Row sums conjecture in iterated Rascal triangles

PETRO KOLOSOV

1. *Introduction*

A Rascal triangle is a Pascal-type numeric triangle developed in 2010 by three middle-school students, Alif Anggoro, Eddy Liu and Angus Tulloch [1]. During their mathematics classes they were challenged to provide the next row for the following sequence:

1

1 1

1 2 1

1 3 3 1

…

The expected answer was the one that matches Pascal's triangle, i.e. 1 4 6 4 1, obtained by applying the recurrence rule of binomial coefficients *South* = *East* + *West*. Instead, the young students suggested that the next row should be 1 4 5 4 1. In contrast to Pascal's triangle rule *South* = *East* + *West*, they produced the new row using the relation they called the *diamond formula*

$$South = \frac{East \cdot West + 1}{North}. \tag{1}$$

By applying the diamond formula (1), the students produced an original triangular sequence known as the Rascal triangle, see Table 1.

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	5	4	1					
5	1	5	7	7	5	1				
6	1	6	9	10	9	6	1			
7	1	7	11	13	13	11	7	1		
8	1	8	13	16	17	16	13	8	1	
9	1	9	15	19	21	21	19	15	9	1

TABLE 1. Rascal triangle R_1. See sequence A077028 in [2].

The fourth row is 1 4 5 4 1 because $4 = \frac{1 \times 3 + 1}{1}$ and $5 = \frac{3 \times 3 + 1}{2}$.

https://doi.org/10.1017/mag.2025.10129 Published online by Cambridge University Press

The Rascal triangle shown at table (1) is the base case of a completely new family of Pascal-type triangles. This new family of triangles, known as *iterated Rascal triangles* R_i, was first mentioned by Jena Gregory in her master's thesis [3].

We denote the k-th term of the n-th row of an iterated Rascal triangle as $\binom{n}{k}_i$ where i indicates the number of iterations. An integer sequence generated by $\binom{n}{k}_i$ is called the *iterated Rascal triangle* R_i, while $\binom{n}{k}_i$ is called the *iterated Rascal number*. Thus, the Rascal triangle of Table (1) is called the iterated Rascal triangle R_1 generated by $\binom{n}{k}_1 = k(n-k)+1$.

Although the iterated Rascal number $\binom{n}{k}_1$ is defined by the diamond rule (1), which is different from the binomial recurrence, it shares a close connection with the binomial coefficients $\binom{n}{k}$. In general,

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{m}. \tag{2}$$

For example, $\binom{7}{4}_3 = 35$, $\binom{12}{7}_5 = 792$ and $\binom{11}{5}_5 = 462$.

Example 1: Rascal triangle R_2 generated by $\binom{n}{k}_2$

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	19	15	6	1			
7	1	7	21	31	31	21	7	1		
8	1	8	28	46	53	46	28	8	1	
9	1	9	36	64	81	81	64	36	9	1

TABLE 2. Rascal triangle R_3. Sequence A374378 in [2].

https://doi.org/10.1017/mag.2025.10129 Published online by Cambridge University Press

Example 2: Rascal triangle R_3 generated by $\binom{n}{k}_3$

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	69	56	28	8	1	
9	1	9	36	84	121	121	84	36	9	1

TABLE 3: Rascal triangle R_3. Sequence A374452 in [2].

The definition of iterated Rascal coefficients (2) is intriguing because it shows a close connection between iterated Rascal triangles R_i and binomial coefficients, making it worthy of further discussion.

