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Combinatorial Structures on van der Waerden sets

KONSTANTINOS TYROS

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
(e-mail: k.tyros@warwick.ac.uk)

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In this paper we provide two results. The first one consists of an infinitary version of the Furstenberg–Weiss theorem. More precisely we show that every subset A of a homogeneous tree T such that

$$\frac{|A \cap T(n)|}{|T(n)|} \geqslant \delta,$$

where $T(n)$ denotes the nth level of T, for all n in a van der Waerden set, for some positive real δ, contains a strong subtree having a level set which forms a van der Waerden set.

The second result is the following. For every sequence $(m_q)_{q\in\mathbb{N}}$ of positive integers and for every real $0 < \delta \leqslant 1$, there exists a sequence $(n_q)_{q\in\mathbb{N}}$ of positive integers such that for every $D \subseteq \bigcup_k \prod_{q=0}^{k-1}[n_q]$ satisfying

$$\frac{|D \cap \prod_{q=0}^{k-1}[n_q]|s}{\prod_{q=0}^{k-1} n_q} \geqslant \delta$$

for every k in a van der Waerden set, there is a sequence $(J_q)_{q\in\mathbb{N}}$, where J_q is an arithmetic progression of length m_q contained in $[n_q]$ for all q, such that $\prod_{q=0}^{k-1} J_q \subseteq D$ for every k in a van der Waerden set. Moreover, working in an abstract setting, we may require J_q to be any configuration of natural numbers that can be found in an arbitrary set of positive density.

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1. Introduction

In 1975 Szemerédi [16], settling in the affirmative an old conjecture of Erdős and Turán, established a density version of van der Waerden's theorem [19]. More precisely, Szemerédi's theorem states the following.

Theorem 1.1. *For every integer $k \geqslant 2$ and every $0 < \delta \leqslant 1$, there exists a positive integer n_0 having the following property. For every $n \geqslant n_0$ and every subset A of $\{0, \dots, n-1\}$ with at least δn elements, there exists an arithmetic progression of length k contained in A. We denote the least such n_0 by $\mathrm{Sz}(k, \delta)$.*

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It is an easy observation that subsets of the natural numbers of positive upper Banach density do not necessarily contain infinite arithmetic progressions. However, using Theorem 1.1 as a pigeonhole principle, one easily derives that such sets are *van der Waerden*, *i.e.*, they contain arbitrarily long arithmetic progressions.

In the present work we provide two examples of infinitary combinatorial structures that can be realized on a van der Waerden set. The first one consists of an infinitary version of a result of Furstenberg and Weiss [8] concerning the existence of 'strong' subtrees having an arithmetic progression as a level set in 'big' subsets of a homogeneous tree. Their result is actually an extension of the finitary version of Szemerédi's theorem (Theorem 1.1). Using the Furstenberg–Weiss theorem as a pigeonhole principle, we construct infinite strong subtrees having a van der Waerden set as a level set inside 'big' subsets of an infinite homogeneous tree.

The second result included in this work consists of an extension of the main theorem of [18], which established a density version of a Ramsey theoretic result proved in [3, 17]. In particular, it was shown that, for every positive real ε and every sequence $(m_q)_{q\in\mathbb{N}}$ of positive integers, there exists a sequence $(n_q)_{q\in\mathbb{N}}$ of positive integers with the following property. For every infinite subset L of the positive integers and every sequence $(D_\ell)_{\ell\in L}$ such that D_ℓ is a subset of $\prod_{q=0}^{\ell-1}\{1,\dots,n_q\}$ of density at least ε for all $\ell\in L$, there exist a sequence $(I_q)_{q\in\mathbb{N}}$ and infinite subset L' of L such that I_q is a subset of $\{1,\dots,n_q\}$ of cardinality m_q for all non-negative integers q and the set $\prod_{q=0}^{\ell-1} I_q$ is a subset of D_ℓ for all $\ell\in L'$. In the present paper we provide a strengthening of this result which is optimal in several aspects. Firstly, we show that the set L' can be chosen to be a van der Waerden set provided that L itself is a van der Waerden set (this is, clearly, a necessary condition). Moreover, the sets I_q can be endowed with additional structure. For instance, each I_q can be chosen to be an arithmetic or a polynomial progression. The construction of the sequence $(I_q)_{q\in\mathbb{N}}$ is effective, avoiding, in particular, compactness arguments as in [18].

