1}{i}$ for $r \leqslant i \leqslant l$, one can deduce by the independence of $X_r, \dots, X_l$ that

$$\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l})| \leqslant \left(\prod_{i=l'+1}^{l}\big(1+\mathbb{E}X_i\big)\right)\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l'})| = \frac{l+1}{l'+1}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,l'})|.$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *For any r, j, k with $1 \leqslant r \leqslant j \leqslant k$, we have*

$$\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|X_j \leqslant \frac{3}{j}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|.$$

Suppose that $j_1, \dots, j_h \in [r,k]$ are distinct integers and that $a_1, \dots, a_h$ are positive integers. Then

$$\mathbb{E}\Big(|\mathscr{L}(\mathbf{X}_{r,k})|X_{j_1}^{a_1}\cdots X_{j_h}^{a_h}\Big) \ll_{a_1,\dots,a_h} \frac{1}{j_1\cdots j_h}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|.$$

Proof. Define $\mathbf{X}'_{r,k}$ by putting $X'_{j_1} = \cdots = X'_{j_h} = 0$ and $X'_j = X_j$ for all other $j \in [r,k]$. By Lemma 3.2, we have

$$\begin{aligned}|\mathscr{L}(\mathbf{X}_{r,k})|X_{j_1}^{a_1}\cdots X_{j_h}^{a_h} &\leqslant |\mathscr{L}(\mathbf{X}'_{r,k})|(1+X_{j_1})\cdots(1+X_{j_h})X_{j_1}^{a_1}\cdots X_{j_h}^{a_h}\\ &= |\mathscr{L}(\mathbf{X}'_{r,k})|\prod_{i=1}^{h}(X_{j_i}^{a_i}+X_{j_i}^{a_i+1}).\end{aligned}$$

Note that $\mathbb{E}|\mathscr{L}(\mathbf{X}'_{r,k})| \leqslant \mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|$. Then by independence of $X_r, \dots, X_k$, we have

$$\mathbb{E}\Big(|\mathscr{L}(\mathbf{X}_{r,k})|X_{j_1}^{a_1}\cdots X_{j_h}^{a_h}\Big) \leqslant \mathbb{E}|\mathscr{L}(\mathbf{X}'_{r,k})|\prod_{i=1}^{h}\big(\mathbb{E}X_{j_i}^{a_i}+\mathbb{E}X_{j_i}^{a_i+1}\big) \leqslant \mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|\prod_{i=1}^{h}\big(\mathbb{E}X_{j_i}^{a_i}+\mathbb{E}X_{j_i}^{a_i+1}\big).$$

When $h = a_1 = 1$ and $j_1 = j$, we have

$$\mathbb{E}\big(X_j+X_j^2\big) = 2\mathbb{E}\binom{X_j}{2} + 2\mathbb{E}X_j = \frac{1}{j^2}+\frac{2}{j} \leqslant \frac{3}{j}.$$

So we have

$$\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|X_j \leqslant \frac{3}{j}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|.$$

Since X_j is Poisson random variable with parameter $1/j$, we have that $\mathbb{E}\binom{X_j}{m} = \frac{(1/j)^m}{m!}$ for $m \in \mathbb{N}_0$. Note that X_j^i is the integral linear combination of $\binom{X_j}{1}, \dots, \binom{X_j}{i}$. Hence we obtain that $\mathbb{E}X_j^i \ll_i 1/j$ for $i \in \mathbb{N}$. So we have

$$\prod_{i=1}^{h}\big(\mathbb{E}X_{j_i}^{a_i}+\mathbb{E}X_{j_i}^{a_i+1}\big) \ll_{a_1,\dots,a_h} \frac{1}{j_1\cdots j_h}.$$

This concludes the proof of Lemma 3.4. □

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. We begin with the lower bound. If $k < 125$, then by Lemma 2.5, we have $\mathcal{R}_r(n,k) \gg 1/k^2 \gg \frac{1}{rk}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|$. So we only need to consider the case that $k \geqslant 125$ in the following. When $k \geqslant 125$, let $t = \lfloor k/20 \rfloor$, so that $t \geqslant 6$ and $r \leqslant 0.04k \leqslant t$. Consider the permutations $\pi = \alpha\tau_1\tau_2\beta \in \mathcal{S}_n$, where τ_1, τ_2 are cycles, $|\alpha| \leqslant 4t < |\tau_1| < |\tau_2| < 16t$, α is of cycle type $\mathbf{c} \in \mathcal{C}_{r,t}$, all cycles in β have length at least $16t$, and α has a fixed set of size $k - |\tau_1| - |\tau_2|$. Then by the size restrictions on α, τ_1, τ_2, we obtain that $k - |\tau_1| - |\tau_2| \in \mathscr{L}(\mathbf{c})$ and $|\tau_1| + |\tau_2| \leqslant k$. So we derive from $n \geqslant 2k$ that $n - |\alpha| - |\tau_1| - |\tau_2| \geqslant \frac{4}{5}k \geqslant 16t$. Fix $\mathbf{c} = (c_1, \dots c_r, \dots, c_t) \in \mathcal{C}_{r,t}$ with $S(\mathbf{c}) \leqslant 4t$,

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and choose l_1, l_2 with $4t < l_1 < l_2 < 16t$ such that $k - l_1 - l_2 \in \mathscr{L}(\mathbf{c})$. By Lemma 2.4, the probability that a random permutation $\pi \in \mathcal{S}_n$ has c_i cycles of length i $(1 \leqslant i \leqslant t)$, one cycle each of length l_1, l_2, and no other cycles of length less than $16t$ is

$$\gg \frac{1}{tl_1l_2\prod_{i=r}^{t} c_i!i^{c_i}} \gg \frac{1}{t^3\prod_{i=r}^{t} c_i!i^{c_i}}.$$