To provide more insights on the definition (2), we review it in terms of generating functions, see [3, p. 4, definition 2.0.1]

$$\sum_{n=0}^{\infty}\binom{n+d}{d}_i q^n = \frac{1}{(1-q)^{i+1}}\sum_{j=0}^{i}\binom{j+d-(i+1)}{d-(i+1)}q^j.$$

By substituting $\binom{n+d}{d}_i = \sum_{m=0}^{i}\binom{n}{m}\binom{d}{m}$ into the equation above, we obtain

$$\sum_{n=0}^{\infty}\binom{n+d}{d}_i q^n = \sum_{n=0}^{\infty}\sum_{m=0}^{i}\binom{n}{m}\binom{d}{m}q^n = \frac{\sum_{j=0}^{i}\sum_{k=0}^{i}(-1)^{j+k}\binom{i-k}{j-k}\binom{d}{k}q^j}{(1-q)^{i+1}}$$

$$= \frac{\sum_{j=0}^{i}q^j(-1)^j\sum_{k=0}^{i}(-1)^k\binom{i-k}{j-k}\binom{d}{k}}{(1-q)^{i+1}}. \quad (3)$$

See [4, eqn. 2.5] for (3).

To show why iterated Rascal numbers $\binom{n}{k}_i$ are given by terms of a double binomial sum, we have to prove that

$$\frac{1}{(1-q)^{i+1}}\sum_{j=0}^{i}\binom{j+d-(i+1)}{d-(i+1)}q^j = \frac{1}{(1-q)^{i+1}}\sum_{j=0}^{i}q^j(-1)^j\sum_{k=0}^{i}(-1)^k\binom{i-k}{j-k}\binom{d}{k}.$$

https://doi.org/10.1017/mag.2025.10129 Published online by Cambridge University Press

Consider the following relation:

$$(-1)^j \sum_{k=0}^{i} (-1)^k \binom{i-k}{j-k}\binom{d}{k} = \binom{j+d-(i+1)}{d-(i+1)}. \tag{4}$$

This is true because

$$\sum_{k=0}^{l} \binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n}. \tag{5}$$

By substituting $l = i$, $m = i-j$, $s = d$ and $n = 0$, the left-hand side of (4) yields

$$\begin{aligned}(-1)^j \sum_{k=0}^{i} (-1)^k \binom{i-k}{i-j}\binom{d}{k} &= (-1)^j\left[(-1)^{2i-j}\binom{d-i+j-i}{j}\right] \\ &= (-1)^{2i}\binom{d-i+j-i}{j}.\end{aligned}$$

Then, by rearranging the terms and by using $(-1)^{2i} = 1$ we reach the desired result

$$(-1)^j \sum_{k=0}^{i} (-1)^k \binom{i-k}{j-k}\binom{d}{k} = \binom{j+d-(i+1)}{d-(i+1)}.$$

Therefore, it is true that

$$\binom{n+d}{d}_i = \sum_{m=0}^{i} \binom{n}{m}\binom{d}{m}.$$

2. *Binomial identities in iterated Rascal triangles*

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [6]

$$\binom{a+b}{r} = \sum_{m=0}^{k} \binom{a}{m}\binom{b}{r-m}.$$

For $a + b = n, r = k$ implies

$$\binom{n}{k} = \sum_{m=0}^{k} \binom{n-k}{m}\binom{k}{k-m}.$$

By (2), the iterated Rascal number is

$$\binom{n}{k}_i = \sum_{m=0}^{i} \binom{n-k}{m}\binom{k}{k-m}.$$

This means that the iterated Rascal number is a partial case of Vandermonde convolution with the upper summation bound equal to i.

https://doi.org/10.1017/mag.2025.10129 Published online by Cambridge University Press

We next consider the extent to which iterated Rascal triangles match Pascal's triangle.

Proposition 1: The iterated Rascal triangle R_i is the same as Pascal's triangle up to the $(2i + 1)$th row. For every $0 \leqslant n \leqslant 2i + 1$ and $0 \leqslant k \leqslant n$

$$\binom{n}{k}_i = \binom{n}{k}.$$

Proof: To prove the proposition we have to show that, for every $i \geqslant 0$ and $0 \leqslant k \leqslant 2i + 1 - n$ having $0 \leqslant n \leqslant 2i + 1$,

$$\binom{2i + 1 - n}{k}_i - \binom{2i + 1 - n}{k} = 0.$$

Therefore we show

$$\sum_{m=0}^{k} \binom{2i + 1 - n - k}{m}_i \binom{k}{m} - \sum_{m=0}^{i} \binom{2i + 1 - n - k}{m}_i \binom{k}{m} = 0.$$

