EXACT ANALYTICAL HYPERBOLIC TEMPERATURE PROFILE IN A THREE-DIMENSIONAL MEDIA UNDER PULSE SURFACE HEAT FLUX

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ABSTRACT

In this paper three-dimensional hyperbolic heat conduction equation in a cubic media with rectangular cross-section under a pulsed heat flux on the upper side has been solved analytically using the method of separation of variables and the Duhamel integral. The closed form solution of both Fourier and non-Fourier profiles are introduced with both modes of steady and pulsed fluxes. The results show the considerable difference between the Fourier and Non-Fourier temperature profiles. Then the answer procedure is used for modeling of interaction of a cubical tissue under a short laser pulse heating. The effects of pulse duration and laser intensity are studied analytically. Furthermore the results can be applied as a verification branch for other numerical solutions or laser treatments of biological tissues.

Keywords: Hyperbolic heat conduction, Time-dependent boundary conditions, Three-dimensional cartesian coordinates, Tissue laser interaction.

1. INTRODUCTION

In recent years Non-Fourier or hyperbolic heat conduction was in focus especially in some cases where we have high temperatures gradients, very large heat fluxes, very short transient duration or in the case of biological tissue heat treatments. Non-Fourier conduction has become a very good replacement for the famous but not physically realistic Fourier theory, because of the infinite speed of propagation. Instead, the hyperbolic theory predicts that heat propagation is not just a diffusion phenomenon but rather a wave one and has finite speed. Considering these explanations the non-Fourier heat conduction theory has found its major use in engineering science fields such as biomedical engineering, electrical engineering like laser-material interaction, temperature control of superconductors, rapid drying *etc*. The hyperbolic equation is based on the constitutive relation of heat flux vector **q** to temperature gradient in solids which was firstly introduced by Cattaneo and Vernotte [1,2]:

$$\mathbf{q}+\tau_q\frac{\partial \mathbf{q}}{\partial t}=-\lambda\nabla\theta \tag{1}$$

where θ is temperature, λ is the thermal conductivity of material and the constant τ_q is called the thermal relaxation time and physically represents the time lag between heat flux and imposed temperature gradient on material points. τ_q is about $10^{-10}\sim 10^{-14}$ for most of the engineering materials, it is reported 16 sec for meat and until 100 sec for sand [3]. The Fourier theory is reasonable for most of engineering problem but the Fourier theory predict an infinite speed for heat flux propagation, and predict that the heat flux is simultaneously established in the media due to the imposed temperature gradient. In contrast, the non-Fourier theory as stated in Eq. (1) predicts that there is a thermal lag between the imposed temperature gradient and the established heat flux. Thus the non-Fourier conduction is more realistic theory and becomes important for heat conduction in engineering phenomena with a fast heat transfer such as laser pulse interaction or biological tissue heat treatment [3]. It is known that $\sqrt{\alpha/\tau_q}$ which $\alpha=\lambda/\rho c$ is the thermal diffusivity and ρc is the material heating capacity denotes the propagation speed of temperature wave. The resulted hyperbolic heat conduction equation from Eq. (1) in absence of internal generation is:

$$\tau_q\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}=\alpha\nabla^2\theta \tag{2}$$

There are some solutions for the above hyperbolic equation in literature. Due to the importance of analytical solutions as verification branch for the numerical ones, solving the Eq. (2) in the Cartesian coordinates is under focus in the past few years. In one dimensional

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DOI : 10.1017/jmech.2015.97
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https://doi.org/10.1017/jmech.2015.97 Published online by Cambridge University Press

problems, Tang and Araki investigated the Non-Fourier heat conduction in a finite medium under periodic surface thermal disturbance [4,5], laser-pulse heating [6] and pulse surface heating [7]. Lewandowska and Malinowski [8] solved the hyperbolic heat conduction in the semi-infinite body with the heat source which capacity linearly depends on temperature in an analytical way. They [9] also found an analytical solution of the hyperbolic heat conduction equation for the case of a finite medium symmetrically heated on both sides using an internal heat source with capacity dependent on coordinate and time. Abdel-Hamid [10] modeled one dimensional non-Fourier heat conduction equation with periodic thermal oscillation using the finite integral transform. Moosaie continued one dimensional equation and solved the non-Fourier heat conduction equation in a finite medium subjected to arbitrary periodic [11] and non-periodic [12] surface disturbance. In these works the principle of superposition along with the Fourier series representation of an arbitrary periodic function was employed. Mikhailov and Cotta [13] also solved the steady-periodic hyperbolic heat conduction in a finite slab using a computer algorithm. Recently Lam and Fong [14] also solved the hyperbolic heat equation analytically in a finite medium obtaining a closed form solution. In three-dimensional coordinate, Barletta and Zanchini [15] solved three-dimensional propagation of hyperbolic thermal waves in a solid bar with rectangular cross section using the Laplace transformation. It can be seen from the literature review that, there is not any pure analytical solution for hyperbolic heat conduction problem in three-dimensional cubic coordinate. Most of the solutions for hyperbolic equation applied the Laplace transform as reference [15] and so need to numerical procedure for Laplace inverse for returning to time space. Thus it is better that tell semi-analytical to the solutions with Laplace transform. All the analytical solutions are said to be exact. The analytical solution here is the exact solution, without using any numerical procedure. In addition there is a closed form solution introduced here, for both hyperbolic and parabolic problem which can be applied as a verification branch with anyone.

