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# A Note on Rough Isometry Invariance of Resistance

ANDRÁS TELCS

IMC Graduate School of Business, Budapest,
Zrínyi u. 14, Budapest, H-1051 Hungary
(e-mail: telcsa@imc.hu)

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In this note it is shown that resistance and resistance dimension are rough isometry invariant.

## 1. Introduction

It was shown by Kanai [7] that random walks on two roughly isometric graphs are of the same type, that is, both recurrent or both transient. In general, recurrence is one of the first questions to be studied when considering a new problem on random walks. This was the case with rough isometry as well. The present paper goes further and shows that the resistance and resistance dimension are invariant under rough isometry between graphs. The presented proof is simple in that it uses only combinatorial and probabilistic methods; it is possible that operator comparison techniques would provide an alternative proof. Recently, rough isometry has been heavily investigated: $p$-harmonic functions have been studied by Holopainen [5] and Holopainen and Soardi [6], and related problems on groups by Pittet and Saloff-Coste [9] (see [8], and [1, 2]). For further reading, Woess's book [12] is a good starting point.

**Definition 1.** Two graphs $G = (V, E)$ and $G' = (V', E')$ are *roughly isometric* (denoted by $G \asymp G'$) if there is a mapping $\varphi : V \to V'$ and constants $L > 0, \lambda > 0$ such that

$$\lambda d(x, y) - L \leqslant d(\varphi(x), \varphi(y)) \leqslant \lambda^{-1} d(x, y) + L$$

for all $x, y \in V$, where $d(\cdot, \cdot)$ is the usual graph distance, and, for all $x' \in V'$,

$$d'\left(x', \varphi(V)\right) \leqslant L.$$

Consider the graph edges as unit resistors and then 'measure' the resistance between vertices, or sets of vertices. The result of the paper can easily be generalized to general

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resistances. The formal definition of the resistance between sets can be given via the following Dirichlet form.

**Definition 2.** For $A, B \subset V, d(A,B) > 0$

$$R(A,B)^{-1} = \inf_f \sum_{(x,y)\in E} (f(x) - f(y))^2,$$

where the infimum is taken over all functions $f$ with $f \equiv 1$ on $A$ and $f \equiv 0$ on $B$.

Let us consider a reference vertex $o \in V$ and, for $N \in \mathbb{N}$, define $G^N = G^{o,N}$ to be the subgraph induced by the set $B(o,N)$ of vertices at distance at most $N$ from $o$. From the monotonicity law (see [3]) it follows that $R^N$, the resistance of $G^N$, is at least $R$, and it is easy to see that

$$\lim_{N\to\infty} R^N = R. \tag{1.1}$$

**Definition 3.** A graph is *locally finite* if there is a $D > 0$ such that, for all $x \in V$, the vertex degree $d(x) = |\{y : (x,y) \in E)\}|$ is at most $D$.

For brevity, $a_n \asymp b_n$ will indicate that there is a constant $c > 1$ for which $c^{-1} < a_n/b_n < c$ for all possible $n$.

Our result is the following.

**Theorem 1.1.** *If $G$ and $G'$ are locally finite graphs with $G \asymp G'$, then there is an $L > 1$ such that for all finite sets $A, B \subset V$ with $d(A,B) > L$,*

$$R(A,B) \asymp R'(\varphi(A), \varphi(B)).$$

**Remark.** The proof of this result is based on some natural and beautiful ideas of Doyle and Snell [3]. In particular, it uses the shortening and cutting method, monotonicity principle and the $K$-fuzz of graphs. These are also the main ingredients of a possible alternative proof, which establishes a general Dirichlet form comparison between roughly isometric graphs [12, Theorem 3.10]. Based on Theorem 3.10 and Corollary 3.6 of [12], the above theorem can be seen easily. One can observe that both approaches are based on the same ideas, while the present proof is slightly more elementary than the proof of Theorem 3.2 of [12] (which implies Corollary 3.6 of [12]).

The proof of the result uses the nice analogy between electrical networks and reversible Markov chains, or more particularly random walks on weighted graphs (see [3]). Here we consider the simplest (unweighted) case of the nearest neighbour random walk $X_n$ on $G$, where $P(X_{n+1} = y | X_n = x) = p_{x,y} = 1/d(x)$ if $(x,y) \in E$.

A special case of the above theorem has particular importance.

Let

$$B_{x,N} := \{z : d(x,z) \leqslant N\}$$

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and

$$S_{x,N} = B_{x,N} \setminus B_{x,N-1}.$$

**Corollary 1.2.** *If $G$ and $G'$ are locally finite graphs with $G \asymp G'$, then there is some constant $K$ such that, for any $x \in V$, and any $N, M > 0$ with $M > N + 2K$,*

$$R(S_{x,N}, S_{x,M}) \asymp R(\varphi(S_{x,N}), \varphi(S_{x,M})). \qquad \square$$

It is well known that the volume growth is rough isometry invariant, but the above result implies that the resistance dimension is also rough isometry invariant.

