Why are there no infinite left-sided decimal expansions?

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Introduction

It was my nine-year-old daughter who got me interested in the title question. As she appreciates, multiplying an integer by a power of 10 is a cinch. To multiply 34 by 100, simply add two zeros at the end: $34 \times 100 = 3400$. Dividing 3400 by 100 is the reverse process: remove two zeros to obtain 34. More generally, to multiply an integer by 10^N, for non-negative N, add N zeros to the end of its decimal notation, and to divide an integer by 10^N remove N zeros from its end — so long as it has them. Easy-peasy; my daughter knows all that.

When the number in question isn't an integer, multiplying it by a power of 10 is a touch trickier. To multiply 3.4 by 100 we can't just add two zeros to the end of its decimal representation, as that would give us 3.400. And to divide by 100 a number that isn't an integer multiple of 100 we can't just remove two zeros from the end of its decimal representation: there aren't two zeros, for example, at the end of 5.3 or 53. To a nine-year-old, that can seem puzzling. But there's a trick that gets round the difficulty whilst respecting the spirit of the method.

The trick exploits the fact that we can, if we like, represent a number as an infinite left and right decimal expansion. For instance, 34 can be written as infinitely many zeros to the left of the digits '3' and '4', followed by a decimal point, followed by infinitely many zeros after the decimal point. Now it's a bit of a nuisance that English makes the full stop carry out so many different mathematical functions. As our notation will use lots of dots, let's circle the decimal point for clarity and emphasis, seeing as its position will be crucial in what follows. So in our expanded notation, we write 34 as

$$\ldots 00034\odot 00000\ldots$$

with the circled full stop representing the decimal point. Using dot notation for recurring numerals (e.g. $\frac{1}{3} = 0.\dot{3}$), this may be written as

$$\dot{0}34\odot\dot{0}$$

more succinctly. Note that the first dotted zero indicates an infinite *leftwards* repetition of zeros and the second one an infinite *rightwards* one. In this notation, 34 is represented as the sum of 3 times 10^1 and 4 times 10^0 and 0 times every other power of 10 — not just *smaller* powers than 10^1 and 10^0 but *larger* ones too.

The number 34 isn't in any way special: every integer can be written in this fashion. Simply add an infinite number of zeros to the leftmost numeral in its base 10 representation and an infinite number of zeros after the decimal point. Negative numbers can be written in the same way with a

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negative sign in front. And there's nothing special about integers either: something similar applies to *all* real numbers. The respective decimal expansions of $\frac{5}{7}$ and π, for example, are $0\odot\dot{7}14285\dot{}$ (meaning 0.714285714285714285…) and $3\odot1415926\ldots$; in expanded notation the first is $\dot{0}\odot\dot{7}14285\dot{}$ and the second $\dot{0}3\odot1415926\ldots$.

From this point of view, a real number's decimal expansion tells only half the story. The expansion presents the infinitely many digits to the right of the decimal point but only a finite part of the infinitely many digits to the left of it. That's understandable, of course, since all of these latter digits are zero except perhaps for a finite segment. Yet when it comes to multiplying or dividing by powers of 10, using the full expansion renders the process transparent. To divide 34 by 100, for instance, we simply shift its digits two places to the right, keeping the decimal point fixed, going from

$$\dot{0}34\odot\dot{0} \qquad \text{to} \qquad \dot{0}\odot34\dot{0}.$$

To multiply 0.53 by 100, we shift the digits of

$$\dot{0}\odot53\dot{0}$$

two places to the left, keeping the decimal point fixed, to obtain

$$\dot{0}53\odot\dot{0}.$$

More generally, when multiplying a number by 10^N for N non-negative, we shift its digits N places to the left, and when dividing it by the same number we shift its digits N places to the right.

Remember how we started down this path: my daughter's puzzlement about where the zero at the start of 0.53 ends up when multiplying 0.53 by 100. On our new way of looking at things, the number 53 in fact has zeros to the left of the numeral 5, which have been moved rather than lost when multiplying 5.3 by 100, since 53 is $\dot{0}53\odot\dot{0}$ and 0.53 is $\dot{0}\odot53\dot{0}$.