To explicitly state our results we need some pieces of notation. By $\mathbb{N}$ we denote the set of the natural numbers $\{0,1,2,\dots\}$ starting from 0. By the term *tree* we mean a partial ordered set $(T,\leqslant)$ satisfying the following.

(i) T is *uniquely rooted*, *i.e.*, T admits a minimum with respect to $\leqslant$.

(ii) For every $t\in T$, the set of predecessors of t in T is finite and linearly ordered.

(iii) For every non-maximal $t\in T$, the set of immediate successors of t in T is finite.

(iv) T is *balanced*, *i.e.*, all maximal chains in T are of the same cardinality.

For a tree T, we define the *height* $h(T)$ of T to be the cardinality of a maximal chain in T, while for every node t in T we set $\ell_T(t)$ to be the cardinality of the set $\{s\in T : s<t\}$. Let us point out that T is infinite if and only if $h(T)=\omega$, where as usual ω stands for the first infinite cardinal number. Finally, for a tree T and an integer n such that $0\leqslant n<h(T)$, we let $T(n)$ denote the nth *level* of T, that is, the set $\{t\in T : \ell_T(t)=n\}$. The substructures that we are interested in are the *strong subtrees*, a notion that was isolated in the 1960s and highlighted in the work of Milliken [13, 14]. A subset S of a tree T is called a strong subtree of T if the following conditions are satisfied.

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(a) S endowed with the induced ordering from T is a tree.

(b) Every level of S is a subset of some level of T.

(c) Every non-maximal s in S has the following property. For every immediate successor t of s in T there exists a unique immediate successor s' of s in S such that $t \leqslant s'$.

The level set of a strong subtree S of a tree T, denoted by $L_T(S)$, is the set $\{n \in \mathbb{N} : n < h(T) \text{ and } T(n) \cap S \neq \emptyset\}$. If b is a positive integer, then a b-homogeneous tree T is a tree such that every non-maximal node t in T has exactly b immediate successors. A tree is called homogeneous if it is b-homogeneous for some positive integer b. The result of Furstenberg and Weiss that we mentioned above is the following.

Theorem 1.2. *Let b,k be positive integers and let $0 < \delta \leqslant 1$. Then there exists a positive integer n_0 satisfying the following property. For every finite b-homogeneous tree T of height at least n_0 and every subset A of T satisfying*

$$\mathbb{E}_{0 \leqslant n < h(T)} \frac{|T(n) \cap A|}{|T(n)|} \geqslant \delta, \tag{1.1}$$

there exists a strong subtree S of T contained in A and having a level set which forms an arithmetic progression of length k. We denote the least such n_0 by $\mathrm{FW}(b,k,\delta)$.

Pach, Solymosi and Tardos [15] gave a combinatorial proof of Theorem 1.2 yielding explicit bounds for the numbers $\mathrm{FW}(b,k,\delta)$. In the present work we establish the following infinitary version of the Furstenberg–Weiss theorem.

Theorem 1.3. *Let T be an infinite homogeneous tree and let A be a subset of T such that*

$$\limsup_{N \to \infty} \mathbb{E}_{0 \leqslant n < N} \frac{|T(n) \cap A|}{|T(n)|} > 0. \tag{1.2}$$

Then there exists a strong subtree S of T contained in A and having a level set which forms a van der Waerden set.

Actually, the above result is an easy consequence of the following theorem, which we will prove in this paper.

Theorem 1.4. *Let T be an infinite homogeneous tree and let A be a subset of T such that there exist a van der Waerden set L and a real $0 < \delta \leqslant 1$ satisfying*

$$\frac{|T(n) \cap A|}{|T(n)|} \geqslant \delta \tag{1.3}$$

for all $n \in L$. Then there exists a strong subtree S of T contained in A and having a level set which forms a van der Waerden set.

In Section 2 we obtain a uniform version of the Furstenberg–Weiss theorem needed for the proof of Theorem 1.4, which is given in Section 3.

To state our second result we need some additional notation. We denote the set of positive integers by $\mathbb{N}_+$. For every set X by $X^{<\omega}$ we denote the set of all finite sequences

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in X. The empty sequence is denoted by $\varnothing$ and is included in $X^{<\omega}$. For every positive integer k, we denote by $[k]$ the set $\{1,\dots,k\}$. By convention, $[0]$ stands for the empty set.