Now we have $\mathscr{L}(\mathbf{c}) \subset \{0\} \bigcup [r, 4t]$. Note that $t = \lfloor k/20 \rfloor$ implies that $20t \leqslant k < 20(t+1)$. Hence, for any l_1 satisfying $4t + 20 \leqslant l_1 \leqslant 8t - 1$ and any $u \in \mathscr{L}(\mathbf{c})$, the unique l_2 with $k - l_1 - l_2 = u$ satisfies $l_2 \leqslant k - l_1 < 20(t+1) - l_1 \leqslant 20(t+1) - (4t+20) = 16t$ and $l_2 \geqslant k - l_1 - 4t \geqslant 20t - (8t - 1) - 4t = 8t + 1 > l_1$. Since

$$\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,t})|\mathbb{1}\big(S(\mathbf{X}_{r,t}) \leqslant 4t\big) = \mathrm{e}^{H_{r-1}-H_t} \sum_{\substack{\mathbf{c}\in\mathcal{C}_{r,t}\\ S(\mathbf{c})\leqslant 4t}} |\mathscr{L}(\mathbf{c})| \prod_{i=r}^{t} \frac{(1/i)^{c_i}}{c_i!}$$

and $8t - 1 - (4t + 20) + 1 = 4t - 20 \geqslant 0.5t$, we derive from (2.1) that

$$\mathcal{R}_r(n,k) \gg \frac{1}{t^2} \sum_{\substack{\mathbf{c}\in\mathcal{C}_{r,t}\\ S(\mathbf{c})\leqslant 4t}} \frac{|\mathscr{L}(\mathbf{c})|}{\prod_{i=r}^{t} c_i!i^{c_i}} \gg \frac{1}{rt}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,t})|\mathbb{1}\big(S(\mathbf{X}_{r,t}) \leqslant 4t\big) \geqslant \frac{1}{rt}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,t})|\left(1 - \frac{S(\mathbf{X}_{r,t})}{4t}\right). \tag{3.2}$$

By Lemma 3.4, we have

$$\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,t})|S(\mathbf{X}_{r,t}) = \sum_{j=r}^{t} \mathbb{E}|\mathscr{L}(\mathbf{X}_{r,t})|jX_j \leqslant 3t\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,t})|.$$

It then follows from (3.2) and Lemma 3.3 that

$$\mathcal{R}_r(n,k) \gg \frac{1}{r}\frac{1}{t}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,t})| \gg \frac{1}{rk}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|$$

as desired.

For the upper bound, we first impose a total ordering on the set of all cycles $\bigcup_{i=r}^{n} \mathcal{C}_i^{(n)}$, first ordering them by length, then imposing an arbitrary ordering of the cycles of a given length. Let $\pi \in \mathcal{S}_n^{(r)}$ have a divisor of size k. Let $k_1 = k$ and $k_2 = n - k$. Then $\pi = \pi_1\pi_2$, where $|\pi_1| = k_1$ and $|\pi_2| = k_2$. For some $j \in \{1, 2\}$, the largest cycle in π, with respect to our total ordering, lies in π_{3-j}. Let σ be the largest cycle in π_j, and note that $|\sigma| \leqslant \min(k_1, k_2) = k$. Write $\pi = \alpha\sigma\beta$, where α is the product of all cycles dividing π which are smaller than σ and have length $\geqslant r$, and β is the product of all cycles which are larger than σ. In particular, $|\beta| \geqslant |\sigma|$ since β contains the largest cycle in π, and thus

$$|\sigma| \leqslant |\beta| = n - |\sigma| - |\alpha|. \tag{3.3}$$

By definition of σ and α, α has a divisor of size $k_j - |\sigma|$. Suppose $|\sigma| = \ell$, $C_j(\alpha) = c_j$ for $j \leqslant \ell$. Then the cycle type of α is $\mathbf{c} = (c_1, c_2, \ldots, c_\ell) \in \mathcal{C}_{r,\ell}$ and $k_j - \ell \in \mathscr{L}(\mathbf{c})$. Hence by using Lemma 2.3, the number of possible pairs α, σ is at most

$$\binom{n}{|\alpha|}|\alpha|!\left(\prod_{i=1}^{\ell}\frac{(1/i)^{c_i}}{c_i!}\right)\binom{n-|\alpha|}{\ell}(\ell-1)! = \frac{n!}{\ell(n-|\alpha|-\ell)!}\prod_{i=r}^{\ell}\frac{(1/i)^{c_i}}{c_i!}.$$

Given α and σ, it follows from Lemma 2.1 and (3.3) that the number of choices for β is at most $(n - |\alpha| - \ell)!/\ell$. So we have

$$\mathcal{R}_r(n,k) \leqslant \sum_{j=1}^{2}\sum_{\ell=r}^{k}\frac{1}{\ell^2}\sum_{\substack{\mathbf{c}\in\mathcal{C}_{r,\ell}\\ k_j-\ell\in\mathscr{L}(\mathbf{c})}}\prod_{i=r}^{\ell}\frac{(1/i)^{c_i}}{c_i!} \leqslant \sum_{j=1}^{2}\sum_{\mathbf{c}\in\mathcal{C}_{r,k}}\prod_{i=r}^{k}\frac{(1/i)^{c_i}}{c_i!}\sum_{\substack{m(\mathbf{c})\leqslant\ell\leqslant k\\ k_j-\ell\in\mathscr{L}(\mathbf{c})}}\frac{1}{\ell^2} \tag{3.4}$$

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where $m(\mathbf{c}) = \max\{\max\{i : c_i > 0\}, 1\}$. From $k_j - \ell \in \mathscr{L}(\mathbf{c})$, we derive that $k_j - \ell \leqslant S(\mathbf{c})$ and $\ell \geqslant k_j - S(\mathbf{c})$. So we have $\ell \geqslant \max\{m(\mathbf{c}), k_j - S(\mathbf{c})\}$ for $\mathbf{c} \in \mathcal{C}_{r,k}$. Note that the number of ℓ satisfying $k_j - \ell \in \mathscr{L}(\mathbf{c})$ is at most $|\mathscr{L}(\mathbf{c})|$. Thus for any given $\mathbf{c} \in \mathcal{C}_{r,k}$, we have

$$\sum_{\substack{m(\mathbf{c})\leqslant \ell \leqslant k \\ k_j-\ell\in\mathscr{L}(\mathbf{c})}} \frac{1}{\ell^2} \leqslant \frac{|\mathscr{L}(\mathbf{c})|}{\max\{m(\mathbf{c}), k_j - S(\mathbf{c})\}^2} \leqslant \frac{|\mathscr{L}(\mathbf{c})|}{\max\{m(\mathbf{c}), k - S(\mathbf{c})\}^2}. \tag{3.5}$$