For the case $k < i$ the proof is given by [3, p. 12, proposition 6.0.1]. The case $k = i$ the proof is trivial. Thus the remaining case $k > i$ assumes that the following is true

$$\sum_{m=i+1}^{\infty} \binom{2i + 1 - n - k}{m} \binom{k}{m} = 0. \tag{6}$$

If for every $k > i$ the equation (6) is true, then its sum over k should also be equal to zero:

$$\sum_{k} \sum_{m=i+1}^{\infty} \binom{2i + 1 - n - k}{m} \binom{k}{m} = 0.$$

[5] gives the following identity in terms of the binomial coefficients

$$\sum_{k=0}^{l} \binom{l - k}{m} \binom{q + k}{n} = \binom{l + q + 1}{m + n + 1}. \tag{7}$$

By setting $l = 2i + 1, n = m, q = 0$ in (7) we get

$$\sum_{k} \binom{2i + 1 - n - k}{m} \binom{k}{m} = \binom{2i + 2 - n}{2m + 1}.$$

So that assumption (6) is equivalent to

$$\sum_{m=i+1}^{k} \binom{2i + 2 - n}{2m + 1} = 0.$$

By substituting $m = i + 1 + m$, we get

$$\sum_{m=0}^{k} \binom{2i + 2 - n}{2(i + 1 + m) + 1} = \sum_{m=0}^{k} \binom{2i + 2 - n}{2i + 3 + 2m} = 0.$$

This is indeed true because $\binom{2i + 2 - n}{2i + 3 + 2m} = 0$ for every $m, n \geqslant 0$.

https://doi.org/10.1017/mag.2025.10129 Published online by Cambridge University Press

Therefore, for every $i \geqslant 0$ and $0 \leqslant k \leqslant 2i + 1 - n$ having $n \leqslant 2i + 1$,

$$\binom{2i + 1 - n}{k}_i = \binom{2i + 1 - n}{k},$$

which implies that for every $0 \leqslant n \leqslant 2i + 1$ and $0 \leqslant k \leqslant n$

$$\binom{n}{k}_i = \binom{n}{k}.$$

This completes the proof.

In particular, (2.1) implies the row sum identity in terms of iterated Rascal numbers. For every $i \geqslant 0$ and $0 \leqslant n \leqslant 2i + 1$,

$$\sum_{k=0}^{2i+1-n} \binom{2i + 1 - n}{k}_i = 2^{2i+1-n}.$$

By setting $n = 0$ in the equation above, for every $i \geqslant 1$ we get

$$\sum_{k=0}^{2i+1} \binom{2i + 1}{k}_i = 2^{2i+1}.$$

Moreover, Proposition 1 gives a Vandermonde-type identity. For every $i \geqslant 0$ and $0 \leqslant n \leqslant 2i + 1$,

$$\binom{2i + 1 - n}{k} = \sum_{m=0}^{i} \binom{2i + 1 - n - k}{m} \binom{k}{m}. \tag{8}$$

By setting $n = 0$ and $n = 1$, (8) gives two more Vandermonde-type identities. For every $i \geqslant 0$:

$$\binom{2i + 1}{k} = \sum_{m=0}^{i} \binom{2i + 1 - k}{m} \binom{k}{m},$$

$$\binom{2i}{k} = \sum_{m=0}^{i} \binom{2i - k}{m} \binom{k}{m}.$$

3. *Row sums conjecture*

As discussed, the iterated Rascal triangles make a family of Pascal-type triangles. It is interesting to notice that row sums in iterated Rascal triangles give powers of 2, similarly to Pascal's triangle.

In [4] the authors make the following conjecture in terms of row sums of iterated Rascal triangles, which is true and proved in this section.

https://doi.org/10.1017/mag.2025.10129 Published online by Cambridge University Press

Conjecture: (Conjecture 7.5 in [4].)

