102.26 An ad hoc limiting process

In this Note we will find a simple expression for the infinite product

$$P(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})\ldots, \qquad (|x| < 1).$$

Now cursory inspection yields

$$P(x) = 1 + x + x^2 + x^3 + x^4 + \ldots,$$

thus prompting the conjecture $P(x) = \dfrac{1}{1 - x}$.

So accepting this challenge, we define

$$\begin{aligned} f(x) = (1-x)P(x) &= (1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ldots \\ &= (1-x^2)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})\ldots \\ &= (1-x^4)(1+x^4)(1+x^8)(1+x^{16})\ldots \\ &= (1-x^8)(1+x^8)(1+x^{16})\ldots \\ &\ldots, \end{aligned}$$

leading to $f(x) = f(x^2) = f(x^4) = f(x^8) = f(x^{16}) = \ldots$.

Now since $|x| < 1$, $x^n \to 0$ and $f(x^n) \to f(0) = 1$ as $n \to \infty$ so $P(x) = \dfrac{1}{1 - x}$ as conjectured (which identity appears as a footnote in the chapter on partitions in [1, p. 366]).

Note finally the series

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \frac{16x^{15}}{1+x^{16}} + \ldots = \frac{1}{1-x}, \quad (|x| < 1),$$

obtained by differentiating $\log P(x)$.

Reference

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, OUP (2008).

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102.27 On the theorem of means for terms in AP and GP

We first define the three means for the well-known inequality $A \geqslant G \geqslant H$.

The arithmetic, geometric and harmonic means for n positive numbers $a_i\ (1 \leqslant i \leqslant n)$ are given by

$$A = \frac{a_1 + a_2 + \ldots + a_n}{n}, \qquad G = \sqrt[n]{a_1 a_2 \ldots a_n},$$

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