Since

$$\begin{aligned}\mathbb{E}\left(\frac{|\mathscr{L}(\mathbf{X}_{r,k})|}{\max\{m(\mathbf{X}_{r,k}), k - S(\mathbf{X}_{r,k})\}^2}\right) &= \sum_{\mathbf{c}\in\mathcal{C}_{r,k}} \frac{|\mathscr{L}(\mathbf{c})|}{\max\{m(\mathbf{c}), k - S(\mathbf{c})\}^2}\left(\prod_{i=r}^{k}\frac{(1/i)^{c_i}}{c_i!}\mathrm{e}^{-1/i}\right)\\ &= \mathrm{e}^{H_{r-1}-H_k}\sum_{\mathbf{c}\in\mathcal{C}_{r,k}} \frac{|\mathscr{L}(\mathbf{c})|}{\max\{m(\mathbf{c}), k - S(\mathbf{c})\}^2}\left(\prod_{i=r}^{k}\frac{(1/i)^{c_i}}{c_i!}\right),\end{aligned}$$

it follows from (2.1), (3.4), and (3.5) that

$$\mathcal{R}_r(n,k) \leqslant 2\mathrm{e}\frac{k}{r}\mathbb{E}\left(\frac{|\mathscr{L}(\mathbf{X}_{r,k})|}{\max\{m(\mathbf{X}_{r,k}), k - S(\mathbf{X}_{r,k})\}^2}\right). \tag{3.6}$$

As in [5], using the inequality $\frac{1}{\max(m,k-S)^2} \leqslant \frac{4}{k^2}(1 + \frac{S^2}{m^2})$, one can derive from (3.6) that

$$\mathcal{R}_r(n,k) \leqslant 8\mathrm{e}\frac{1}{rk}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})| + 8\mathrm{e}\frac{1}{rk}\mathbb{E}\frac{|\mathscr{L}(\mathbf{X}_{r,k})|S(\mathbf{X}_{r,k})^2}{m(\mathbf{X}_{r,k})^2}. \tag{3.7}$$

Since $\mathbb{1}\big(X_m \geqslant 1\big) \leqslant X_m$, it follows from (2.1) and conditioning on $m = m(\mathbf{X}_{r,k})$ that

$$\begin{aligned}&\mathbb{E}\frac{|\mathscr{L}(\mathbf{X}_{r,k})|S(\mathbf{X}_{r,k})^2}{m(\mathbf{X}_{r,k})^2}\\ &= \sum_{m=r}^{k}\frac{1}{m^2}\sum_{\substack{\mathbf{c}\in\mathcal{C}_{r,k},c_m\geqslant 1\\ c_{m+1}=\cdots=c_k=0}} |\mathscr{L}(\mathbf{c})|S(\mathbf{c})^2\mathbb{P}\big(\mathbf{X}_{r,m} = (c_r,\ldots,c_m)\big)\prod_{t=m+1}^{k}\mathbb{P}\big(X_t = c_t\big)\\ &= \sum_{m=r}^{k}\frac{1}{m^2}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,m})|S(\mathbf{X}_{r,m})^2\mathbb{1}\big(X_m\geqslant 1\big)\mathrm{e}^{H_m-H_k}\\ &\leqslant \frac{\mathrm{e}}{k}\sum_{m=r}^{k}\frac{1}{m}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,m})|S(\mathbf{X}_{r,m})^2X_m.\end{aligned}$$

Hence we deduce by expanding $S(\mathbf{X}_{r,m})^2 = (rX_r + \cdots + mX_m)^2$ that

$$\mathbb{E}\frac{|\mathscr{L}(\mathbf{X}_{r,k})|S(\mathbf{X}_{r,k})^2}{m(\mathbf{X}_{r,k})^2} \ll \frac{1}{k}\sum_{m=r}^{k}\frac{1}{m}\sum_{i_1,i_2=r}^{m} i_1i_2\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,m})|X_{i_1}X_{i_2}X_m. \tag{3.8}$$

Using Lemma 3.4, we obtain the estimate on the innermost sum as follows.

$$i_1i_2\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,m})|X_{i_1}X_{i_2}X_m \ll \begin{cases}\frac{1}{m}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,m})|, & \text{if } r\leqslant i_1\neq i_2<m,\\ m\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,m})|, & \text{if } i_1=i_2=m,\\ \mathbb{E}|\mathscr{L}(\mathbf{X}_{r,m})|, & \text{otherwise.}\end{cases} \tag{3.9}$$

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By Lemma 3.3, we obtain that $\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,m})| \leqslant \mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|$ for $r \leqslant m \leqslant k$. It then follows from (3.8) and (3.9) that

$$\mathbb{E}\frac{|\mathscr{L}(\mathbf{X}_{r,k})|S(\mathbf{X}_{r,k})^2}{m(\mathbf{X}_{r,k})^2} \ll \mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|.$$

So we derive from (3.7) that

$$\mathcal{R}_r(n,k) \ll \frac{1}{rk}\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})|$$

as desired. The Proof of Proposition 3.1 is completed. □

4. Proof of Theorem 1.1

Let r_0 be a sufficiently large integer, and let $\varepsilon > 0$ be a sufficiently small constant. If either $k \leqslant r_0$, or $k > r_0$ and $r > \varepsilon k$, then we obtain the desired result in Theorem 1.1 by using Lemmas 2.5 and 2.6 directly. In the following, we only need to deal with the case that $k > r_0$ and $r \leqslant \varepsilon k$. In this case, we can use the global-to-local principle – Proposition 3.1.

