CORRELATION PROBLEMS IN EVOLUTIONARY STOCHASTIC PROCESSESS

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ABSTRACT. In a previous contribution to these *Proceedings* (Ramakrishnan (1)) the concept of product density was introduced to describe the statistical distribution of a discrete number of particles in a continuous space E, corresponding to a single point t, where t is the parameter with respect to which the stochastic process evolves. This is extended to densities corresponding to n points on the t axis and correlation problems associated with these density functions are studied with particular reference to electron-photon cascades.

1. *Introduction.* An evolutionary stochastic process† is characterized by the variation of an aggregate of random variables $N_1(t), N_2(t), \ldots, N_n(t)$ denoted by $\langle N_i(t)\rangle$ with a one-dimensional parameter (say t) in a manner such that the 'realization' of the stochastic process at $t = t_0$ need not imply its 'realization' at $t > t_0$. This definition though brief completely contains the concept of 'evolution'. In a deterministic process representing the evolution of a system with time t it is possible to conceive of the existence of the system at time t without reference to its existence at a later time. We carry over this definition to random processes by applying this idea to 'realization' which is employed in the conventional statistical sense of a particular sample function.

There are two types of correlation problems that arise immediately from this concept.

(i) *Instant correlation.* This is the correlation between random variables corresponding *to the same value of* t, i.e. the study of the expectation value

$$\mathscr{E}\{N_1(t)\, N_2(t) \ldots N_n(t)\}.$$

The nth moment $\mathscr{E}\{N_i^n(t)\}$ can be interpreted as the nth order correlation of $N_i(t)$ with itself.

(ii) *Sequent correlation.* It is the correlation between the random variables corresponding *to m time points,* i.e. the expectation value $\mathscr{E}\{N_{i_1}(t_1)\, N_{i_2}(t_2) \ldots N_{i_m}(t_m)\}$, where $i_1, i_2, \ldots, i_m$ can take values from 1 to n. This correlation function is of fundamental importance in stochastic theory especially if it is determined for values of m as large

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† M. S. Bartlett in his paper (2) has defined an evolutionary process as one which is non-stationary. But this is only a residuary definition and is intuitively satisfactory when t refers to time. In quantum mechanical systems stationarity implies a time dependence of wave functions of the type e^{iEt} and the wave function can still be considered evolutionary in the sense of our definition though it is stationary.

The parameter t is referred to as 'time' for convenience. In cosmic ray cascades it denotes the thickness of matter traversed.

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as we please, not only because of its physical significance but also due to its mathematical properties. It includes within its definition the correlation functions corresponding to the same 'time' under suitably defined limiting process $t_1 \to t_2 \to \ldots \to t_n$.

2. *Correlation in point processes.* In this paper we shall concern ourselves with the two types of correlation functions in point processes (for definition see Bartlett (3)) with particular reference to electron-photon cascades in cosmic radiation. For point processes correlation of type (i) is studied as is now well known (see Bharucha-Reid (4)) through the (instant) product densities in the E space and type (ii) needs an adaptation to different values of t, i.e. through the sequent product densities. In any physical problem the determination of the correlation functions is possible if the process characterized by these functions is Markovian with respect to t and consequently the stochastic differential equation with respect to t can be written down on postulating what happens to a random variable as t varies by an infinitesimal quantity Δ. However, even if the process is non-Markovian it may be possible to redefine the process and introduce a new set of random variables in such a manner that the new process is Markovian. More precisely if the random variables are distinguished by the energy parameter E we shall write $N(E; t)$ as the number of particles with energy greater than E. We can also speak of different kinds of particles and in such a case, we define $N_i(E; t)$ as the random variable representing the number of particles of type i with energy greater than E. It is not possible to obtain a differential equation for the probability frequency function $\pi(\langle n_i \rangle E; t)$, defining the probability that $N_i(E; t) = n_i$, since the process is non-Markovian if we assume the multiplicative behaviour of the individual particles as E-dependent. In such a case it has been shown in a previous contribution to these *Proceedings* by one of us (A. R.) that we have to define product densities of these particles in the E-space and the process is Markovian with respect to these product densities.