The presented paper here, introduces an exact closed form analytical solution for the hyperbolic heat conduction Eq. (2) in three-dimensional Cartesian coordinates under time-dependent boundary heat flux. The method of solution is separation of variables with Duhamel integral, similar to the works of Atefi and Talaee [16] and Talaee and Atefi [17] in cylindrical coordinates. Then the results are used for modeling of a sample laser interaction of a tissue under heating of a short laser pulse for studying the effect of some parameters such as laser intensity, pulse duration and relaxation time. The innovations of the solutions are summarized as:

- It doesn't need any numerical solution unlike the inverse procedure in Laplace method;
- This method enables us to apply any time-dependent function of boundary flux and can be extended for non-uniform flux;
- Using this analytical procedure it is possible to introduce closed form answers for modeling both steady and periodic heat fluxes on boundaries in Fourier and non-Fourier problems;
- The procedure can be extended to cover imposing heat flux on all sides of cube, using superposition rule.
- It gives us a good foundation in order to verify the other numerical solution in three-dimensional models or estimation of temperature profiles in some physical phenomena such as laser pulse interaction of biological tissues *etc*.

2. MATHEMATICAL MODEL

Let's take into account an isotropic finite cubic tissue as shown in Fig. 1. The governing hyperbolic heat equation for this bar with homogeneous properties is:

$$\tau_q \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} = \alpha \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right] \tag{3}$$

The upper surface of the tissue is subjected to a time-dependent heat flux $q(t)$ while the interiors of the bar are at constant temperature. So the boundary conditions of the problem will be:

$$\begin{cases} \theta(0, y, z, t) = \theta(a, y, z, t) = 0 \\ \theta(x, 0, z, t) = \theta(x, b, z, t) = 0 \\ \theta(x, y, 0, t) = 0\,, \quad \lambda \left. \dfrac{\partial \theta}{\partial z} \right|_{x,y,c,t} = q_0 \left[F(t) + \tau_q \dfrac{\partial F(t)}{\partial t} \right] \end{cases} \tag{4}$$

where $F(t)$ is an arbitrary function of time and q_0 is a constant. Furthermore the initial conditions of the problem are considered to be:

$$\begin{cases} \theta(x, y, z, 0) = 0 \\ \left. \dfrac{\partial \theta}{\partial t} \right|_{x,y,z,0} = 0 \end{cases} \tag{5}$$

Solving the problem in this state directly with separation of variables is impossible because of the time-dependent non-homogeneous boundary conditions, unless the problem is divided into two problems [17]:

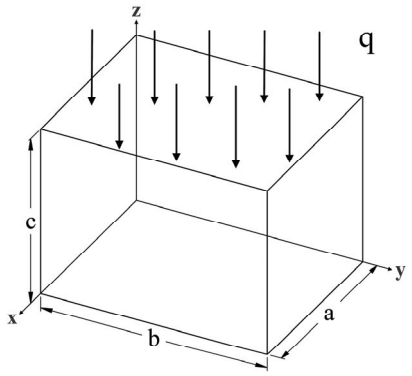


Fig. 1 The cubic biological tissue under uniform flux distribution.

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- First the problem is considered without the restraint of time-dependency in boundary conditions
  (1) The steady-state equation $\theta_0(x, y, z)$ is solved with non-homogeneous boundary conditions;
  (2) The transient equation $\theta_1(x, y, z, t)$ is solved with homogeneous boundary conditions and changed initial conditions;
  (3) The solution of the problem with time- independent boundary flux using the superposition of steady and transient solution will become:

$$\theta(x,y,z,t) = \theta_0(x,y,z) + \theta_1(x,y,z,t) \tag{6}$$

- At the end, the complete solution of the problem with time-dependent boundary conditions will be achieved applying the Duhamel's super-position integral on the later solution.

These four steps for the problem are explained as follows:

2.1 Steady-State Problem

The first step as said before is the steady-state problem with time-independent non-homogeneous boundary conditions. The problem is:

$$\frac{\partial^2\theta_0}{\partial x^2} + \frac{\partial^2\theta_0}{\partial y^2} + \frac{\partial^2\theta_0}{\partial z^2} = 0 \tag{7}$$

$$\begin{cases} \theta_0(0, y, z) = \theta_0(a, y, z) = 0 \\ \theta_0(x, 0, z) = \theta_0(x, b, z) = 0 \\ \theta_0(x, y, 0) = 0\,, \ \lambda \dfrac{\partial\theta_0}{\partial z}_{x,y,c} = q_0 \end{cases} \tag{8}$$