Taking $t = k$ in (3.1), we derive by (2.1) that

$$\mathbb{E}|\mathscr{L}(\mathbf{X}_{r,k})| = \sum_{\mathbf{c}\in\mathcal{C}_{r,k}} |\mathscr{L}(\mathbf{c})| \prod_{i=r}^{k} \frac{(1/i)^{c_i}}{c_i!} \mathrm{e}^{-(\frac{1}{r}+\cdots+\frac{1}{k})} = \mathrm{e}^{H_{r-1}-H_k} \sum_{\mathbf{c}\in\mathcal{C}_{r,k}} \frac{|\mathscr{L}(\mathbf{c})|}{\prod_{i=r}^{k} c_i! i^{c_i}} \asymp \frac{r}{k} \sum_{\mathbf{c}\in\mathcal{C}_{r,k}} \frac{|\mathscr{L}(\mathbf{c})|}{\prod_{i=r}^{k} c_i! i^{c_i}}.$$

Hence by Proposition 3.1, we have

$$\mathcal{R}_r(n,k) \asymp \frac{1}{k^2} \sum_{\mathbf{c}\in\mathcal{C}_{r,k}} \frac{|\mathscr{L}(\mathbf{c})|}{\prod_{i=r}^{k} c_i! i^{c_i}}. \tag{4.1}$$

As in [5], for all $\mathbf{c} = (0,\dots 0, c_r, \dots, c_k) \in \mathcal{C}_{r,k}$ with $t = c_r + \cdots + c_k$, we have

$$\sum_{\substack{\mathbf{c}\in\mathcal{C}_{r,k} \\ c_r+\cdots+c_k=t}} \frac{|\mathscr{L}(\mathbf{c})|}{\prod_{i=r}^{k} c_i! i^{c_i}} = \frac{1}{t!} \sum_{a_1,\dots,a_t=r}^{k} \frac{|\mathscr{L}^*(\mathbf{a})|}{a_1\cdots a_t}, \tag{4.2}$$

where $\mathbf{a} = (a_1, \dots, a_t)$ and

$$\mathscr{L}^*(\mathbf{a}) = \left\{ \sum_{i\in I} a_i : I \subseteq [t] \right\}. \tag{4.3}$$

For convenience, let

$$J_1 = \left\lceil \frac{\log r}{\log 2} \right\rceil + 1, J_2 = \left\lfloor \frac{\log k}{\log 2} \right\rfloor, \ t_0 = J_2 - J_1 + 1. \tag{4.4}$$

4.1 The lower bound in Theorem 1.1

For a vector $\mathbf{b} = (b_1, \dots, b_{J_2})$ of nonnegative integers, let $\mathscr{A}(\mathbf{b})$ be the set of permutations $\pi \in S_n$ composed of exactly b_j cycle factors with length lying in $[2^{j-1}, 2^j - 1]$ for $1 \leqslant j \leqslant J_2$.

Suppose that $b_1, \dots, b_{J_2}$ are arbitrary nonnegative integers with sum t and there are exactly b_j of the a_i in each interval $[2^{j-1}, 2^j - 1]$, where $b_j = 0$ for $j < J_1$. Writing

$$\mathscr{D}(\mathbf{b}) = \prod_{j=J_1}^{J_2} \{2^{j-1}, \dots, 2^j - 1\}^{b_j},$$

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We then have by (4.2) and (4.3) that

$$\frac{1}{t!}\sum_{a_1,\dots,a_t=2^{J_1-1}}^{2^{J_2}-1}\frac{|\mathscr{L}^*(\mathbf{a})|}{a_1\cdots a_t}=\sum_{b_{J_1}+\cdots+b_{J_2}=t}\frac{1}{b_{J_1}!\cdots b_{J_2}!}\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b})}\frac{|\mathscr{L}^*(\mathbf{d})|}{d_1\cdots d_t}. \tag{4.5}$$

Combining (4.1), (4.2), and (4.5), we derive that

$$\mathcal{R}_r(n,k)\gg\frac{1}{k^2}\sum_{t}\sum_{b_{J_1}+\cdots+b_{J_2}=t}\frac{1}{b_{J_1}!\cdots b_{J_2}!}\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b})}\frac{|\mathscr{L}^*(\mathbf{d})|}{d_1\cdots d_t}. \tag{4.6}$$

We now give the estimate on $\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b})}\frac{|\mathscr{L}^*(\mathbf{d})|}{d_1\cdots d_t}$ as follows.

Lemma 4.1. *Let t be a positive integer. Then for any vector $\mathbf{b}=(b_1,\dots,b_{J_2})$ of nonnegative integers such that $b_j=0$ for $j<J_1$ and $b_{J_1}+\cdots+b_{J_2}=t$, we have*

$$\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b})}\frac{|\mathscr{L}^*(\mathbf{d})|}{d_1\cdots d_t}\gg\frac{(2\log 2)^t}{\max\Big(\sum_{i=J_1}^{J_2}2^{b_{J_1}+\cdots+b_i-i},1\Big)}.$$

Proof. As in the proof of Lemma 4.1 in [5], for given $\ell\in\mathbb{N}$, let $R(\mathbf{d},\ell)$ be the number of $I\subseteq[t]$ with $\ell=\sum_{i\in I}d_i$. Also, define $\lambda_i=\sum_{j=2^{i-1}}^{2^i-1}\frac{1}{j}$ for $J_1\leqslant i\leqslant J_2$. One then has $\log 2\leqslant\lambda_i\leqslant 1$ for $J_1\leqslant i\leqslant J_2$. Since $\sum_{\ell}R(\mathbf{d},\ell)=2^t$, we obtain by Cauchy-Schwarz that

$$\begin{aligned}2^{2t}\prod_{j=J_1}^{J_2}\lambda_j^{2b_j}&=\Big(\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b})}\frac{1}{d_1\cdots d_t}\sum_{\ell}R(\mathbf{d},\ell)\Big)^2\\&\leqslant\Big(\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b})}\frac{\big(\sum_{\ell\in\mathscr{L}^*(\mathbf{d})}R(\mathbf{d},\ell)^2\big)^{\frac12}}{\sqrt{d_1\cdots d_t}}\frac{|\mathscr{L}^*(\mathbf{d})|^{\frac12}}{\sqrt{d_1\cdots d_t}}\Big)^2\\&\leqslant\Big(\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b}),\ell}\frac{R(\mathbf{d},\ell)^2}{d_1\cdots d_t}\Big)\Big(\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b})}\frac{|\mathscr{L}^*(\mathbf{d})|}{d_1\cdots d_t}\Big).\end{aligned} \tag{4.7}$$