For every i

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}.$$

Proof: Rewrite the conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^{i} \binom{4i+3-k}{m}\binom{k}{m} = 2^{4i+2}.$$

Rearranging sums and omitting summation bounds for k yields

$$\sum_{m=0}^{i} \sum_{k} \binom{4i+3-k}{m}\binom{k}{m} = 2^{4i+2}. \tag{9}$$

Now we apply the identity (7)

$$\sum_{k=0}^{l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1}$$

to (9). Then setting $l = 4i+3, q = 0, n = m$ yields

$$\sum_{k} \binom{4i+3-k}{m}\binom{k}{m} = \binom{4i+4}{2m+1}.$$

Therefore the conjecture is equivalent to

$$\sum_{m=0}^{i} \binom{4i+4}{2m+1} = 2^{4i+2}.$$

Now

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+3}$$

because

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^{2i+1} \left[\binom{4i+3}{2m+1} + \binom{4i+3}{2m}\right]$$

$$= \sum_{m=0}^{2i+1} \binom{4i+3}{m} = 2^{4i+3}.$$

Hence

$$\sum_{m=0}^{i} \binom{4i+4}{2m+1} = \frac{1}{2} \sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = 2^{4i+2}.$$

This implies

$$\sum_{k=0}^{2i+3} \binom{4i+3}{k}_i = 2^{4i+2}.$$

Therefore the conjecture is true.

https://doi.org/10.1017/mag.2025.10129 Published online by Cambridge University Press

Moreover, identity (7) implicates a more general relation in row sums of iterated Rascal triangles. We consider this proposition

Proposition 2 (Iterated Rascal triangles row sum).

For every i

$$\sum_{k=0}^{n} \binom{n}{k}_i = \sum_{m=0}^{2i+1} \binom{n}{m}.$$

Proof. By definition,

$$\sum_{k=0}^{n} \binom{n}{k}_i = \sum_{k=0}^{n} \sum_{m=0}^{i} \binom{n-k}{m} \binom{k}{k-m}.$$

By identity (7)

$$\sum_{k=0}^{n} \binom{n-k}{m} \binom{k}{k-m} = \binom{n+1}{2m+1}.$$

Therefore

$$\sum_{k=0}^{n} \binom{n}{k}_i = \sum_{m=0}^{i} \binom{n+1}{2m+1} = \sum_{m=0}^{i} \left[\binom{n}{2m} + \binom{n}{2m+1} \right] = \sum_{m=0}^{2i+1} \binom{n}{m}.$$

4. *Acknowledgements*

The author is grateful to Oleksandr Kulkov, Markus Scheuer and Amelia Gibbs for their valuable feedback and suggestions regarding the row sums conjecture.

References

1. Alif Anggoro, Eddy Liu and Angus Tulloch, The Rascal Triangle. *The College Mathematics Journal*, **41**(5) (2010) pp. 393-395.
2. The On-Line Encyclopedia of Integer Sequences, https://oeis.org.
3. Jena M. Gregory, Iterated Rascal triangles, *Theses and Dissertations*. 1050 (2022) available at https://scholarworks.utrgv.edu/etd/1050/.
4. Jena Gregory, Brandt Kronholm and Jacob White, Iterated Rascal triangles. *Aequationes mathematicae* (2023) pp. 1-18.
5. Ronald L. Graham, Donald E. Knuth and Oren Patashnik, Concrete mathematics: a foundation for computer science (2nd edn.) Addison-Wesley (1994).
6. George E. Andrews, Richard Askey and Ranjan Roy, *Special Functions*, Cambridge (1999),

10.1017/mag.2025.10129 © The Authors, 2025
Published by Cambridge University Press
on behalf of The Mathematical Association

PETRO KOLOSOV
Jawornicka 14,
Poznan,
Poland, 60–161
e-mail: *kolosovp94@gmail.com*

https://doi.org/10.1017/mag.2025.10129 Published online by Cambridge University Press