with the separation of variables method, exerting. $\theta_0(x, y, z) = X_0(x).\ Y_0(y).\ Z_0(z)$ into Eq. (7) produces the equations as:

$$\begin{cases} \dfrac{d^2X_0}{dx^2} + k_x^2 X_0(x) = 0 \\ \dfrac{d^2Y_0}{dy^2} + k_y^2 Y_0(y) = 0 \\ \dfrac{d^2Z_0}{dz^2} - k_z^2\, Z_0(z) = 0 \end{cases} \tag{9}$$

where $k_x = m\pi / a\,,\quad k_y = n\pi / b\,,\quad k_z = k_{mn} = \sqrt{(m\pi/a)^2 + (n\pi/b)^2}$. The resulted steady-state solution after imposing the boundary conditions (8(a), 8(b)) is:

$$\theta_0(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh\left(\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, z\right) \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{10}$$

where A_{mn} is constant and can be determined using the heat flux boundary condition Eq. 8(c) In order to find A_{mn} it's necessary to expand the function q_0 in terms of $\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$ Using the orthogonality relation and doing the integration, A_{mn} is determined zero when m or n are even numbers and for odd number of m and n:

$$A_{mn} = \frac{16 q_0}{mn\pi^2 k_{mn} \cosh(k_{mn} c)} \tag{11}$$

At last, the steady-state solution of the problem can be written as:

$$\theta_0(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \eta_{mn}(z) \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{12}$$

where $\eta_{mn}(z) = A_{mn} \sinh(k_{mn} z)$.

2.2 Transient Problem

Now the transient part of the problem with homogeneous boundary conditions and transformed initial conditions will be displayed as follows:

$$\tau_q \frac{\partial^2\theta_1}{\partial t^2} + \frac{\partial\theta_1}{\partial t} = \alpha\left[\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_1}{\partial y^2} + \frac{\partial^2\theta_1}{\partial z^2}\right] \tag{13}$$

$$\begin{cases} \theta_1(0, y, z, t) = \theta_1(a, y, z, t) = 0 \\ \theta_1(x, 0, z, t) = \theta_1(x, b, z, t) = 0 \\ \theta_1(x, y, 0, t) = 0\,, \quad \lambda \dfrac{\partial\theta_1}{\partial z}\Big|_{x,y,c,t} = 0 \end{cases} \tag{14}$$

$$\begin{cases} \theta_1(x, y, z, 0) = -\theta_0(x, y, z) \\ \dfrac{\partial\theta_1}{\partial t}\Big|_{x,y,z,0} = 0 \end{cases} \tag{15}$$

Like the steady-state step of the problem the separation of variables method will be used also for the transient problem as, $\theta_1(x\ y\ z\ t) = X_1(x)\ Y_1(y)\ Z_1(z)\ T_1(t)$ Exerting this function to the Eq. (14) leads to:

$$\begin{cases} \dfrac{d^2X_1}{dx^2} + k_x^2 X_1(x) = 0 \\ \dfrac{d^2Y_1}{dy^2} + k_y^2 Y_1(y) = 0 \\ \dfrac{d^2Z_1}{dz^2} + k_z^2 Z_1(z) = 0 \\ \tau_q \dfrac{\partial^2 T}{\partial t^2} + \dfrac{\partial T}{\partial t} = -(k_x^2 + k_y^2 + k_z)\,\alpha T = -\omega^2 \alpha T \end{cases} \tag{16}$$

And the solutions are [17]:

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$$\begin{cases} T(t) = e^{\frac{-t}{2\tau_q}}\left(E_1\cos(\Gamma t) + E_2\sin(\Gamma t)\right) \\ X(x) = A_1\cos(k_x x) + B_1\sin(k_x x) \\ Y(x) = A_2\cos(k_y y) + B_2\sin(k_y y) \\ Z(z) = A_3\cos(k_z z) + B_3\sin(k_z z) \end{cases} \quad (17)$$

where $\Gamma = \sqrt{4\omega^2\tau_q\,\alpha - 1}\,/\,2\tau_q$ and $k_x = m\pi/a$, $k_y = n\pi/b$, $k_z = (k\pi/c + \pi/2c)$, $A1$, $B1$, $A2$, $B2$, $A3$, $B3$, $E1$ and $E2$ are constants and can be determined with applying the homogenous boundary conditions in Eq. (14). The resulted product solution is:

$$\theta(x, y, z, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left[e^{\frac{-t}{2\tau_q}}\left(E_{mnk}\cos(\Gamma_{mnk}t) + F_{mnk}\sin(\Gamma_{mnk}t)\right)\right] \cdot\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right] \quad (18)$$

where $\Gamma_{mnk} = \dfrac{\sqrt{4\omega_{mnk}^2\tau_q\alpha - 1}}{2\tau_q}$ and $\omega_{mnk}^2 = \left[\left(\dfrac{m\pi}{a}x\right)^2 + \left(\dfrac{n\pi}{b}y\right)^2 + \left(\dfrac{k\pi}{c}+\dfrac{\pi}{2c}\right)^2\right]$. The coefficient E_{mnk} will be found utilizing the initial conditions (15a) as:

$$\sum_{k=0}^{\infty} E_{mnk}\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right] = -\eta_{mn}(z) \quad (19)$$