Let Y be a (possibly infinite) set. If X is a non-empty finite subset of Y and A is an arbitrary subset of Y, then the *density of A relative to X*, denoted by $\mathrm{dens}_X(A)$, is the quantity defined by

$$\mathrm{dens}_X(A) = \frac{|A \cap X|}{|X|}. \tag{1.4}$$

We will also need the following slight variant of the standard notion of a density regular family (see, *e.g.*, [11, 12]).

Definition 1.5. A family $\mathcal{F}$ of non-empty finite subsets of $\mathbb{N}_+$ is called *uniformly density regular* if for every $0 < \varepsilon \leqslant 1$ there exists an integer n_0 such that, for every interval I of $\mathbb{N}_+$ of length at least n_0 and every subset A of I with $\mathrm{dens}_I(A) \geqslant \varepsilon$, the set A contains an element of $\mathcal{F}$. The least such n_0 will be denoted by $B(\mathcal{F},\varepsilon)$. Finally, the set of all uniformly density regular families will be denoted by $\mathcal{R}$.

There are several examples of uniformly density regular families. The simplest one is the set of all subsets of the positive integers with exactly k elements, where k is a fixed positive integer. By the famous theorem of Szemerédi [16], the set AP_k of all arithmetic progressions of length k is also a uniformly density regular family; see [9] for the best known upper bounds for the numbers $B(\mathrm{AP}_k,\varepsilon)$. Moreover, for every choice of polynomials $p_1,\dots,p_k$ taking integer values on the integers and zero on zero, the family $\{\{a+p_1(n),\dots,a+p_k(n)\} : a \in \mathbb{Z} \text{ and } n \in \mathbb{Z}\}$ is uniformly density regular. This is a consequence of the work of Bergelson and Leibman [1]. More examples of uniformly density regular families can be found in [7, 2].

We are now ready to state our result.

Theorem 1.6. *Let $0 < \delta \leqslant 1$. Then there exists a map $V_\delta : \mathcal{R}^{<\omega} \times \mathbb{N}_+^{<\omega} \to \mathbb{N}$ with the following property. For every sequence $(n_q)_{q\in\mathbb{N}}$ of positive integers, every sequence $(\mathcal{F}_q)_{q\in\mathbb{N}}$ of uniformly regular families, every van der Waerden subset L of $\mathbb{N}_+$ and every sequence $(D_k)_{k\in L}$ such that*

(a) $n_0 \geqslant V_\delta\big(\mathcal{F}_0,\varnothing\big)$,

(b) $n_q \geqslant V_\delta\big((\mathcal{F}_p)_{p=0}^{q},(n_p)_{p=0}^{q-1}\big)$ *for all positive integers q, and*

(c) D_ℓ *is a subset of* $\prod_{q=0}^{\ell-1}[n_q]$ *of density at least δ for all $\ell \in L$,*

there exist a sequence $(I_q)_{q\in\mathbb{N}}$ and a van der Waerden subset L' of L such that

(i) I_q *is an element of $\mathcal{F}_q$ contained in $[n_q]$ for all $q \in \mathbb{N}$, and*

(ii) $\prod_{q=0}^{\ell-1} I_q \subseteq D_\ell$ *for all $\ell \in L'$.*

The definition of the map V_δ is based on an auxiliary map T that we will define in Section 5. The proof of Theorem 1.6 as well as the definition of the map V_δ are given in Section 6. Some preliminary work is contained in Section 4.

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2. Uniform version of the Furstenberg–Weiss theorem

The aim of this section is to obtain a uniform version of the Furstenberg–Weiss theorem. The machinery that we present in this section was developed in [4]. First, let us recall the definition of the Furstenberg–Weiss measures. For every tree T, subset A of T and finite subset P of $\mathbb{N}$ with $\max P < h(T)$, we define the Furstenberg–Weiss measure

$$d_T^{\mathrm{FW}}(A|P) = \mathbb{E}_{p\in P}\,\mathrm{dens}_{T(p)}(A), \tag{2.1}$$

where $\mathrm{dens}_{T(p)}(A)$ is as defined in (1.4). If T is finite and P is equal to $\{0,\dots,h(T)-1\}$, then we simply write $d_T^{\mathrm{FW}}(A)$ instead of $d_T^{\mathrm{FW}}(A|P)$. Moreover, for a subset A of a tree T and T' a strong subtree of T, we set $d_{T'}^{\mathrm{FW}}(A) = d_{T'}^{\mathrm{FW}}(A\cap T')$.