Applying the argument on pages 6725–6726 in [5], one can derive that

$$\prod_{j=J_1}^{J_2}\lambda_j^{-b_j}\sum_{\mathbf{d}\in\mathscr{D}(\mathbf{b}),\ell}\frac{R(\mathbf{d},\ell)^2}{d_1\cdots d_t}\ll 2^t+2^t\sum_{i=J_1}^{J_2}2^{b_{J_1}+\cdots+b_i-i}\ll 2^t\max\Big(\sum_{i=J_1}^{J_2}2^{b_{J_1}+\cdots+b_i-i},1\Big).$$

Comparing with (4.7), and using again the fact that $\lambda_i\geqslant\log 2$, we obtain the desired result. □

If $k>r_0$, $r\leqslant\varepsilon k$ and $r\leqslant\sqrt{r_0}$, then $J_2-J_1+1\asymp J_2$. Consider those vectors $\mathbf{b}=(b_1,\dots,b_{J_2})$ of nonnegative integers such that $b_j=0$ for $j<J_1$ and $b_{J_1}+\cdots+b_{J_2}=t_0$. Setting $x_i=2^{b_{J_1+i-1}-1}$, we then have $x_1\cdots x_{t_0}=1$. And hence by Lemma 2.7, we obtain that

$$\frac{1}{\sum_{i=J_1}^{J_2}2^{b_{J_1}+\cdots+b_i-i}}=\frac{2^{J_1-1}}{\sum_{i=1}^{t_0}x_1\cdots x_i}=\frac{2^{J_1-1}}{t_0}.$$

Using multinomial theorem and Stirling's formula, we deduce from (4.4) that

$$\sum_{b_{J_1}+\cdots+b_{J_2}=t_0}\frac{1}{b_{J_1}!\cdots b_{J_2}!\sum_{i=J_1}^{J_2}2^{b_{J_1}+\cdots+b_i-i}}=\frac{t_0^{t_0}}{t_0!}\frac{2^{J_1-1}}{t_0}\asymp\frac{t_0^{t_0}\mathrm{e}^{t_0}}{\sqrt{2\pi t_0}t_0^{t_0}}\frac{2^{J_1-1}}{t_0}\asymp\frac{k^{\frac{1}{\log 2}}}{(\log k)^{3/2}}.$$

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It then follows from Lemma 4.1 and (4.6) that

$$\mathcal{R}_r(n,k) \gg \frac{1}{k^2}(2\log 2)^{t_0}\frac{k^{\frac{1}{\log 2}}}{(\log k)^{3/2}} \gg \frac{1}{k^{\mathcal{E}}(1+\log k)^{3/2}} \asymp i(n,k)$$

as desired.

In the following, we assume that $k > r_0$ and $\sqrt{r_0} < r \leqslant \varepsilon k$.

Take

$$M = \frac{\log r_0}{200} \tag{4.8}$$

such that M is a sufficiently large constant because r_0 is. Choose ε such that

$$J_1 \geqslant 200M, \quad J_2 - J_1 \geqslant 200M. \tag{4.9}$$

Let $\mathscr{B}_t$ be the set of vectors $\mathbf{b} = (b_1, \ldots, b_{J_1}, \ldots, b_{J_2})$ which satisfy the following:

(a) $b_1 = \cdots = b_{J_1-1} = 0$;
(b) $b_{J_1} + \cdots + b_{J_2} = t$;
(c) $\sum_{j=J_1}^{J_2} 2^{b_{J_1}+\cdots+b_j-j} \leqslant 2^{-M}$;
(d) $b_{J_1+i-1} \leqslant M + i^2 \quad (i \geqslant 1)$;
(e) $b_{J_2-i+1} \leqslant M + i^2 \quad (i \geqslant 1)$.

We then obtain by Lemma 4.1 that

$$\mathcal{R}_r(n,k) \gg \frac{1}{k^2}\sum_{t_1 \leqslant t \leqslant t_2}\sum_{\mathbf{b}\in\mathscr{B}_t}\frac{(\log 4)^t}{b_{J_1}!\cdots b_{J_2}!} \tag{4.10}$$

for any two positive integers t_1, t_2 with $t_1 \leqslant t_2$. Take

$$t_2 = \left\lfloor \min\left(\frac{\log k}{\log 2}, 2(\log k - \log r)\right) - 2M \right\rfloor. \tag{4.11}$$

By (4.9), we have $t_2 \geqslant 100M$. Choose t_1 such that

$$10M \leqslant t_1 \leqslant t_2. \tag{4.12}$$

Now let

$$s = J_1 - 2 - M. \tag{4.13}$$

Setting

$$g_i = b_{J_1+i-1}$$

for $i \geqslant 1$, we obtain by (c) that

$$\sum_{i=1}^{t_0} 2^{-i+g_1+\cdots+g_i} \leqslant 2^{J_1-1}\cdot 2^{-M} = 2^{s+1},$$

where t_0 is defined as in (4.4).

By (d) and (e) in the definition of $\mathscr{B}_t$, $g_i \leqslant M + i^2$ and $g_{t_0+1-i} \leqslant M + i^2$ for every $i \geqslant 1$. Applying the argument on top of Page 419 of [10], we deduce that

$$\sum_{\mathbf{b}\in\mathscr{B}_t}\frac{1}{b_{J_1}!\cdots b_{J_2}!} \gg t_0{}^t \mathrm{Vol}(Y_t(s,t_0)) \tag{4.14}$$

for $t_1 \leqslant t \leqslant t_2$, where $Y_t(s,t_0)$ is defined as in Lemma 2.8.

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Since r_0 is sufficiently large (implying that M is sufficiently large) and ε is sufficiently small, we obtain by (4.4), (4.9), (4.11), (4.12) and (4.13) that

$$\begin{aligned} t_0 &= J_2 - J_1 + 1 \geqslant 200M \geqslant 1, \\ 10M \leqslant t_1 \leqslant t_2 &\leqslant (\log 4)(J_2 - J_1) = (\log 4)(t_0 - 1) \leqslant 100(t_0 - 1), \\ s &\geqslant \log r - M \geqslant M/2 + 1, \\ 0 \leqslant t_2 - t_0 &\leqslant (J_2 - 2M) - (J_2 - J_1 + 1) = s - M + 3 \leqslant s - M/3 - 1. \end{aligned}$$

Therefore, it follows from (4.10), (4.14), and Lemma 2.8 that

$$\mathcal{R}_r(n,k) \gg \frac{1}{k^2} \sum_{t=t_1}^{t_2} \frac{(t_0 \log 4)^t}{t!} \left(\frac{t - t_0 + 1}{t+1} \right). \tag{4.15}$$

We now consider the following two cases.