From the concept of orthogonality E_{mnk} is found as:

$$E_{mnk} = -\frac{2}{c}\int_0^c \eta_{mn}(z)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right]dz \quad (20)$$

Also F_{mnk} can be determined using the second initial condition (15.b) as:

$$F_{mnk} = \frac{1}{2\tau_q\Gamma_{mnk}}E_{mnk} \quad (21)$$

2.3 Superposition

In order to find the solution for the problem with time-independent boundary conditions the superposition concept will be used [18]. Summing the steady-state and the transient solution the result is:

$$\theta(x, y, z, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\eta_{mn}(z) + \sum_{k=0}^{\infty}E_{mnk}G_{mnk}(t)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right]\right\} \cdot\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \quad (22)$$

where

$$G_{mnk}(t) = e^{\frac{-t}{2\tau_q}}\left(\cos(\Gamma_{mnk}t) + \frac{1}{2\tau_q\,\Gamma_{mnk}}\sin(\Gamma_{mnk}t)\right) \quad (23)$$

2.4 Complete Solution

Now for applying the time-dependency of boundary conditions as mentioned before, the Duhamel's theorem will be used. If $f(t)$ is considered as a continuous time function of boundary condition, the complete solution of the problem with time-dependent boundary conditions, will be derived using the solution of the problem with time-independent boundary conditions according to the Duhamel's theory as [18]:

$$\bar{\theta}(x, y, z, t) = f(0)\,.\,\theta(x, y, z, t) + \int_0^t \theta(x, y, z, t-\tau)\frac{\partial f(\tau)}{\partial\tau}d\tau \quad (24)$$

Also $f(t)$ considered as time-dependent part of the boundary heat flux function, whichis:

$$f(t) = \left[F(t) + \tau_q\frac{\partial F(t)}{\partial t}\right] \quad (25)$$

In order to use $f(t)$ in the simplest way in the Duhamel's relation (24), $f(t)$ is shown with the Fourier expansion of 1 as $f(t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{mn}(t)\sin\left(\dfrac{m\pi}{a}x\right)\sin\left(\dfrac{n\pi}{b}y\right)$ with the coefficient of:

$$C_{mn}(t) = \frac{4}{ab}\int_0^a\int_0^b f(t)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dxdy \quad (26)$$

Thus, the Duhamel relation will give the complete solution as:

$$\bar{\theta}(x, y, z, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\eta_{mn}(z) + \sum_{k=0}^{\infty}E_{mnk}G_{mnk}(t)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right]\right\}C_{mn}(0)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t\left\{\eta_{mn}(z) + \sum_{k=0}^{\infty}E_{mnk}G_{mnk}(t-\tau)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right]\right\}\frac{\partial C_{mn}(\tau)}{\partial\tau}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)d\tau \quad (27)$$

In order to reduce this equation the below relation will be used:

$$C_{mn}(0) + \int_0^t\frac{\partial C_{mn}(\tau)}{\partial\tau}d\tau = C_{mn}(t) \quad (28)$$

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which will give

$$\bar{\theta}(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\eta_{mn}(z)C_{mn}(t)+\sum_{k=0}^{\infty}E_{mnk}\left(G_{mnk}(t)C_{mn}(0)+\int_0^t G_{mnk}(t-\tau)\frac{\partial C_{mn}(\tau)}{\partial\tau}d\tau\right)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right]\right\}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{29}$$

By solving the integration term in Eq. (29) with the method of integration by part and considering $G_{mnk}(0) = 1$, Eq. (29) will reduce to:

$$\bar{\theta}(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\eta_{mn}(z)C_{mn}(t)+\sum_{k=0}^{\infty}E_{mnk}\left(C_{mn}(t)+\int_0^t\frac{\partial G_{mnk}(t-\tau)}{\partial\tau}C_{mn}(\tau)\,d\tau\right)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right]\right\}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{30}$$

Replacing $\eta_{mn}(z)$ and $G_{mnk}(t)$ respectively from Eqs. (19) and (23) Into Eq. (30) will give the complete or closed-form solution of the problem with time dependent boundary flux, as:

$$\begin{aligned}\bar{\theta}(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}E_{mnk}&\left\{\int_0^t\left(e^{\frac{-(t-\tau)}{2\tau_q}}\left(\Gamma_{mnk}+\frac{1}{(2\tau_q)^2\Gamma_{mnk}}\right)\sin\left(\Gamma_{mnk}(t-\tau)\right)C_{mn}(\tau)\right)d\tau\right\}\\&\times\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right]\end{aligned} \tag{31}$$

Also by the same procedure, the Fourier solution for this problem will be:

$$\bar{\theta}(x,y,z,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}E_{mnk}\left\{\int_0^t\alpha\omega_{mnk}^2e^{-\alpha\omega_{mnk}^2(t-\tau)}C_{mn}(\tau)\,d\tau\right\}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left[\left(\frac{k\pi}{c}+\frac{\pi}{2c}\right)z\right] \tag{32}$$