Theorem 2.1. *Let b,k be positive integers and $0<\delta\leqslant 1$. Then there exists a positive integer n_0 with the following property. For every infinite b-homogeneous tree T, every subset A of T and every arithmetic progression P of length at least n_0 satisfying*

$$d_T^{\mathrm{FW}}(A|P) \geqslant \delta, \tag{2.2}$$

there exists a strong subtree S of T contained in A and having a level set which forms an arithmetic progression of length k. We denote the least such n_0 by $\mathrm{UFW}(b,k,\delta)$.

Theorem 2.1 will follow from Theorem 1.2 if we can find a strong subtree T' of T having P as a level set and satisfying $d_{T'}^{\mathrm{FW}}(A)\geqslant\delta$. In order to prove the existence of such a T' we need some notation. First we describe a finite b-homogeneous tree as the set of all sequences over the set $[b]$ of length less than $h(T)$. More precisely, we define the following. Let b a positive integer and let n be a non-negative integer. We set $[b]^n$ to be the set of all maps taking values in $[b]$ and having domain the set $\{0,\dots,n-1\}$ (if $n=0$, then $[b]^0=\{\varnothing\}$, where $\varnothing$ denotes the empty sequence). We also set $[b]^{<n}=\cup_{0\leqslant m<n}[b]^m$. It is easy to see that for every pair of positive integers b,n the set $[b]^{<n}$ endowed with the end-extension ordering $\sqsubseteq$ is a finite b-homogeneous tree of height n. Conversely, every finite b-homogeneous tree T is isomorphic to $[b]^{<h(T)}$. Recall that we denote by $[b]^{<\omega}$ the set of all finite sequences over $[b]$. It is immediate that $[b]^{<\omega}$ endowed with the end-extension ordering is an infinite b-homogeneous tree and that every infinite b-homogeneous tree is isomorphic to $[b]^{<\omega}$.

We also need to recall the notion of the convolution operation introduced in [4]. To this end we need some additional notation. Let $P=\{p_0<p_1<\cdots<p_{n-1}\}$ be a non-empty finite subset of $\mathbb{N}$ and let b be a positive integer. We denote by $[b]^P$ the set of all maps taking values in $[b]$ and having domain the set P. The *canonical isomorphism* associated to P is the bijection $\mathrm{I}_P:[b]^n\to[b]^P$ defined by the rule

$$\mathrm{I}_P(x)(p_i)=x(i) \tag{2.3}$$

for every $i\in\{0,\dots,n-1\}$. Moreover, for every $x_1\in[b]^{P_1}$ and $x_2\in[b]^{P_2}$, where P_1 and P_2 are disjoint finite subsets of $\mathbb{N}$, we denote by (x_1,x_2) the unique element $x\in[b]^{P_1\cup P_2}$ such that $x(i)=x_1(i)$ for all $i\in P_1$ and $x(i)=x_2(i)$ for all $i\in P_2$. Finally, for every pair of non-negative integers $m<n$ and every $x\in[b]^n$, we denote by $R_m(x)$ the initial segment of x of length m.

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Definition 2.2. Let b be a positive integer and let $P = \{p_0 < p_1 < \cdots < P_{|P|-1}\}$ be a non-empty finite subset of $\mathbb{N}$. For every $i \in \{0, \ldots, |P|-1\}$ we set

$$P_i = \{p \in P : p < p_i\} \quad \text{and} \quad \overline{P}_i = \{n \in \mathbb{N} : n < p_i \text{ and } n \notin P_i\}. \tag{2.4}$$

Also, let $n_P = \max(P) - |P| + 1$ and set

$$X_P = [b]^{n_P}. \tag{2.5}$$

We define the *convolution operation* $\mathrm{c}_P : [b]^{<|P|} \times X_P \to [b]^{<\omega}$ associated to P as follows. For every $i \in \{0, \ldots, |P|-1\}$, every $t \in [b]^i$ and every $x \in X_P$ we set

$$\mathrm{c}_P(t, x) = \big(\mathrm{I}_{P_i}(t), \mathrm{I}_{\overline{P}_i}(R_{|\overline{P}_i|}(x))\big) \in [b]^{p_i}. \tag{2.6}$$

The following fact is immediate from the definitions above.