In the case when there is only one type of particle if

$$f_n(E_1, E_2, \ldots, E_n; t)\, dE_1 dE_2 \ldots dE_n$$

represents the joint probability that there lies one particle in dE_1, one in dE_2, ..., and one in dE_n at the particular thickness t, then f_n is called the product density of degree n. If $N(E)$ is the stochastic variable representing the number of particles with energy above E then

$$\mathscr{E}\{[N(E) - N(E_0)]^r\} = \sum_{s=1}^{r} C_s^r \int_E^{E_0} \int_E^{E_0} \ldots \int_E^{E_0} f_s(E_1, E_2, \ldots, E_s)\, dE_1 dE_2 \ldots dE_s,$$

where C_s^r are coefficients defined by one of us (A. R.) by

$$M^r = \sum_{s=1}^{r} C_s^r M(M-1) \ldots (M-s+1)$$

M being a positive integer. The coefficients are obtained by putting $M = 1, 2, \ldots$. The new sequent product density functions $\mathfrak{F}_n$ corresponding to different thicknesses are defined such that

$$\mathfrak{F}_n(E_1, E_2, \ldots, E_n; t_1, t_2, \ldots, t_n)\, dE_1 dE_2 \ldots dE_n$$

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represents the joint probability that there exists a particle in dE_1 at t_1, one in dE_2 at t_2, ..., and one in dE_n at t_n.* Taking one type of particle only, the nth order sequent density corresponding to different thicknesses is connected to all the instant densities of order m less than or equal to n. We first establish in the case of cascade processes the connexion for the second order. For $t_2 > t_1$

$$\mathfrak{F}_2(E_1, E_2; t_1, t_2|E_0) = \int_{E_2}^{\infty} f_2(E_1, E_2'|E_0; t_1) f_1(E_2|E_2'; t_2 - t_1)\, dE_2' + f_1(E_1|E_0; t_1) f_1(E_2|E_1; t_2 - t_1), \quad (1)$$

where the parameter E_0 merely indicates that at $t = 0$ there was a primary of energy E_0. This equation is obtained by the following argument. The particle of energy E_2 at t_2 must belong to a cascade generated either by a particle of energy E_2' at t_1 or by the particle of energy E_1 ($> E_2$) at t_1. The two terms in (1) refer to these two possibilities. It is to be noted that $\mathfrak{F}_2$ is not symmetric in t_1 and t_2 as should be expected from the evolutionary nature of the Markovian process.

Defining $N(E; t_1)$ and $N(E; t_2)$ as the random variables representing the number of particles above energy E at t_1 and t_2, respectively, the expectation value of their product is given by

$$\mathscr{E}\{N(E; t_1)\, N(E; t_2)\} = \int_E^{E_0} \int_E^{E_0} \mathfrak{F}_2(E_1, E_2; t_1, t_2)\, dE_1\, dE_2. \quad (2)$$

The variation of $\mathscr{E}\{N(E; t_1)\, N(E; t_2)\}$ as t_1 varies from 0 to t_2 is not only of considerable mathematical but also of physical interest. For, if we assume the entire cascade to be generated by a single particle of energy E_0, the second-order density at $t_1 = 0$ is zero and $f_1(E_1; 0) = \delta(E_0 - E_1)$. Hence when $t_1 \to 0$ the first term of (1) vanishes and $\mathfrak{F}_2$ reduces to

$$\mathfrak{F}_2(E_1, E_2; 0, t_2) = \delta(E_1 - E_0) f_1(E_2|E_1; t_2). \quad (3)$$

As $t_1 \to t_2$, $\mathfrak{F}_2(E_1, E_2; t_1, t_2)$ does not reduce just to $f_2(E_1, E_2; t_2)$ but to

$$\begin{aligned} \mathfrak{F}_2(E_1, E_2; t_2, t_2) &= \int f_2(E_1, E_2'; t_2)\, \delta(E_2 - E_2')\, dE_2' + f_1(E_1; t_2)\, \delta(E_2 - E_1) \\ &= f_2(E_1, E_2; t_2) + f_1(E_1; t_2)\, \delta(E_2 - E_1). \end{aligned} \quad (4)$$