3. DEMONSTRATION AND RESULTS

3.1. Step Pulse

In order to demonstrate the results of this study it is considered a step pulsed function with four different widths $\tau_p = 0.05, 0.1, 0.15, 0.2$ shown in Fig. 2, which causes a thermal disturbance on the upper surface of the cube, while the other surfaces remain at the same initial temperature. The step function considered here, is $F(t) = H(t) - H(t - \tau_p)$. So the boundary condition in z direction is converted to:

$$\begin{cases}\theta(x,y,0,t)=0\\ k\dfrac{\partial\theta}{\partial z}\Big|_{x,y,c,t}=q_0\left[H(t)-H(t-\tau_p)+\tau_q\left(\delta(t)-\delta(t-\tau_p)\right)\right]\end{cases} \tag{33}$$

The resulted temperature profiles for $q_0 = 1$, $a = 0.1$, $b = 0.1$, $c = 0.1$ are shown in Figs. 3 ~ 10. The dimensionless parameters Fo, Ve, $\ominus$, X, Y, Z are defined as:

$$\mathrm{Fo}=\frac{\alpha t}{c^2},\ \mathrm{Ve}=\sqrt{\frac{\alpha\tau_q}{c^2}},\ \ominus=\frac{\theta}{q_0c/\lambda},\ \mathrm{X}=\frac{x}{a},\ \mathrm{Y}=\frac{y}{b},\ \mathrm{Z}=\frac{z}{c} \tag{34}$$

Temperature variations of center point of cube (Z = 0.5, Y = 0.5, X = 0.5) under heat pulse flux with time-step duration of $\tau_p = 0.1$ have been plotted for different Ve due to Fo for pulsed heating in Fig. 3, and profiles of constant heating have been shown in Fig. 4. One-dimensional temperature profiles in middle Z plain (X = 0.5 and Y = 0.5) for Ve = 0.5 and different Fo have been

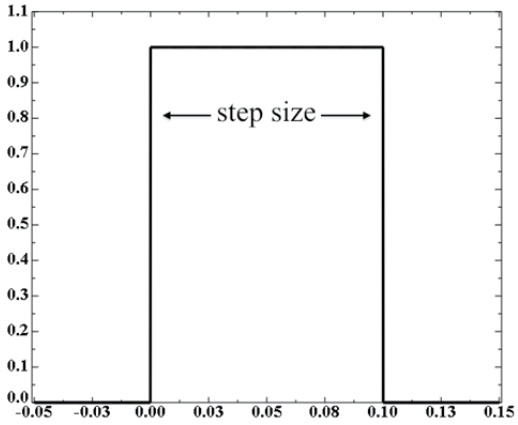


Fig 2 Step function configuration with step size = 0.1.

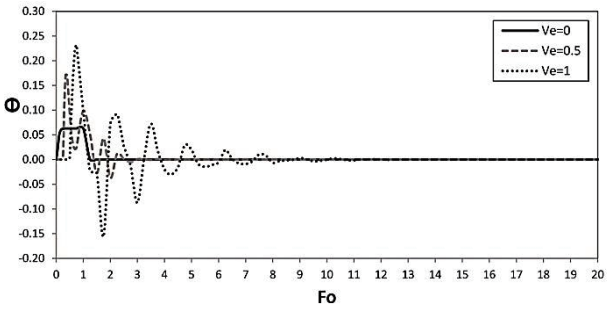


Fig. 3 Temperature variation of middle point of cube due to Fo with $\tau_p = 0.1$ for various Ve (time dependent heating).

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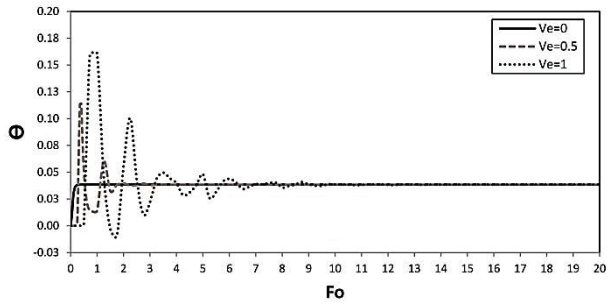


Fig. 4 Temperature variation of middle point of cube due to Fo for various Ve (constant heating).

shown for periodic heating in Fig. 5 and constant heating in Fig. 6. Three-Dimensional temperature profiles in X-Y and X-Z plains are shown in Figs. 7 and 8. The effect of duration of pulse heat flux step τ_p on the temperature profiles is studied in Figs. 9 and 10. By studying the figures these results can be deduced:

- The oscillatory behavior of non-Fourier profile before reaching stability compared with Fourier one can be seen in Figs. 3 and 4, which predict higher temperature change
- Increasing the Vernotte number leads to the increase of the oscillations amplitude and time lag of temperature initiation, in Figs. 3 and 4, which shows the increase of inertial effect of temperature profiles due to the increase of relaxation time;
- The difference of temperature profiles under pulse and constant heat fluxes are distinguished spatially on Fourier ones, where the final steady temperature is zero under a pulsed heat flux despite of the constant heating;
- The main behavior of non-Fourier profiles which is the finite speed of penetration of heat flux to the solid body can be seen from Figs. 5, 6. This may lead to the phase difference of temperature change from the first motivation.
- With the derived solution, it is possible to show the three-dimensional temperature profiles in all plain, such as Figs. 7, 8.
- Variation of temperature profiles due to Fo with different duration of pulsed heat flux are showed in Figs. 9 and 10. It can be seen that by increasing the time-step size the amplitude changes are negligible but the steady reaching time differs, due to the induced phase difference in non-Fourier profiles.

3.2 Laser Pulse Modeling

For the second demonstration, the ability of this solution to analyze the distributed boundary flux is shown too. Thus the interaction of cubic tissue under a pulse laser is modeled with the derived procedure, as shown in Fig. 11. The tumor thermal interaction with laser is a new way for cancer therapy. The laser pulse incident to tumor tissue was modeled as a heat absorption or as a surface flux in literature [3,17,19]. Here it is considered that a laser probe is set on the tissue surface and heating is considered as a surface flux. The laser pulse function has a Gaussian distribution in a circular point on the upper surface of the cubic tissue, and has exponential pulse in time zone, as in Eq. (35) [3,17]:

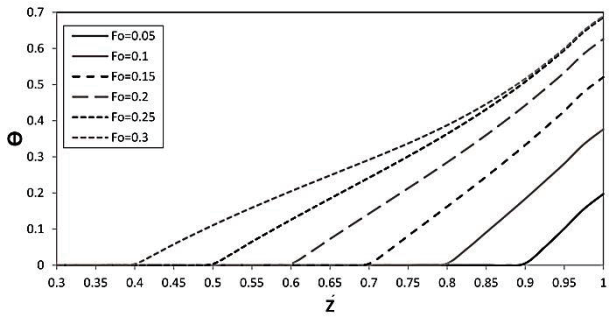


Fig. 5 Temperature variation in Z direction with Ve = 0.5 and τ_p = 0.1 in various Fo (time dependent heating).

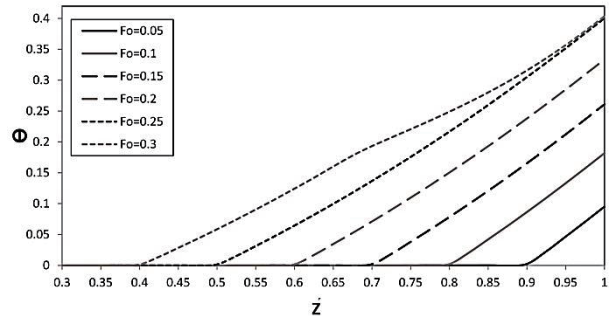


Fig. 6 Temperature variation in Z direction and Ve = 0.5 in various Fo (constant heating).

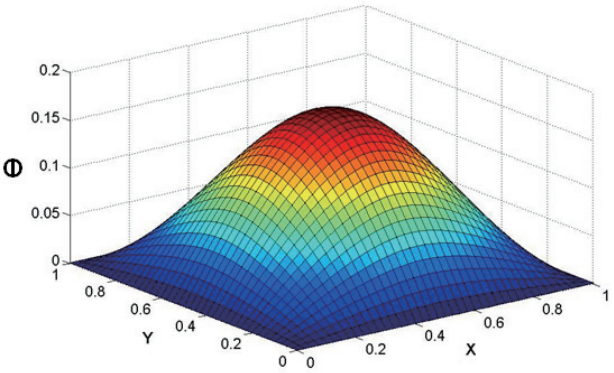


Fig. 7 Three dimensional temperature profile at Z = 0.5 for Ve = 0.5 and τ_p = 0.1.

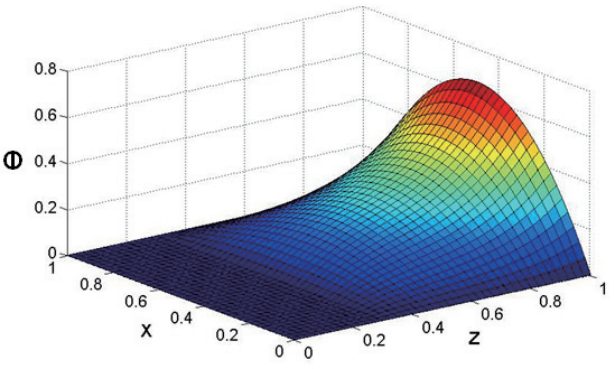


Fig. 8 Three dimensional temperature profile at Y = 0.5 for Ve = 0.5 and τ_p = 0.1.