Fact 2.3. *Let b be a positive integer and let P be a non-empty finite subset of $\mathbb{N}$. Also, let X_P be as defined in* (2.5)*. Then, for every $x \in X_P$, we have that the set*

$$\{\mathrm{c}_P(t, x) : t \in [b]^{<|P|}\} \tag{2.7}$$

forms a strong subtree of $[b]^{<\omega}$ having P as a level set.

Actually, the set defined in (2.7) is a Carlson–Simpson tree (see [4] for the relative definition), but this additional structure is not needed for the purposes of the present work. We will need the following fact.

Fact 2.4. *Let b be a positive integer and let P be a non-empty finite subset of $\mathbb{N}$. Also, let A be a subset of $[b]^{<\omega}$. Then*

$$\mathbb{E}_{p \in P} \mathrm{dens}_{[b]^p}(A) = \mathbb{E}_{i \in \{0, \ldots, |P|-1\}} \mathbb{E}_{t \in [b]^i} \mathbb{E}_{x \in X_P} \chi_A\big(\mathrm{c}_P(t, x)\big), \tag{2.8}$$

where χ_A denotes the characteristic function of A and X_P is as defined in (2.5)*.*

Proof. Let $P = \{p_0, \ldots, p_{|P|-1}\}$. We will prove that for every $i \in \{0, \ldots, |P|-1\}$

$$\mathrm{dens}_{[b]^{p_i}}(A) = \mathbb{E}_{t \in [b]^i} \mathbb{E}_{x \in X_P} \chi_A\big(\mathrm{c}_P(t, x)\big). \tag{2.9}$$

Obviously the result follows from (2.9). Let us fix some $i \in \{0, \ldots, |P|-1\}$. First observe that for every $s \in [b]^{p_i}$ there exists a unique $t_s \in [b]^i$ such that there exists $x \in X_P$ such that $\mathrm{c}_P(t_s, x) = s$. For every $s \in [b]^{p_i}$, we set $\Omega_s = \{x \in X_P : \mathrm{c}_P(t_s, x) = s\}$. Also, observe that there exists a unique $w_s \in [b]^{p_i - i}$ such that $\Omega_s = \{x \in X_P : w_s \sqsubseteq x\}$, and therefore

$$\mathbb{E}_{x \in X_P} \chi_{\{s\}}\big(\mathrm{c}_P(t_s, x)\big) = b^{i - p_i}. \tag{2.10}$$

Thus

$$\mathbb{E}_{t \in [b]^i} \mathbb{E}_{x \in X_P} \chi_{\{s\}}\big(\mathrm{c}_P(t, x)\big) = b^{-p_i}. \tag{2.11}$$

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Finally, let us observe that c_P restricted to $[b]^i \times X_P$ is onto $[b]^{p_i}$. Hence

$$\begin{aligned}\mathbb{E}_{t\in[b]^i}\mathbb{E}_{x\in X_P}\chi_A\big(\mathrm{c}_P(t,x)\big) &= \mathbb{E}_{t\in[b]^i}\mathbb{E}_{x\in X_P}\sum_{s\in A}\chi_{\{s\}}\big(\mathrm{c}_P(t,x)\big)\\ &= \sum_{s\in A\cap[b]^{p_i}}\mathbb{E}_{t\in[b]^i}\mathbb{E}_{x\in X_P}\chi_{\{s\}}\big(\mathrm{c}_P(t,x)\big)\\ &\overset{(2.11)}{=} \frac{|A\cap[b]^{p_i}|}{|[b]^{p_i}|} = \mathrm{dens}_{[b]^{p_i}}(A). \end{aligned} \tag{2.12}$$

The proof is complete. □

By Facts 2.3 and 2.4 we have the following corollary.