Hence $\mathscr{E}\{N(E; t_1)\, N(E; t_2)\}$ in the two limiting cases are given by

$$\begin{aligned} \mathscr{E}\{N(E; 0)\, N(E; t_2)\} &= \int_E^{E_0} \int_E^{E_0} \delta(E_1 - E_0) f_1(E_2|E_1; t_2)\, dE_1\, dE_2 \\ &= \int_E^{E_0} f_1(E_2|E_0; t_2)\, dE_2 \end{aligned} \quad (5)$$

and

$$\mathscr{E}\{N^2(E; t_2)\} = \int_E^{E_0} \int_E^{E_0} f_2(E_1, E_2; t_2)\, dE_1\, dE_2 + \int_E^{E_0} f_1(E_1; t_2)\, dE_1. \quad (6)$$

In the case of instant product densities the rth moment is connected to all moments of order less than r in view of the degeneracies that occur in the density of degree r

* This should not be confused with the product densities in the product space of E and t defined in the paper 'New approach to the cascade theory', by Alladi Ramakrishnan and S. K. Srinivasan, *Proc. Indian Acad. Sci.* **44** (1956).

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when the energy variables become equal. It turns out that this is implied in the definition of sequent product density and hence the limiting process $t_1 \to t_2$ in the equation (2) gives directly the mean square number *and involves the first moment also* according to (6).

3. *Application to electron-photon cascades.* Now we shall apply the method of sequent product densities to the electron-photon cascade. Then Equation (1) is modified as

$$\begin{aligned}\mathfrak{F}_2^{(1)}(E_1, E_2; t_1, t_2) = &\int f_2^{(1)}(E_1, E_2'; t_1) f_1^{(1)}(E_2|E_2'; t_2 - t_1)\, dE_2' \\ &+ \int f g_{1,1}^{(1)}(E_1, E_2'; t_1) f_1^{(2)}(E_2|E_2'; t_2 - t_1)\, dE_2' \\ &+ f_1^{(1)}(E_1; t_1) f_1(E_2|E_1; t_2 - t_1), \end{aligned} \tag{7}$$

where f_n and g_n are product densities of degree n of electrons and photons respectively and $fg_{1,1}$ the mixed product density of degree (1, 1) of electrons and photons and the superscripts (1) and (2) refer to electron and photon generated showers, respectively. The second term represents the probability that there was an electron and a photon of energy E_1 and E_2' respectively at t_1 and that the photon generates the electron with energy E_2 at t_2. For the case when f_1 is homogeneous in the energy variables the Mellin transform of $\mathfrak{F}_2$ is obtained in terms of the Mellin transforms of $f_2, fg_{1,1}$ and f_1. Defining the transform P of $\mathfrak{F}_2$ as

$$P(r, s; t_1, t_2) = \int_0^\infty \int_0^\infty \mathfrak{F}_2(E_1, E_2; t_1, t_2)\, E_1^{r-1} E_2^{s-1}\, dE_1\, dE_2 \tag{8}$$

we have

$$\begin{aligned} P(r, s; t_1, t_2) = &\nu_2^{(1)}(r, s; t_1)\, \nu_1^{(1)}(s; t_2 - t_1) \\ &+ \nu\gamma^{(1)}(r, s; t_1)\, \nu_1^{(2)}(s; t_2 - t_1) \\ &+ \nu_1^{(1)}(r + s - 1; t_1)\, \nu_1^{(1)}(s; t_2 - t_1), \end{aligned} \tag{9}$$

where ν_2, ν_1 and $\nu\gamma$ are the Mellin transforms of f_2, f_1 and $fg_{1,1}$, respectively. Since ν_2, $\nu\gamma$ and ν_1 have already been explicitly solved we can obtain $\mathfrak{F}_2$ by inverting P and $\mathscr{E}\{N(E; t_1) N(E; t_2)\}$ can be evaluated by the saddle-point method.*