Would it be highly impractical to write numbers in this way, e.g. as $\dot{0}53\odot\dot{0}$ instead of 53? Not really. We could simply deploy our existing recurring decimal notation, as above, prefacing each numeral with a dotted zero (with negative numbers adding a negative sign in front). True, taking a few milliseconds more to write a number adds up to billions of person-hours lost per year. So perhaps our expanded notation — with infinitely many zeros to the left — is how we should *think* of real-number representations, even if we don't *write* them that way, for labour-saving reasons.

At any rate, when I explained this method to my daughter, she caught on immediately. It's the method she now implicitly uses to multiply and divide by powers of 10. Once she'd got used to it, though, there came a killer question. For she asked (paraphrasing a little): *Why must all numerals to the left in the expanded representation be 0 from some point onwards, whereas numerals to the right can be any digit from 0 to 9*?

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Expanded sums

A good question. As we know, every real number can be written in decimal notation as

$$\pm \sum_{-\infty < i \leqslant N} a_i 10^i$$

where N is an integer, a_i is one of 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9, and ± is + (usually suppressed) if the number is positive, and − if the number is negative (and either + or − if the number is 0, also usually suppressed). As we also know, this notation does not correspond one-to-one to real numbers, as a consequence of the fact that

$$\sum_{-\infty < i \leqslant N} 9 \times 10^i = \sum_{-\infty < i \leqslant N} 0 \times 10^i + 1 \times 10^{N+1},$$

and similarly for negative numbers. For this reason, and so as not to privilege base 10, we should strictly speaking think of a decimal expansion not as a real number but as its representation, a point that will be pertinent later.

Let's call an *expanded sum* a formal sum

$$\pm \sum_{-\infty < i < \infty} a_i 10^i$$

with each a_i a whole number from 0 to 9 inclusive. Expanded sums restore left-right symmetry. These sums are almost, but not quite, expanded reals. (I'll explain why in the next section.) Here, then, is another way to phrase my daughter's question: what's wrong with expanded sums? If real numbers are OK, why aren't such sums? In other words, why the asymmetry between left-sided and right-sided decimal expansions?

Perhaps the first answer that springs to mind is that infinite left-sided decimal expansions represent numbers that are infinitely big (when not equivalent to real numbers). So they don't correspond to 'real-world' quantities, whence the asymmetry between infinite left- and right-sided expansions.

But at this point we have to tread very carefully. Are there *any* 'real-world' infinite right-sided decimal expansions — infinitely precise physical quantities? Physics and many other natural sciences, not to mention the social sciences, use real numbers. All these uses, however, seem to involve idealisations. Are there *perfect* circles in nature, the ratio of whose circumference to their diameter is *exactly* π? Do the chambers in molluscs' shells really exhibit a precise logarithmic order? You may think not: after all, at the molecular level or below, boundaries are fuzzy. Moreover, all measurements are finite. Thermometers, barometers, rulers and any other measuring device you care to mention aren't infinitely precise; they use finite multiples of some unit as their numerical scale. Even if these instruments could somehow be made infinitely precise, humans' perceptual

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capacities are inherently finite, meaning that we cannot discriminate between readings within ε of one another for small but positive ε. Finite discriminators like us cannot read off infinitely precise information from hypothetical infinitely accurate instruments, still less from our actual and approximate instruments.

In short, it simplifies scientific theory to use real numbers, but how exactly these real quantities correspond to quantities out there in nature remains unclear. The need for circumspection is perhaps clearest in economics, which also makes copious use of real numbers, but in which many of the quantities (e.g. goods or money) are evidently discrete.

Scientific applications might be able to break the asymmetry between left-sided and right-sided expansions. But since the grounds for employing real numbers in science are *theoretical* and not in any straightforward way *perceptual*, for all that has been said there might be similarly theoretical scientific grounds for using infinite quantities. In any case, even if expanded sums have no obvious scientific application, why should that be an obstacle to their mathematical use? According to set theory, there are infinitely many types of infinity, although investigations of these infinite cardinals haven't resulted in strictly scientific applications.* So we need to think about expanded sums' *mathematical* utility.

Operations on expanded sums/expanded reals

The real number system ($\mathbb{R}$) has been extended by the complex numbers, as well as the quaternions and the octonions. If we think about its most familiar extension, *viz*. the complex numbers ($\mathbb{C}$), we observe that it preserves the four operations of addition, subtraction, multiplication and division, in the sense that all these operations are definable in $\mathbb{C}$, with their restrictions to real numbers (complex numbers $a + bi$ in the usual notation with $b = 0$) coinciding with real addition, subtraction, multiplication and division. In moving from $\mathbb{R}$ to $\mathbb{C}$, however, we lose the ability to order the field: there's no way to naturally order the complex numbers.†

Since the key field-theoretic operations on the reals are addition and multiplication (as well as their respective inverses, subtraction and division), that hardly matters. The most important thing is that we can add and multiply complex numbers in a natural way that extends the corresponding real operations (and similarly for subtraction and division).