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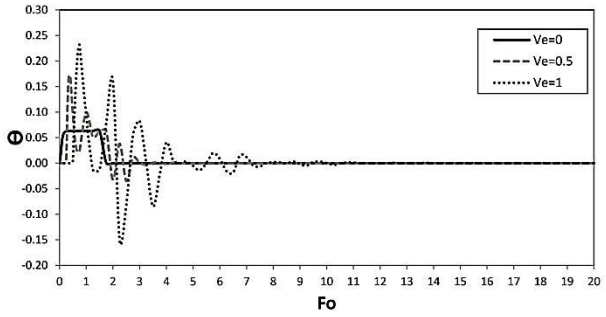


Fig. 9 Temperature variation of middle point of cube due to Fo with for various Ve (time dependent heating).

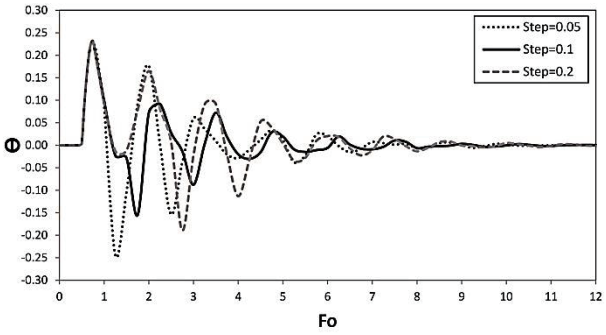


Fig. 10 Temperature variation of middle point of cube due to Fo with Ve = 1 for varioustime step of pulse flux (τ_p).

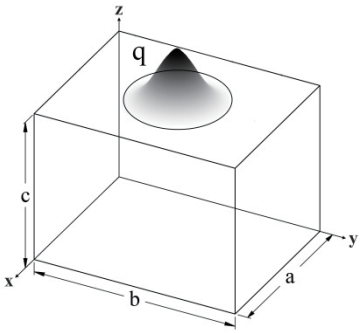


Fig. 11 Cubictissue under the laser flux with Gaussian distribution.

$$q(x,y,t)=q_0.e^{-\frac{(x^2+y^2)}{2\sigma^2}}.\left[\left(\frac{t}{t_p}\right)^2 e^{-\frac{t}{t_p}}\right] \tag{35}$$

where σ is concentration factor of laser pulse which show the intensity of laser pulse, and t_p shows the time pulse duration. The smaller value of σ shows the more intense laser flux and larger value of t_p means the larger duration of laser pulse [16,17] as shown in Figs. 12, 13. The laser fluxes distribution in Fig. 12 are considered such that the total energy of flux become equal to the constant flux distribution. The solution of the problem in non-Fourier and Fourier modes are as in Eqs. (31), (32) in which the coefficients of A_{mn} in Eq. (11) must be determined with the relation:

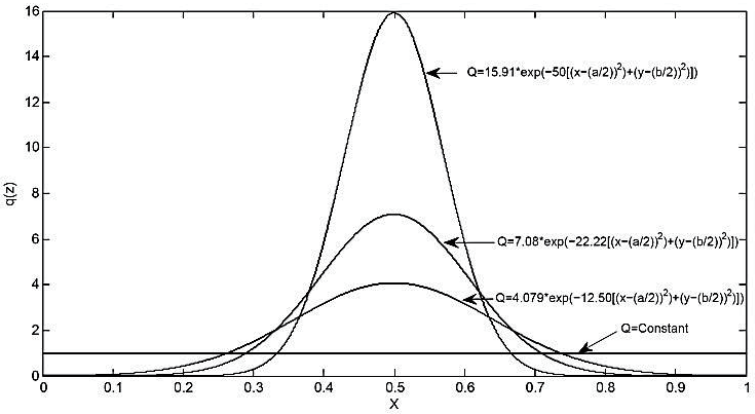


Fig. 12 Laser pulse distribution profiles with the same total energy.

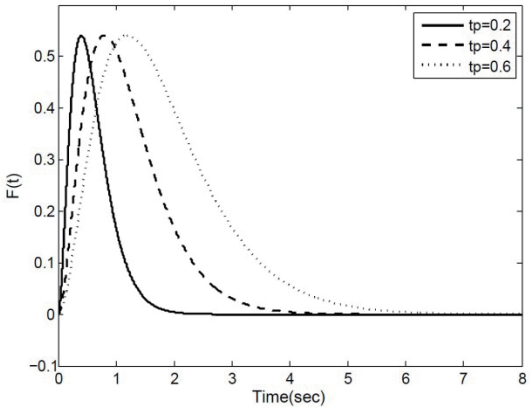


Fig. 13 Laser time-pulse function.

$$A_{mn}=\frac{4q_0}{abk_{mn}\cosh(k_{mn}c)} \int_0^a\int_0^b e^{-\frac{\left(\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2\right)}{2\sigma^2}} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dydx \tag{36}$$

where, the integration in space can be simplified and determined numerically. The result of temperature profiles are shown in Figs. 14 ~ 16. Figure 14 shows that increasing the intensity of the laser pulse leads to the increasing of the maximum impact temperature compared to the constant flux. The oscillation amplitude of the temperature profiles are increased with increasing the intensity of laser pulses as shown in Fig. 15. The effect of time-pulse t_p of exponential pulse on temperature profiles is shown in Fig. (16). It can be seen that the temperature profiles follow the behavior of the motivation function time pulse so that the maximum temperature is shifted with increasing the exponential time pulse duration. Also, increasing t_p leads to decrease of temperature amplitudes because of the temperature inertia of non-Fourier profiles, which resist against the temperature variations.