4. *Integral equations for* $\mathfrak{F}_n$. We shall now write the integral equations for general $\mathfrak{F}_n$ when there is only one type of particle not merely for the sake of completeness, but also to illustrate how the degeneracies in the product density are implied in the definition of $\mathfrak{F}_n$. For this purpose it is necessary to adopt the following notation in view of the large number of energy and 'time' variables used. By $\langle l; t_1, t_2, \ldots, t_n\rangle$ we mean a set of any l 'time' points chosen from the set $t_1, t_2, \ldots, t_n$ and by $\langle E\rangle$ a set of energies which when taken in association with $\langle n; t_1, t_2, \ldots, t_n\rangle$ implies that the set of energies correspond to 'time' points $\langle n; t_1, t_2, \ldots, t_n\rangle$, understanding of course that E_1 is associated with t_1, E_2 with t_2, ... and E_n with t_n. Thus $\mathfrak{F}_n(E_1, E_2, \ldots, E_n; t_1, t_2, \ldots, t_n|E_0)$ can be written as $\mathfrak{F}_n(\langle E\rangle\langle n; t_1, t_2, \ldots, t_n\rangle|E_0)$ and it satisfies the equation

$$\begin{aligned}&\mathfrak{F}_n(\langle E\rangle\langle n; t_1, t_2, \ldots, t_n\rangle|E_0) \\ &\quad = \int_{E_l'} \sum_l f_l(E_1, E_2', \ldots, E_l'; t_1) \sum_c \prod_{i=2}^{l} \mathfrak{F}_{n_i}(\langle E\rangle\langle n_i; t_2, \ldots, t_n\rangle|E_i'; t_1) \end{aligned} \tag{10}$$

* Numerical calculations on this work are in progress.

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such that $n_1+n_2+\ldots+n_l=n-1$ and $\sum_c$ implies summation over all modes of division of n into $(l-1)$ parts. This equation is obtained by merely arguing that the $(n-1)$ particles at the corresponding 'time' points must have been generated by l particles at t_1 of which the particle of energy E_1 may be one. This equation amounts to a recursion relation for $\mathfrak{F}$ which is only of formal use for large n. We are interested in the limit when the 'time' points within each group become the same and equal to t_1 and hence $\mathfrak{F}_n(\langle E\rangle\langle n_1; t_2, \ldots, t_n\rangle|E_i'; t_1)$ reduces to $\Pi\delta(E_i'-E_{\alpha i})$, where $E_{\alpha i}$ represents the energies $E_{\alpha_1}, E_{\alpha_2}, \ldots$ belonging to $\langle E\rangle$ associated with $\langle n_i; t_2, \ldots, t_n\rangle$ and need not necessarily be $E_1, E_2, \ldots$. But these energies should belong to one of the energies $E_1, E_2, \ldots, E_n$. On integration the n_i energies in each group become equal and the number of terms in the summation is the number of ways in which the n energies degenerate to a set of l different energies and hence we obtain the coefficient C_l^n in equation (10). Thus as $t_1 \to t_2 \to \ldots \to t_n$ we have

$$\int \mathfrak{F}_n(E_1, E_2, \ldots, E_n; t_n, t_n, \ldots, t_n)\, dE_1 dE_2 \ldots dE_n = \sum_{l=1}^{n} C_l^n f_l(E_1, E_2, \ldots, E_l; t_n)\, dE_1 dE_2 \ldots dE_l.$$

The concept of sequent product density function can be carried over into quantum mechanics, and application of this to a system of Fermions yields an alternative method of deducing the Feynman propagator for such systems. This will form the subject matter of a later contribution.

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REFERENCES

(1) Ramakrishnan, A. Stochastic processes relating to particles distributed in a continuous infinity of states. *Proc. Cambridge Philos. Soc.* 46 (1960), 595–602.
(2) Bartlett, M. S. Some evolutionary stochastic processes. *J. Roy. Statist. Soc. Ser.* B, 11 (1949), 211–29.
(3) Bartlett, M. S. *Introduction to stochastic processes* (Cambridge, 1955).
(4) Bharucha-Reid, A. T. *Elements of the theory of Markov processes and their applications* (McGraw-Hill: New York, 1960).

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