Case 1. $\delta \geqslant 1 - 1/\log 4$. In this case, we have $\log r \geqslant (1 - 1/\log 4) \log k$. By (4.11), we have

$$t_2 = \Big\lfloor 2(\log k - \log r) - 2M \Big\rfloor.$$

Take

$$t_1 = \frac{3t_2}{4}.$$

It is easy to check that $t_1 \geqslant 10M$. With these choices and (4.4), we obtain for $t_1 \leqslant t \leqslant t_2$ that

$$\frac{t - t_0 + 1}{t+1} \asymp 1.$$

Applying Lemma 2.2 to the sum in (4.15), we obtain

$$\mathcal{R}_r(n,k) \gg \frac{(t_0 \log 4)^{t_2}}{t_2!} \sqrt{t_0 \log 4} \gg \frac{1}{k^2} \mathrm{e}^{t_0 \log 4} \gg \frac{1}{k^2} \mathrm{e}^{2(\log k - \log r)} = \frac{1}{r^2}$$

as desired.

Case 2. $0 < \delta < 1 - 1/\log 4$. By (4.11), we have

$$t_2 = \Big\lfloor \frac{\log k}{\log 2} - 2M \Big\rfloor.$$

We now divide the proof of Case 2 into the following two subcases.

Subcase 2.1. $\frac{1}{10} < \delta < 1 - 1/\log 4$. In this subcase, we take

$$t_1 = \frac{19t_2}{20},$$

so that $t - t_0 + 1 \asymp t \asymp \log k$ for $t_1 \leqslant t \leqslant t_2$. Thus for $t_1 \leqslant t \leqslant t_2$, we have

$$\frac{t - t_0 + 1}{t+1} \asymp 1 \asymp \delta.$$

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Hence, recalling the definition of $\mathcal{E}$ and $B(r,k)$ (i.e., (1.1) and (1.4)), we obtain by applying Lemma 2.2 and Stirling's formula to the sum in (4.15) that

$$\begin{aligned}\mathcal{R}_r(n,k) &\gg k^{-2}\frac{(t_0\log 4)^{t_2}}{t_2!}\min\left((t_0\log 4)^{1/2}, \frac{t_0\log 4}{t_0\log 4 - t_2}, t_2 - t_1 + 1\right)\\ &\gg k^{-\mathcal{E}+\log(1-\delta)/\log 2}(\log k)^{-1/2}\min\left((\log k)^{1/2}, \frac{1-\delta}{1-\delta-1/\log 4} + O(1/\log k)\right)\\ &\gg k^{-\mathcal{E}+\log(1-\delta)/\log 2}(\log k)^{-1/2}(\log k)^{1/2}\min\left(1, \frac{1}{(1-\delta)\log 4 - 1}(\log k)^{-1/2}\right)\\ &\gg \delta B(r,k)k^{-\mathcal{E}+\log(1-\delta)/\log 2}\end{aligned}$$

as required.

Subcase 2.2. $0 < \delta < \frac{1}{10}$. In this case, we take

$$t_1 = \left\lfloor \frac{\log k}{\log 2} - 2M \right\rfloor = t_2.$$

We now have

$$t_2 - t_0 + 1 \asymp J_1, \quad \frac{t_2 - t_0 + 1}{t_2 + 1} \asymp \frac{\log r}{\log k} = \delta, \quad B(r,k) \asymp (\log k)^{-1/2}.$$

Applying Stirling's formula to the sum in (4.15), we obtain that

$$\mathcal{R}_r(n,k) \gg \frac{\delta}{k^2}\frac{(t_0\log 4)^{t_2}}{t_2!} \gg \delta B(r,k)k^{-\mathcal{E}+\log(1-\delta)/\log 2}$$

as desired. This completes the proof of the lower bound in Theorem 1.1.

4.2 The upper bound in Theorem 1.1

By the discussion in the first paragraph of this section, we only need to deal with the case that $k > r_0$ and $r \leqslant \varepsilon k$. If $\delta = \frac{\log r}{\log k} \geqslant (1 - 1/\log 4)$, then by Lemma 2.6, one can get

$$\mathcal{R}_r(n,k) \ll \frac{1}{r^2},$$

which is just the desired upper bound in Theorem 1.1.

From now on, we assume that

$$\delta = \frac{\log r}{\log k} < 1 - 1/\log 4.$$

We divide our proof into the following two cases.

Case 1. $1/10 < \delta < 1 - 1/\log 4$. By the definition of $\mathscr{L}^*(\mathbf{a})$ (see (4.3)), we have

$$\left|\mathscr{L}^*(\mathbf{a})\right| \leqslant \min(2^t, a_1 + a_2 + \cdots a_t + 1). \tag{4.16}$$

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It then follows from (4.1), (4.2), and (4.16) that

$$\begin{aligned}\mathcal{R}_r(n,k) &\ll \frac{1}{k^2}\sum_{t\geqslant 1}\frac{1}{t!}\sum_{a_1,\dots,a_t=r}^{k}\frac{|\mathscr{L}^*(\mathbf{a})|}{a_1\cdots a_t}\\ &\leqslant \frac{1}{k^2}\sum_{t\leqslant J_2}\frac{1}{t!}\sum_{a_1,\dots,a_t=r}^{k}\frac{2^t}{a_1\cdots a_t}+\frac{1}{k^2}\sum_{t>J_2}\frac{1}{t!}\sum_{a_1,\dots,a_t=r}^{k}\frac{a_1+\cdots+a_t+1}{a_1\cdots a_t}\\ &\ll \frac{1}{k^2}\sum_{t\leqslant J_2}\frac{(2\log\frac{k}{r})^t}{t!}+\frac{1}{k^2}\sum_{t>J_2}\frac{t(k-r+1)(\log\frac{k}{r})^{t-1}}{t!}.\end{aligned}\tag{4.17}$$