Things are different with expanded sums. Here, I have some good news and some bad news. The good news first: we can *add* and *subtract* expanded

* Naturally, infinitary set theory has applications in other parts of pure mathematics, which in turn have consequences for applied mathematics and science.

† What's meant by a *natural* order? We would like to preserve the rule that if $a, b > 0$ then $ab > 0$, the usual order relation on the reals, the condition that $a > 0$ if, and only if, $-a < 0$, and finally the fact that $a > 0$ or $a < 0$ for non-zero a. But if $i > 0$ then $i^2 = -1 > 0$ and if $i < 0$ then $-i > 0$, so that $(-i)^2 = -1 > 0$. Either way, we can't satisfy all the desired constraints.

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sums. The bad news: we can't, in general, multiply or divide them (in a natural way); nor can we order them (ditto).

The natural way to define operations on expanded sums is to define them on their real-number trunctations and then 'take the limit'. This is admittedly a stipulative definition of 'natural', but there are no other candidates. As an illustration, take $2\dot{4}\odot 2\dot{4}$ and $3\dot{5}\odot 3\dot{5}$. Their real-number truncations $24\odot 2\dot{4}$ and $35\odot 3\dot{5}$ sum to $59\odot 5\dot{9}$; the truncations $424\odot 2\dot{4}$ and $535\odot 3\dot{5}$ sum to $959\odot 5\dot{9}$; $2424\odot 2\dot{4}$ and $3535\odot 3\dot{5}$ sum to $5959\odot 5\dot{9}$; and so on. Taking the limit, the sum of $2\dot{4}\odot 2\dot{4}$ and $3\dot{5}\odot 3\dot{5}$ therefore has to be $5\dot{9}\odot 5\dot{9}$. There's simply no other choice for the sum of the expanded sums $2\dot{4}\odot 2\dot{4}$ and $3\dot{5}\odot 3\dot{5}$, if we define it as the limit of the truncated sums.*

More generally, consider two expanded sums $\sum_{-\infty < i < \infty} a_i 10^i$ and $\sum_{-\infty < i < \infty} b_i 10^i$. Their sum is defined by

$$\sum_{-\infty < i < \infty} a_i 10^i + \sum_{-\infty < i < \infty} b_i 10^i = \sum_{-\infty < i < \infty} c_i 10^i$$

where c_N is determined by the decimal representation of

$$\sum_{-\infty < i \leqslant N} a_i 10^i + \sum_{-\infty < i \leqslant N} b_i 10^i,$$

and more generally by the decimal representation of

$$\sum_{-\infty < i \leqslant k} a_i 10^i + \sum_{-\infty < i \leqslant k} b_i 10^i$$

for every $k \geqslant N$. It should be clear that this determination is unique up to the equation

$$\sum_{-\infty < i \leqslant N} 9 \times 10^i = \sum_{-\infty < i \leqslant N} 0 \times 10^i + 1 \times 10^{N+1},$$

just as it is with the real numbers. And it shouldn't be too hard to see that defining the sum of

$$a = \sum_{-\infty < i < \infty} a_i 10^i \qquad \text{and} \qquad b = \sum_{-\infty < i < \infty} b_i 10^i$$

as the 'limit of the truncated sums' will yield the same result however we truncate

$$a = \sum_{-\infty < i < \infty} a_i 10^i \qquad \text{and} \qquad b = \sum_{-\infty < i < \infty} b_i 10^i.$$

* The full argument for this conclusion is straightforward but longer than indicated, since we need to check that all ways of truncating the expanded sum yield the same limit.

