$2b^2 - a^2 - c^2 = ab + bc - 2ac$. Together these imply that $c^2 = 3\left(b^2 - a^2\right)$ and then eliminating $c$ leads via

$$3\left(b^2 - a^2\right) = c^2 = \frac{\left(2a^2 - b^2 - ab\right)^2}{(b - 2a)^2} = \frac{(b - a)^2(2a + b)^2}{(2a - b)^2}$$

to $3(a+b)(2a-b)^2 = (b-a)(2a+b)^2$ and $8a^3 - 6ab^2 + b^3 = 0$. Setting $\frac{a}{b} = \cos\theta$ for an auxiliary angle $\theta$ then gives $\cos 3\theta = -\frac{1}{2}$ so that $\theta = 40°$, $\frac{a}{b} = \cos 40°$ and $c^2 = 3\left(b^2 - a^2\right) = 3b^2 \sin^2 40°$. We thus have $a : b : c = \cos 40° : 1 : \sqrt{3}\sin 40°$. Moreover, since $s$ is proportional to $1 + \cos 40° + \sqrt{3}\sin 40° = 1 + 2\cos 20°$, we see that the common difference $d$ is characterised by

$$a : d = \frac{1}{s} : \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s}\right) = \frac{1}{2\cos 20° + 1} : \frac{1}{4\cos^2 20° - 1} = 2\cos 20° - 1 : 1.$$

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## 99.20 A projective Simson line

*Introduction*

Let $ABC$ be a triangle with circumcircle $\Gamma$. Let $P$ be a point in the plane of the triangle. Simson's theorem [1] is that the feet of the perpendiculars dropped from $P$ to the side-lines of the triangle are collinear if, and only if, $P$ is on $\Gamma$, in which case this line meets the line segment joining $P$ to the orthocentre $H$ of $ABC$ at its midpoint. It seems that this result was misattributed to Robert Simson by Jean-Victor Poncelet, and the result was first published by William Wallace.

Instead of working with the feet of the perpendiculars, we can recast the statement as being that the three reflections of $P$ in the sides of the triangle are collinear if, and only if, $P$ lies on $\Gamma$, in which case this line passes through $H$.

It is known that the three reflections $H_A$, $H_B$ and $H_C$ of $H$ in the sides of $ABC$ are on $\Gamma$, so we can restate the result again as being that the three lines which are the reflection of $PH_A$ in $BC$, the reflection of $PH_B$ in $CA$ and the reflection of $PH_C$ in $AB$, are the same line if, and only if, $P$ lies on $\Gamma$.

*Result*

We are now in a position to state a generalisation of this result which is a theorem of projective geometry. Note that we introduce the points $A''$, $B''$ and $C''$ to obviate the need for Euclidean reflections, and the arbitrary point $Q$ plays the role of $H$. To see how this theorem specializes to the classical Simson configuration, first enlarge the Simson line with scale factor 2 from the distinguished point on the circumcircle, to obtain the doubled Simson line which passes through $H$, and then consider the reflections of the doubled Simson line in each of the three sides of triangle $ABC$. The points

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$A''$, $B''$ and $C''$ are where the doubled Simson line meets $BC$, $CA$ and $AB$ respectively.

*Theorem*: Let $ABC$ be a triangle and let $P$ and $Q$ be other points in the plane of this triangle. Let $\Gamma$ be a conic which passes through $A$, $B$ and $C$. Let $AQ$ meet $\Gamma$ again at $A'$, and let $PA'$ meet $BC$ at $A''$. Points $B'$, $B''$, $C'$ and $C''$ are similarly defined. Then $A''Q$, $B''Q$ and $C''Q$ are the same line if, and only if, $P$ lies on $\Gamma$.

*Proof*: When the six points $A$, $A'$, $P$, $C'$, $C$ and $B$ lie on the conic $\Gamma$, Pascal's Hexagram Theorem asserts that opposite sides of this figure meet in three collinear points: $AA'$ meets $CC'$ at $Q$, $A'P$ meets $BC$ at $A''$, and finally $C'P$ meets $AB$ at $C''$ so $A''$, $C''$ and $Q$ are collinear. By cyclic change of letters (or another hexagram) it follows that $A''$, $B''$, $C''$ and $Q$ are collinear.

Pascal's Hexagram Theorem is reversible, so if $P$ does not lie on $\Gamma$, then $A''$, $C''$ and $Q$ will not be collinear.

The non-projective variation of this result, where $\Gamma$ is a circle and $A''$, $B''$ and $C''$ are instead the reflections of $A'$, $B'$ and $C'$ in $BC$, $CA$ and $AB$ respectively, is the classical theorem of Hagge [2] that $A''$, $B''$, $C''$ and $H$ lie on a circle.

*History*

Several authors have written papers about generalisations of the Simson line, including the Bradleys [3], Geiring [4] and Pech [5], but this author has not been able to find the stated result there, nor in the standard reference works [6] and [7]. The geometric literature is so diverse and scattered, that it would be very rash to insist that a result of elementary geometry is new. Since submitting this Note, I have been contacted by Dao Thanh Oai of Viet Nam. He discovered this projective result indpendently and before me [8]. He has also located another independent prior discovery by Luis Gonzélez of Venezula [9]. It would seem that a projective proof of the existence of the Simson line is the *Zeitgeist*.

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## 99.21 The simplest proof ever of the pons asinorum theorem

The pons asinorum, or the bridge of asses, refers to Proposition 5 of Book I of Euclid's *Elements*, and it states that if the sides $AB$ and $AC$ of a triangle $ABC$ are equal, then the angles $\angle B$ and $\angle C$ are equal. Speculations on why it was so named can be found in [1, pp. 415-416], [2, p. 6], [3, p. 54] and [4, p. 89]. We give below what we think of as the simplest proof ever of the theorem. It runs as follows:

Let $ABC$ be a triangle in which

$$AB = AC. \tag{1}$$

We are to prove that $\angle B = \angle C$. For this we place triangle $ABC$ in the Euclidean plane in such a way that $A$ does not lie at the origin. Thus we may assume without any loss in generality that $A \neq 0$. Multiplying (1) by $A^{-1}$ from the left, we obtain

$$B = C. \tag{2}$$

Multiplying (2), again from the left, by $\angle$, we obtain

$$\angle B = \angle C,$$

as desired.

Instructors of geometry, especially the mean ones, are invited to run this proof on a class of students in a geometry course to test how awake they are.

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