Since $J_2=\lfloor\frac{\log k}{\log 2}\rfloor$ and $1/10\leqslant\delta=\frac{\log r}{\log k}$, we have $\sqrt{2}\log\frac{k}{r}\leqslant J_2\leqslant 2\log\frac{k}{r}$. So we derive from Lemma 2.2 and Stirling's formula that

$$\begin{aligned}\frac{1}{k^2}\sum_{t\leqslant J_2}\frac{(2\log\frac{k}{r})^t}{t!} &\ll \frac{1}{k^2}\frac{(2\log\frac{k}{r})^{J_2}}{J_2!}\min\left(\sqrt{2\log\frac{k}{r}},\frac{2\log\frac{k}{r}}{2\log\frac{k}{r}-J_2},J_2\right)\\ &\ll k^{-\mathcal{E}+\frac{\log(1-\delta)}{\log 2}}\min\left(1,(\log k)^{-\frac{1}{2}}\big((1-\delta)\log 4-1\big)^{-1}\right).\end{aligned}\tag{4.18}$$

Since $J_2>\sqrt{2}\log\frac{k}{r}$, we obtain by Stirling's formula that

$$\frac{1}{k^2}\sum_{t>J_2}\frac{(k-r+1)t(\log\frac{k}{r})^{t-1}}{t!}\ll\frac{k}{k^2}\sum_{t\geqslant J_2}^{\infty}\frac{(\log\frac{k}{r})^t}{t!}\ll\frac{1}{k}\frac{(\log\frac{k}{r})^{J_2}}{J_2!}\ll k^{-\mathcal{E}+\frac{\log(1-\delta)}{\log 2}}(\log k)^{-\frac{1}{2}}.\tag{4.19}$$

So we derive from (4.17), (4.18), and (4.19) that

$$\mathcal{R}_r(n,k)\ll\delta B(r,k)k^{-\mathcal{E}+((\log(1-\delta))/\log 2)}$$

as required in this case.

Case 2. $0<\delta\leqslant 1/10$. For any positive integer t, let

$$G_t=\sum_{a_1,\dots,a_t=r}^{k}\frac{|\mathscr{L}^*(\mathbf{a})|}{a_1\cdots a_t},\tag{4.20}$$

where $\mathscr{L}^*(\mathbf{a})$ is defined as in (4.3). Then by (4.1) and (4.2), we have

$$\mathcal{R}_r(n,k)\ll\frac{1}{k^2}\sum_{t=1}^{\infty}\frac{G_t}{t!}.\tag{4.21}$$

In this case, we bound G_t in a manner similar to that in [5]. Let $\widetilde{a_1},\widetilde{a_2},\dots,\widetilde{a_t}$ be the increasing rearrangement of the sequence $\{a_i\}_{i=1}^t$, so that $\widetilde{a_1}\leqslant\widetilde{a_2}\leqslant\cdots\leqslant\widetilde{a_t}$. For any j satisfying $0\leqslant j\leqslant t$, we have

$$\mathscr{L}^*(\mathbf{a})\subseteq\left\{m+\sum_{i\in I}\widetilde{a_i}:r\leqslant m\leqslant\sum_{i=1}^{j}\widetilde{a_i},I\subseteq\{j+1,\dots,t\}\right\},$$

which implies that

$$|\mathscr{L}^*(\mathbf{a})|\leqslant F(\mathbf{a}),\tag{4.22}$$

where

$$F(\mathbf{a})=\min_{0\leqslant j\leqslant t}2^{t-j}(\widetilde{a_1}+\cdots+\widetilde{a_j}+1).\tag{4.23}$$

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It is reasonable to expect that

$$\sum_{a_1,\dots,a_t=r}^{k}\frac{F(\mathbf{a})}{a_1\cdots a_t}\sim\int_r^k\cdots\int_r^k\frac{F(\mathbf{x})}{x_1\cdots x_t}d\mathbf{x}=\big(\log\frac{k}{r}\big)^t\int_0^1\cdots\int_0^1 F(re^{\xi_1\log\frac{k}{r}},\dots,re^{\xi_t\log\frac{k}{r}})d\boldsymbol{\xi}. \tag{4.24}$$

We may also prove an approximate version of (4.24) as in [5].

Lemma 4.2. *Suppose that k, r are sufficiently large integers, with $\log r\leqslant(1-1/\log 4)\log k$. Let*

$$v_0=\Big\lfloor\frac{\log k-\log r}{\log 2}\Big\rfloor,\ u_0=\Big\lfloor\frac{\log r}{\log 2}\Big\rfloor. \tag{4.25}$$

Then for any $t\geqslant 1$, we have

$$G_t\ll\Big(2\log\frac{k}{r}\Big)^t t!U_t(v_0,u_0),$$

where

$$U_t(v,u):=\int_{\Omega_t}\min_{0\leqslant j\leqslant t}2^{-j}\Big(2^{v\xi_1+u}+\cdots+2^{v\xi_j+u}+1\Big)d\boldsymbol{\xi} \tag{4.26}$$

with

$$\Omega_t=\{\boldsymbol{\xi}=(\xi_1,\dots,\xi_t)\in\mathbb{R}^t{:}0\leqslant\xi_1\leqslant\xi_2\leqslant\cdots\leqslant\xi_t\leqslant 1\}.$$

Proof. As in [5], define the product sets

$$R(\mathbf{a})=\prod_{i=1}^{t}[\exp(H_{a_i-1}),\exp(H_{a_i})].$$

Then we obtain by (4.22) that

$$G_t\leqslant\sum_{a_1,\dots,a_t=r}^{k}\frac{F(\mathbf{a})}{a_1\cdots a_t}=\sum_{a_1,\dots,a_t=r}^{k}F(\mathbf{a})\int_{R(\mathbf{a})}\frac{d\mathbf{x}}{x_1\cdots x_t}.$$

For $\mathbf{x}\in R(\mathbf{a})$, we write $\widetilde{x}_1\leqslant\widetilde{x}_2\leqslant\cdots\leqslant\widetilde{x}_t$ for the non-decreasing rearrangement of the components of the vector $\mathbf{x}$. We then derive from (2.1) that

$$\widetilde{a}_i\leqslant\exp(H_{\widetilde{a}_i-1})\leqslant\widetilde{x}_i\leqslant\exp(H_{\widetilde{a}_i})\text{ for }1\leqslant i\leqslant t.$$