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Before moving on to subtraction, we note a key result: for any expanded sum $a = \sum_{-\infty < i < \infty} a_i 10^i$,

$$a + \dot{0}\odot\dot{0} = a + \dot{9}\odot\dot{9}.$$

Clearly, $a + \dot{0}\odot\dot{0} = a$, since all real-number truncations of $\dot{0}\odot\dot{0}$ are equal to 0. As for $a + \dot{9}\odot\dot{9}$, its truncation up to the units place is

$$\sum_{-\infty < i \leqslant 0} a_i 10^i + \sum_{-\infty < i \leqslant 0} 9 \times 10^i = \sum_{-\infty < i \leqslant 0} a_i 10^i + 10,$$

so that all coefficients of powers of 10 smaller than 10^1 are the same as those of a; its truncation up to the tens place is $\sum_{-\infty < i \leqslant 1} a_i 10^i + 100$, so that all coefficients of powers of 10 smaller than 10^2 are the same as those of a; and so on.

What this shows is that, given our definition of addition (the only natural one) we *must* identify $\dot{0}\odot\dot{0}$ and $\dot{9}\odot\dot{9}$. We should take great care, then, not to identify expanded reals with expanded sums. Just as a real number should not be identified with a decimal expansion, because, for example, the distinct decimal expansions $1\odot\dot{0}$ and $0\odot\dot{9}$ represent the same real number, likewise expanded reals should *not* be defined as the formal sum $\pm \sum_{-\infty < i < \infty} a_i 10^i$. Instead, we should think of expanded reals as such sums *modulo* $\dot{9}\odot\dot{9}$. In other words, if $X + \dot{9}\odot\dot{9} = Y$ then $X = Y$. We could articulate this revised definition much more formally, but there's no need.

The basic idea is that expanded reals are expanded sums '*modulo* 9 recurring' just as real numbers are decimal expansions '*modulo* 9 recurring'. The only difference is that in the first case, 9's recurrence is infinite in both directions whereas in the second case it is only infinite to the right.*

We note another odd consequence of the fact that we are forced, given the only natural definition of addition going, to identify $\dot{9}\odot\dot{9}$ and $\dot{0}\odot\dot{0}$. When a is an apparently negative expanded real, adding $\dot{9}\odot\dot{9}$ to it yields an apparently positive expanded real. For example, $-\dot{5}\dot{6}\odot\dot{5}\dot{6} = \dot{4}\dot{3}\odot\dot{4}\dot{3}$, since, as the reader may check, $\dot{5}\dot{6}\odot\dot{5}\dot{6} + \dot{4}\dot{3}\odot\dot{4}\dot{3} = \dot{9}\odot\dot{9} = \dot{0}\odot\dot{0}$. So $-\dot{5}\dot{6}\odot\dot{5}\dot{6}$ and $\dot{4}\dot{3}\odot\dot{4}\dot{3}$ are distinct expanded sums that correspond to the same expanded real. Though counterintuitive, this fact just has to be accepted if we wish to work with expanded reals. It's forced upon us by countenancing these objects and the only natural way of defining addition on them.

Turning to subtraction, we cannot in the general case subtract an expanded sum from another to produce a *unique* answer, at least not using the truncation method. But we obtain a unique answer *modulo* the identity $\dot{0}\odot\dot{0} = \dot{9}\odot\dot{9}$. Subtraction, although not uniquely defined on expanded sums,

* Here's another, quick, argument for $\dot{9}\odot\dot{9} = 0$. Setting $x = \dot{9}\odot\dot{9}$, we notice that $10x = x$, so that $9x = 0$ whence $x = 0 = \dot{0}\odot\dot{0}$. This argument might be thought suspicious because it uses multiplication by 10, subtraction, and division by 9, and we haven't shown that these are valid operations on expanded sums yet. In fact though, they *are* valid because 10 and 9 are integers, even if in the general case one cannot (naturally) define multiplication and division on expanded reals.

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is uniquely defined on expanded reals. The only method with any claim to naturalness is once again the truncation method. As an illustration, let's subtract $2\dot{4}\odot 2\dot{4}$ from $4\dot{2}\odot 4\dot{2}$. We note that $42\odot 4\dot{2} - 24\odot 2\dot{4} = 18\odot 1\dot{8}$; that $242\odot 4\dot{2} - 424\odot 2\dot{4} = -181\odot 8\dot{1}$; that $4242\odot 4\dot{2} - 2424\odot 2\dot{4} = 1818\odot 1\dot{8}$; that $24242\odot 4\dot{2} - 42424\odot 2\dot{4} = -18181\odot 8\dot{1}$; and so on. All these equations just involve real numbers. Continuing in this way, the truncation method thus yields two limits for $4\dot{2}\odot 4\dot{2} - 2\dot{4}\odot 2\dot{4}$, namely $1\dot{8}\odot 1\dot{8}$ and $-8\dot{1}\odot 8\dot{1}$. *Two* limits? Well, these two expanded sums are in fact the *same* expanded real, since $1\dot{8}\odot 1\dot{8} = -8\dot{1}\odot 8\dot{1}$. So we have a well-defined result for $4\dot{2}\odot 4\dot{2} - 2\dot{4}\odot 2\dot{4}$, where the numbers are conceived as expanded reals (rather than expanded sums).