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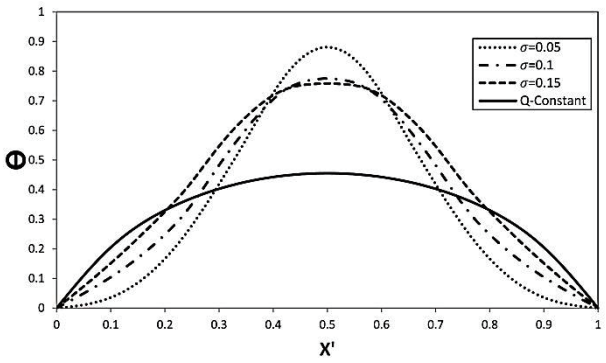


Fig. 14 Temperature profile in longitudinal direction in $Y = 0.8$ and $Z = 0.8$ for Fo = 0.5 and Ve = 2 and various Gaussian distribution factor compared with uniform distribution.

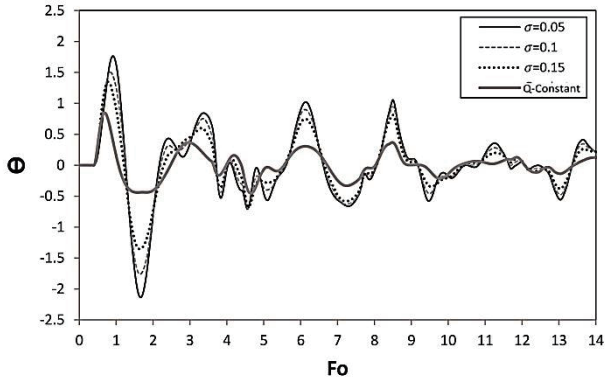


Fig. 15 Temperature change with Fo in $Y = 0.8$ and $Z = 0.8$ and Ve = 2 for various Gaussian distribution factor.

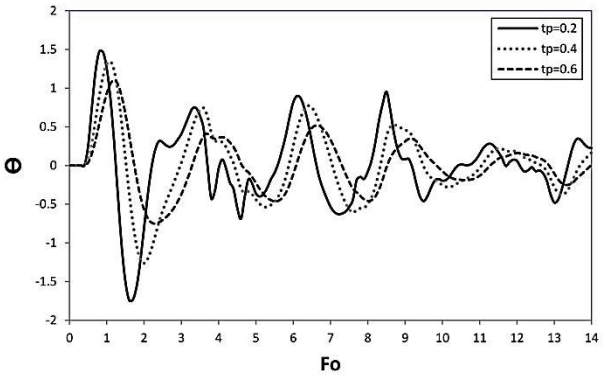


Fig. 16 Temperature change with Fo in $Y = 0.8$ and $Z = 0.8$ and Ve = 2 for various time pulse duration.

4. CONCLUSIONS

A perfect analytical solution of the three-dimensional hyperbolic heat conduction equation in a cubic tissue is derived under pulsed and constant surface heat flux. Also the procedure is applied for classical Fourier heat conduction too. The implemented method is separation of variables and Duhamel integral. A straight forward solution is derived which unlike the inverse procedure in Laplace method doesn't need any numerical calculation. The results of this paper can be applied for studying the parameters of intensity and duration of a laser pulse interaction of a tissues or it can be utilized as a verification branch for other numerical solutions.

ACKNOWLEDGEMENTS

Dedicated to the pure spirit of Dr. Gholamali Atefi, the lights of road who stayed off.

NOMENCLATURES

a	Length of the cube
b	Width of the cube
c	Height of the cube
Fo	Fourier number
$F(t)$	Time function of heat flux
H	Heaviside function
λ	Thermal conductivity
t	time
Ve	Vernotte number
x, y, z	Spatial coordinate
X, Y, Z	Non-dimensional coordinates of x, y, z respectively
$\boldsymbol{q}$	Heat flux vector
k_x, k_y, k_z	Eigenvalues
m, n, k	Summation counter

Greek Symbols

α	Thermal diffusivity
ρ	Mass density
Δ	Laplace's differentialOperator
∇	Gradient operator
τ	Integrand in Duhamel relation
τ_q	Thermal relaxation time
t_p	Time pulse of laser
θ_0	Steady Temperature
θ_1	Transient Temperature
θ	Total Temperature
$\ominus$	Dimensionles temperature
σ	Laser concentration
δ	Dirac Function

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(Manuscript received February 25, 2015,
accepted for publication September 3, 2015.)

https://doi.org/10.1017/jmech.2015.97 Published online by Cambridge University Press