**Definition 4.** The resistance dimension is $d_\Omega$ if there is some $A > 1$ such that

$$2 - d_\Omega = \limsup_{R \to \infty} \frac{R(S_{x,R}, S_{x,AR})}{\log R}. \tag{1.2}$$

**Corollary 1.3.** *If $G \asymp G'$ then the resistance dimensions $d_\Omega$, $d'_\Omega$ of the two graphs are equal.* $\square$

It can be seen (see [4, 11]) that these results have far-reaching consequences for random walks on graphs. The proof of this corollary is straightforward from Corollary 1.2, with the proper choice of $A'$ in the definition of $d'_\Omega$ depending on $A$, $\lambda$ and $K$.

## 2. Proof of the result

**Definition 5.** $\overline{G}^K = (V, \overline{E}^K)$ is the $K$-fuzz of a graph $G$ if the vertex set is the same and $(x, y) \in \overline{E}^K$ if $d(x, y) \leqslant K$.

**Lemma 2.1.** *If $G \asymp G'$ and $\varphi$ is the mapping from $G$ to $G'$ then there are constants $K > 1$, $c > 0$ such that*

$$R(A, B) \geqslant c\overline{R'}^K(\varphi(A), \varphi(B)) \tag{2.1}$$

*for any $A, B \subset V$.*

**Proof.** Let us choose $N$ sufficiently large to ensure that

$$A \cup B \subset B(o, N - 2L),$$

and restrict our attention to $G^N$. We shall prove the statement for $R^N$ which implies (2.1) using (1.1). To avoid nested superscripts we suppress $N$ in the notation. Let us first consider a minimal set $W \subset V$ of vertices which represent the equivalence classes of vertices projected to the same image. This means that $\varphi(W) = \varphi(V)$ and, for all $x, y \in W$, $\varphi(x) \neq \varphi(y)$. Let us introduce $W' = \varphi(V)$ and decompose $\varphi = P \circ C$, such that $C : V \to W$ collapses each equivalence class onto its representative and $P : W \to W'$ is a bijection. Thanks to the definition of rough isometry and the boundedness of the vertex degrees, each equivalence class contains at most $M = \lfloor \lambda^{-1} L \rfloor^{D-1}$ vertices. Let us define a

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new graph $G_W = (W, E_W)$ on $W$, where $(x, y) \in E_W$ if there is some $(s, t) \in E$ such that $\varphi(s) = x, \varphi(t) = y$. Let $R_W$ be the resistance on $G_W$. It is clear that, in this way, at most $M^2$ edges project to $(x, y)$. From the monotonicity law it follows that

$$R(A, B) \geqslant \frac{1}{M^2} R_W(C(A), C(B)).$$

If $(x, y) \in E_W$ then

$$\begin{aligned} d'(P(x), P(y)) &\leqslant \lambda^{-1} \max_{s \in C^{-1}(x), t \in C^{-1}(y)} d(s, t) + L \\ &\leqslant \lambda^{-1} \max_{s \in C^{-1}(x), t \in C^{-1}(y)} (d(s, x) + d_W(x, y) + d(y, t)) + L \\ &\leqslant \lambda^{-1}(2\lambda^{-1} L + 1)) + L =: K, \end{aligned}$$

which means that $(P(x), P(y))$ is an edge in the $K$-fuzz of $G_{W'}$. From this it follows that

$$R_W(C(A), C(B)) \geqslant \overline{R}^K_{W'}(\varphi(A), \varphi(B)) \geqslant \overline{R'}^K(\varphi(A), \varphi(B)). \qquad \square$$

**Lemma 2.2.** *For a locally finite graph $G$ there are constants $K, c > 1$ such that, for any infinite sets $A, B \subset V, d(A, B) > K$,*

$$R(A, B) \leqslant c\overline{R}^K(A, B). \tag{2.2}$$

**Proof.** We modify $G$ by contracting $A$ and $B$ into single vertices $a$ and $b$. The new graph has the edges (on $V \backslash (A \cup B)$ ) of the old one which are disjoint from $A$ and $B$, and $a$ and $b$ connected to all vertices which had neighbours in $A$ and $B$ respectively. It is clear that this operation cannot increase the resistance between any two sets (*cf.* the monotonicity law in [3]). It is evident that $R(A, B) = R(a, b)$, since the resistances are influenced only by the edges between the sets. The random walk on the new graph is again the simple nearest neighbour walk on the graph. With a slight abuse of notation, we shall use $P$ for the probabilities of the random walk reflected on $A$ and killed in $B$ on the finite graph $G^{o,N}$ (*cf.* the remark in the proof of Lemma 2.1 and assume that $N$ is sufficiently large). Let us start with an observation borrowed from [3] or [10]:

$$R(A, B) = R(a, b) = \frac{1}{d(a)} \sum_i P(X_i = a, i < T_b),$$

where $T_b = \min\{i : X_i = b\}$. Following [3], the random walk on the $K$-fuzz $\overline{G}^K$ of $G$ evolves by the law

$$\overline{P}^K = \frac{P + P^2 + \cdots + P^K}{K},$$

that is, it takes steps of length $1, 2, \ldots, K$ with equal probability with respect to the appropriate power of $P$. From this it follows that, for a given trajectory of $X_n$, a random time decimation can define $\overline{X}_k = \overline{X}^K_k$, the fuzz walk. Formally $\overline{X}_0 = X_0$, the uniform waiting time is $\tau_1$ and $\overline{X_1} = X_{\tau_1}$, and then $\tau_{,}\overline{X_2}, \ldots$, are defined in the same way. One should observe that, for $i \in [\tau_j, \tau_{j+1})$,

$$\{X_i = 0\} \subset \{\overline{X}^K_j \in B_{0,K}\},$$

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and

$$\begin{aligned} P(X_i = a, i < T_b) &= P\left(X_i = a, \overline{X}_j \in B_{a,K}, i < T_b, i \in [\tau_j, \tau_{j+1})\right) \\ &= \sum_{z \in B_{a,K}} P\left(X_i = a, \overline{X}_j = z, i < T_b, i \in [\tau_j, \tau_{j+1})\right) \\ &= \sum_{z \in B_{a,K}} P\left(X_i = a, \overline{X}_j = z, i < T_b, i \in [\tau_j, \tau_{j+1})\right) \frac{\overline{P}^K(z,a)}{\overline{P}^K(z,a)} \\ &\leqslant C\overline{P}^K\left(\overline{X}_{j+1} = a, j+1 < \overline{T_b}, i \in [\tau_j, \tau_{j+1})\right), \end{aligned} \tag{2.3}$$

where the estimate $\overline{P}^K(z,a) \geqslant c$ was used. Summing (2.3) over $i$,

$$\sum_{i=k}^{\infty} P(X_i = a, i < T_b) \leqslant C \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P\left(\overline{X}_{j+1} = a, j+1 < \overline{T_b}, i \in [\tau_j, \tau_{j+1})\right),$$

which gives the result

$$\sum_{i=0}^{\infty} P\left(X_i = a, i < T_b\right) \leqslant KC \sum_{j=0}^{\infty} P\left(\overline{X}_j = a, j < \overline{T_b}\right).$$

Finally

$$\begin{aligned} R(A,B) &= \frac{1}{d(a)} \sum_{i=0}^{\infty} P(X_i = a, i < T_b) \\ &\leqslant \frac{C}{d(a)} \sum_{j=0}^{\infty} P(\overline{X}_j = a, j < \overline{T_b}) \\ &= C \frac{\overline{d}^K(a)}{d(a)} \overline{R}^K(A,B) \\ &\leqslant C\overline{R}^K(A,B), \end{aligned}$$

where the last inequality follows from the local finiteness of the graph. □

**Proof of Theorem 1.1.** One direction of the statement follows by applying Lemma 2.1 to $G$ and $G'$, then Lemma 2.2 to $G'$. For the given rough isometry $\varphi$ we obtain

$$R(A,B) \geqslant cR'(\varphi(A), \varphi(B)).$$

The opposite direction can be seen as follows. One can see that for two sets $A, B$ and given $\varphi$ with constants $\lambda$ and $K$, an 'inverse' rough isometry $\psi$ can be constructed, (if $d(A,B) > L$) with constants $\lambda, 3K$ satisfying

$$\psi \circ \varphi(A) \subset A$$

and

$$\psi \circ \varphi(B) \subset B.$$

The trick of the construction is easy: one should simply select $y \in A \cap \varphi^{-1}(z)$, if $A \cap \varphi^{-1}(z) \neq \varnothing$ (or for $B$ respectively), and set $\psi(z) = y$. This construction and the resistance estimates

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require that the sets and their images are disjoint, which holds if $L \geqslant \frac{1}{\lambda^2}(2\lambda K + 6K)$. Having this $\psi$, we apply the above estimate again to conclude that

$$R'[\varphi(A), \varphi(A)] \geqslant c'' R[\psi \circ \varphi(A), \psi \circ \varphi(B)] \geqslant c'' R(A, B),$$

which gives the desired comparison. □

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