The general argument for subtraction's well-definedness is similar. The key observation is that, for any positive integer k,

$$\sum_{-\infty<i\leqslant N+k} a_i 10^i - \sum_{-\infty<i\leqslant N+k} b_i 10^i = \left(\sum_{-\infty<i\leqslant N} a_i 10^i - \sum_{-\infty<i\leqslant N} b_i 10^i\right) \pm M\times 10^{N+1}$$

where M is a non-negative integer. Elaborating this a little shows that the truncation-method-difference of two expanded reals is uniquely defined; or if you like, that the difference of two expanded sums $\pm \sum_{-\infty<i<\infty} a_i 10^i$ is defined up to the identity $0\dot{\odot}0 = 9\dot{\odot}9$. The details of the argument are straightforward and omitted.

Now comes the bad news: we can't *order*, *multiply* or *divide* expanded reals in a natural way. Starting with order, the only natural account must be based on the truncation method; for instance we would like $2\dot{\odot}2 > 1\dot{\odot}1$ since $2\dot{\odot}2$ truncated up to N is greater than $1\dot{\odot}1$ truncated up to N. But how might we go about comparing say $6\dot{3}\dot{\odot}0\dot{}$ and $7\dot{2}\dot{\odot}0\dot{}$? Their truncations up to N alternate in size: $3\dot{\odot}0\dot{} > 2\dot{\odot}0\dot{}$, $63\dot{\odot}0\dot{} < 72\dot{\odot}0\dot{}$, $363\dot{\odot}0\dot{} > 272\dot{\odot}0\dot{}$, … . However, the problem is deeper than that. For the incomparability of some elements, such as $6\dot{3}\dot{\odot}0\dot{}$ and $7\dot{2}\dot{\odot}0\dot{}$, is compatible with the order being a partial one.* But assuming we want the order to be natural, we can show that expanded reals cannot even be partially ordered. To see why, note that zero should be taken as $0\dot{\odot}0\dot{}\,(= 9\dot{\odot}9\dot{})$, as explained, and that we would like it to be the case that if $X, Y > 0$ then $X + Y > 0$. But then countless pairs of apparently positive expanded reals sum to 0, for example,

$$3\dot{6}\dot{\odot}6\dot{3} + 6\dot{3}\dot{\odot}3\dot{6} = 9\dot{\odot}9\dot{} = 0\dot{\odot}0\dot{}.$$

The problem is that, since $0\dot{\odot}0\dot{} = 9\dot{\odot}9\dot{} = -9\dot{\odot}9\dot{}$, every intuitively 'positive' expanded real X (with a leading plus sign) is equal to an intuitively 'negative' expanded real (with a leading minus sign), *viz*. the expanded real obtained by subtracting $9\dot{\odot}9\dot{}$ from X. For example, $3\dot{6}\dot{\odot}6\dot{3} = -6\dot{3}\dot{\odot}3\dot{6}$, since as we've seen $3\dot{6}\dot{\odot}6\dot{3}$ and $6\dot{3}\dot{\odot}3\dot{6}$ sum to zero. Our definition of an expanded real number identifies $3\dot{6}\dot{\odot}6\dot{3}$ and $6\dot{3}\dot{\odot}3\dot{6}$, as it should. Yet that leaves no way to define positive or negative naturally for expanded reals. Order has gone out the window.

* For the *cognoscenti*: moreover, the Order Extension Theorem tells us that every partial order can be extended to a linear order.