So we have

$$F(\mathbf{a})\leqslant\min_{0\leqslant j\leqslant t}2^{t-j}(\widetilde{x}_1+\cdots+\widetilde{x}_j+1)=F(\mathbf{x})$$

for all $\mathbf{x}\in R(\mathbf{a})$. It follows that

$$\sum_{a_1,\dots,a_t=r}^{k}F(\mathbf{a})\int_{R(\mathbf{a})}\frac{d\mathbf{x}}{x_1\cdots x_t}\leqslant\sum_{a_1,\dots,a_t=r}^{k}\int_{R(\mathbf{a})}\frac{F(\mathbf{x})}{x_1\cdots x_t}d\mathbf{x}=\int_{\exp(H_{r-1})}^{\exp(H_k)}\cdots\int_{\exp(H_{r-1})}^{\exp(H_k)}\frac{F(\mathbf{x})}{x_1\cdots x_t}d\mathbf{x}.$$

We then obtain by making the change of variables $x_i=re^{\xi_i(H_k-H_{r-1})}$ that

$$\begin{aligned}G_t&\leqslant(2(H_k-H_{r-1}))^t t!\int_{\Omega_t}\min_{0\leqslant j\leqslant t}2^{-j}\Big(re^{\xi_1(H_k-H_{r-1})}+\cdots+re^{\xi_j(H_k-H_{r-1})}+1\Big)d\boldsymbol{\xi}\\&\ll\Big(2\log\frac{k}{r}\Big)^t t!\int_{\Omega_t}\min_{0\leqslant j\leqslant t}2^{-j}\Big(r\Big(\frac{k}{r}\Big)^{\xi_1}+\cdots+r\Big(\frac{k}{r}\Big)^{\xi_j}+1\Big)\mathrm{d}\boldsymbol{\xi}\\&\ll\Big(2\log\frac{k}{r}\Big)^t t!U_t(v_0,u_0)\end{aligned}$$

as desired. □

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To bound $U_t(v_0, u_0)$, we apply the following estimate established by Ford [13].

Lemma 4.3. *Suppose that t, u, v are integers satisfying $1 \leqslant t \leqslant 10v$ and $u \geqslant 1$. Then*

$$U_t(v, u) \ll \frac{u(1 + (t - u - v)^2)}{(t + 1)!(2^{t-v-u} + 1)}.$$

Since $v_0 = \lfloor \frac{\log k - \log r}{\log 2} \rfloor$, we have

$$\Big(\frac{k}{r}\Big)^{\xi_i} \asymp 2^{v_0 \xi_i}.$$

If $0 < \delta \leqslant 1/10$, then by (4.4) and (4.25), we have $J_2 = v_0 + u_0 + O(1)$ and $8J_2 \leqslant 10v_0$. Hence $2 \log \frac{k}{r} = 2(\log k - \log r) \geqslant 2 \times 0.9 \log k > 1.2J_2$. Note that $\frac{\log k - \log r}{J_2} < \frac{5}{7}$. Hence, using Lemmas 4.2 and 4.3, we obtain by (4.25) that

$$\begin{aligned}
\sum_{1 \leqslant t \leqslant 8J_2} \frac{G_t}{t!} &\ll \sum_{1 \leqslant t < J_2} \Big(2 \log \frac{k}{r}\Big)^t U_t(v_0, u_0) + \sum_{J_2 \leqslant t \leqslant 8J_2} \Big(2 \log \frac{k}{r}\Big)^t U_t(v_0, u_0) \\
&\ll \sum_{1 \leqslant t < J_2} \frac{u_0(1 + (t - J_2)^2)}{(t + 1)!(2^{t-J_2} + 1)} \Big(2 \log \frac{k}{r}\Big)^t \\
&\quad + \sum_{J_2 \leqslant t \leqslant 8J_2} \frac{u_0(1 + (t - J_2)^2)}{(t + 1)!(2^{t-J_2} + 1)} \Big(2 \log \frac{k}{r}\Big)^t \\
&\ll u_0 \frac{\big(2 \log \frac{k}{r}\big)^{J_2 - 1}}{J_2!} + u_0 \sum_{l=0}^{7J_2} \frac{1 + l^2}{(J_2 + 1 + l)! 2^l} \Big(2 \log \frac{k}{r}\Big)^{J_2 + l} \\
&\ll u_0 \frac{(2 \log \frac{k}{r})^{J_2 - 1}}{J_2!} + u_0 \frac{(2 \log \frac{k}{r})^{J_2}}{(J_2 + 1)!} \\
&\ll (\log r) \frac{(2 \log \frac{k}{r})^{J_2}}{(J_2 + 1)!}.
\end{aligned} \tag{4.27}$$

If $t > 8J_2$, one can infer from (4.26) that $U_t(v_0, u_0) \leqslant \int_{\Omega_t} 1 d\boldsymbol{\xi} = \frac{1}{t!}$. Moreover, one can derive from $2 \log \frac{k}{r} \leqslant 2 \log k$ and $J_2 = \lfloor \frac{\log k}{\log 2} \rfloor$ that $\frac{2 \log \frac{k}{r}}{8J_2} \leqslant 1/4$. Hence we have by Lemma 4.3 that

$$\sum_{t > 8J_2} \frac{G_t}{t!} \ll \sum_{t > 8J_2} \frac{\big(2 \log \frac{k}{r}\big)^t}{t!} \ll \frac{(2 \log \frac{k}{r})^{8J_2}}{(8J_2)!} \ll \frac{(2 \log \frac{k}{r})^{J_2}}{(J_2 + 1)!}. \tag{4.28}$$

If $0 < \delta \leqslant \frac{1}{10}$, then by (1.4) we have $B(r, k) \asymp (\log k)^{-1/2}$. So we derive from (4.21), (4.27), (4.28), and Stirling's formula that

$$\mathcal{R}_r(n, k) \ll \frac{\log r}{k^2} \frac{(2 \log \frac{k}{r})^{J_2}}{(J_2 + 1)!} \ll \frac{\log r}{(\log k)^{3/2}} k^{-\mathcal{E} + ((\log(1-\delta))/\log 2)} \ll \delta B(r, k) k^{-\mathcal{E} + ((\log(1-\delta))/\log 2)}$$

as desired if $0 < \delta \leqslant \frac{1}{10}$.

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