Corollary 2.5. *Let T be an infinite homogeneous tree, let P be a non-empty finite subset of $\mathbb{N}$ and let A be a subset of T. Then there exists a strong subtree T' of T having P as a level set and satisfying*

$$d^{\mathrm{FW}}_{T'}(A) \geqslant d^{\mathrm{FW}}_{T}(A|P). \tag{2.13}$$

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We show that $\mathrm{UFW}(b,k,\delta) \leqslant \mathrm{FW}(b,k,\delta)$. Indeed, let us fix an infinite b-homogeneous tree T, a subset A of T and an arithmetic progression P of length at least $\mathrm{FW}(b,k,\delta)$ such that

$$d^{\mathrm{FW}}_{T}(A|P) \geqslant \delta. \tag{2.14}$$

Applying Corollary 2.5, we pass to a strong subtree T' of T having P as a level set and satisfying

$$d^{\mathrm{FW}}_{T'}(A) \geqslant d^{\mathrm{FW}}_{T}(A|P) \geqslant \delta. \tag{2.15}$$

Application of Theorem 1.2 completes the proof. □

Theorem 2.1 has the following immediate consequence.

Corollary 2.6. *Let b,k be positive integers and $0 < \delta \leqslant 1$. Then, for every infinite b-homogeneous tree T, every subset A of T and every arithmetic progression P in $\mathbb{N}$ of length at least $\mathrm{UFW}(b,k,\delta)$ satisfying*

$$\frac{|T(n)\cap A|}{|T(n)|} \geqslant \delta \tag{2.16}$$

for all $n \in P$, there exists a strong subtree S of T contained in A and having a level set which forms an arithmetic progression of length k.

Remark 2.7. Actually, Theorem 1.2 and Corollary 2.6 are equivalent. In order to establish that Corollary 2.6 implies Theorem 1.2, let us fix a pair of positive integers b,k and

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$0 < \delta \leqslant 1$. Also, let T be a b-homogeneous tree of height at least $\mathrm{Sz}\big(\mathrm{UFW}(b,k,\delta/2),\delta/2\big)$ and let A be a subset of T such that (1.1) is satisfied. Setting

$$L = \{n \in \{0,\dots,h(T)-1\} : \mathrm{dens}_{T(n)}(A) \geqslant \delta/2\},$$

by (1.1) we have that $|L| \geqslant (\delta/2)h(T)$ and therefore, by Theorem 1.1, there exists an arithmetic progression P contained in L of length $\mathrm{UFW}(b,k,\delta/2)$. Application of Corollary 2.6 provides us with a strong subtree S of T contained in A and having a level set which forms an arithmetic progression of length k.

3. Proof of Theorem 1.4

Let us start by introducing some additional pieces of notation. Let T be a tree and let t be a node of T. We set $\mathrm{Succ}_T(t) = \{t' \in T : t \leqslant t'\}$. Observe that $\mathrm{Succ}_T(t)$ forms a strong subtree of T. Also, let n be a non-negative integer with $n < h(T)$ and let A be a subset of T. We define

$$\mathrm{dens}_T(A|n) = \mathrm{dens}_{T(n)}(A), \tag{3.1}$$

and if $\ell_T(t) \leqslant n$ we set

$$\mathrm{dens}_T(A|t,n) = \mathrm{dens}_{\mathrm{Succ}_T(t)}(A|n-\ell_T(t)). \tag{3.2}$$

Moreover, we will need the following invariants. Let b,k be positive integers and $0 < \rho \leqslant 1$. We inductively define a sequence $(f_q(b,k,\rho))_{q=1}^{\infty}$ as follows:

$$\begin{aligned} f_1(b,k,\rho) &= \mathrm{UFW}(b,k,\rho),\\ f_{q+1}(b,k,\rho) &= \mathrm{UFW}(b,f_q(b,k,\rho),\rho). \end{aligned} \tag{3.3}$$

Lemma 3.1. *Let b,k,q be positive integers and $0 < \rho \leqslant 1$. Also, let $T_1,\dots,T_q$ be infinite b-homogeneous trees, let P be an arithmetic progression of length $f_q(b,k,\rho)$ and let $B_1,\dots,B_q$ be subsets of $T_1,\dots,T_q$ respectively, such that $\mathrm{dens}_{T_j}(B_j|p) \geqslant \rho$ for all $j \in [q]$ and $p \in P$. Then there exist an arithmetic progression Q of length k contained in P and strong subtrees $S_1,\dots,S_q$ of $T_1,\dots,T_q$ respectively, such that S_j is contained in B_j and $L_{T_j}(S_j) = Q$ for all $j \in [q]$.*

Proof. We set $P_0 = P$ and we inductively construct arithmetic progressions $P_1,\dots,P_q$ and strong subtrees $S_1',\dots,S_q'$ of $T_1,\dots,T_q$ respectively, satisfying the following for every $j = 1,\dots,q$:

(i) P_j is contained in P_{j-1}, and

(ii) $L_{T_j}(S_j) = P_j$.