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As mentioned earlier, the fact that we can't naturally order an expansion of the real numbers (such as the complex numbers) doesn't prevent us from defining the four operations of addition, subtraction, multiplication and division on it. So can we, in the general case, multiply two expanded reals? Unfortunately, we can't. For what would $\dot{1}\odot\dot{1} \times 1\odot\dot{1}$ be? The problem is that any given power of 10, 10^i say, is equal to the product of infinitely many products of powers of 10: $10^i = 10^{(i+N)} \times 10^{-N}$ for each N. So the coefficient of 10^N in the convolution $\dot{1}\odot\dot{1} \times \dot{1}\odot\dot{1}$ is determined by the sum of infinitely many terms of the form 1×1, and by the overspill from the previous sum. Coefficients of arbitrarily large powers of 10 combine with coefficients of arbitrarily small powers of 10 to undermine any stable definition of 10^N's coefficient. Moral: there is *no* natural way to define multiplication for expanded reals. For the same sort of reason — the simple argument is left to the reader — division is also not definable in general. Notice by the way that if division were definable, so would multiplication, since $a \times b = c$ if, and only if, $c \div b = a$ (when $b \neq 0$).*

An unnatural method

There are of course various *unnatural* ways to define multiplication on double-sided decimals. Describing one such will help clarify the distinction beween natural and unnatural methods. This particular method maps the two-sided decimal

$$\ldots a_3a_2a_1\odot b_1b_2b_3\ldots$$

to the real number

$$0\odot a_1b_1a_2b_2a_3b_3\ldots$$

and likewise the negative two-sided decimal $-\ldots a_3a_2a_1\odot b_1b_2b_3\ldots$ to the real number $-0\odot a_1b_1a_2b_2a_3b_3\ldots$. Call this map m. To find the product of the two-sided decimals A and B, multiply their values $m(A)$ and $m(B)$, and then take the inverse under m, i.e. define $A \times B$ by $m^{-1}\left(m(A).m(B)\right)$, where the dot represents real multiplication. Since the product of two reals whose absolute value is at most 1 is also a real with absolute value at most 1, this is a well-defined operation.

* The *cognoscenti* may be reminded of p-adic numbers. A real number may be represented as $\pm \sum_{-\infty < i \leqslant N} a_i 10^i$, whereas a p-adic number may be represented as $\pm \sum_{N \leqslant i < \infty} a_i p^i$. But actually p-adic numbers are similar to the reals: there aren't infinitely large ones of these either. From the point of view of expansions, the p-adic field's metric is the 'opposite' of the real one: the larger the a_N at which two p-adic numbers first diverge (for a given p), the closer they are to one another. So one could consider expanded p-adic numbers, i.e. formal representations $\pm \sum_{-\infty < i < \infty} a_i p^i$ with the p-adic metric and come to the same conclusion as ours about expanded representations' mathematical uselessness. There is also a parallel to be drawn here with Laurent series.

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Observe that this definition is well-defined for expanded sums but not expanded reals. For instance, $\dot{9}\odot\dot{9}$ and $\dot{0}\odot\dot{0}$ have different images under m, *viz*. 1 and 0 respectively, with knock-on effects for their products with other expanded two-sided decimal expansions. An analogous definition also can't be used for addition, since the sum of two reals whose absolute value is at most 1 need not be a real with absolute value at most 1. A yet further problem with multiplication thus defined is that, when restricted to reals, it's no longer equivalent to real multiplication. For example, we would like $\dot{0}8\odot\dot{0}$ multiplied by itself to be $\dot{0}64\odot\dot{0}$, since $8 \times 8 = 64$, but using the given method the result turns out to be $\dot{0}6\odot4\dot{0}$, i.e. 6.4. This is all to say that the unnatural definition won't do. These sorts of considerations justify our search for *natural* operations on expanded reals.

Moral

The moral of the story: it's probably best *not* to think of a real number as $\pm \sum_{-\infty < i < \infty} a_i 10^i$ with all the a_i equal to 0 from some point onwards, i.e. with $a_i = 0$ for all $i > N$ for some N. For thinking that way naturally leads to expanded sums and thence to expanded reals, which are a 'wrong' generalisation of the real numbers. Mathematically speaking, they don't get us very far, as we can't recover multiplication or division, two key operations on real numbers. At least we can't do so using the natural truncation method described above. Nor can we define a natural order on them.

To be clear, we haven't ruled out *all* ways of multiplying and dividing expanded reals. But we've ruled out what appear to be the only natural ones, obtained by taking the limit of the corresponding real operations.

Perhaps then it's best to shelve the expanded reals, and not even use them as a heuristic for multiplying and dividing by powers of 10. I had better tell my daughter that.

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