The inductive step consists of a straightforward application of Corollary 2.6. We set $Q = P_q$, and for every $j \in [q]$ we pick a strong subtree S_j of S_j' such that $L_{T_j}(S_j) = P_j$. It is immediate that $S_1,\dots,S_q$ are as desired. □

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We will also need the following fact, essentially stating that the collection of all van der Waerden sets forms a coideal. It is an easy consequence of the finitary version of van der Waerden's theorem [19].

Fact 3.2. *For every finite colouring of a van der Waerden set, one of the colours forms a van der Waerden set.*

Lemma 3.3. *Let $0 < \delta \leqslant 1$, let T be an infinite homogeneous tree, let A be a subset of T, let L be a van der Waerden set and let $t \in T$ be such that $\mathrm{dens}_T(A|t,n) \geqslant \delta$ for all $n \in L$ with $n \geqslant \ell_T(t)$. Then, for every integer m with $m \geqslant \ell_T(t)$, there exist $t' \in \mathrm{Succ}_T(t) \cap T(m)$ and a van der Waerden subset L' of L such that $\min L' \geqslant m$ and $\mathrm{dens}_T(A|t',n) \geqslant \delta$ for all $n \in L'$.*

Proof. Let m be an integer with $m \geqslant \ell_T(t)$. Passing to a final segment of L, if necessary, we may assume that $\min L \geqslant m$. Then, for every $n \in L$, we have the following. By the homogeneity of T, we have that

$$\mathbb{E}_{t' \in \mathrm{Succ}_T(t) \cap T(m)} \mathrm{dens}_T(A|t',n) = \mathrm{dens}_T(A|t,n) \geqslant \delta. \tag{3.4}$$

Thus there exists $t'_n \in \mathrm{Succ}_T(t) \cap T(m)$ such that $\mathrm{dens}_T(A|t'_n,n) \geqslant \delta$. Since $\mathrm{Succ}_T(t) \cap T(m)$ is finite, by Fact 3.2 there exist a van der Waerden subset L' of L and $t' \in \mathrm{Succ}_T(t) \cap T(m)$ such that $t'_n = t$ for all $n \in L'$. The proof is complete. □

We will need the following elementary fact.

Fact 3.4. *Let m be positive integer, δ, ε reals with $0 < \delta, \varepsilon \leqslant 1$ and $a_1, \ldots, a_m$ non-negative reals such that $\mathbb{E}_{j \in [m]} a_j \geqslant \delta$. If $a_j \leqslant \delta + \varepsilon^2$ for all $j \in [m]$, then setting $I = \{j \in [m] : a_j \geqslant \delta - \varepsilon\}$, we have that $|I|/m \geqslant 1 - \varepsilon$.*

Moreover, for our convenience, for every van der Waerden set L, we set $\mathrm{Wd}(L)$ to be the collection of all van der Waerden sets contained in L.

Lemma 3.5. *Let b, k, q be positive integers and $0 < \eta \leqslant 1$. Also, let $T_1, \ldots, T_q$ be infinite b-homogeneous trees, let L be a van der Waerden set and let $A_1, \ldots, A_q$ be subsets of $T_1, \ldots, T_q$ respectively, such that*

$$\mathrm{dens}_{T_j}(A_j|n) \geqslant \eta \tag{3.5}$$

for all $j \in [q]$ and $n \in L$. Then there exist an arithmetic progression Q of length k contained in L, a van der Waerden subset L' of L with $\min L' \geqslant \max Q$ and strong subtrees $S_1, \ldots, S_q$ of $T_1, \ldots, T_q$ respectively, such that for every $j \in [q]$ the following are satisfied:

(i) *S_j is contained in A_j,*

(ii) *$L_{T_j}(S_j) = Q$, and*

(iii) *$\mathrm{dens}_{T_j}(A_j|t,n) \geqslant \eta/2$ for all $t \in S_j$ and $n \in L'$.*

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Proof. For every $j \in [q]$, $t \in T_j$ and $N \in \mathrm{Wd}(L)$ with $\min N \geqslant \ell_{T_j}(t)$, we set

$$\begin{aligned} d_{j,t,N} &= \inf_{n \in N} \mathrm{dens}_{T_j}(A_j|t,n), \quad \text{and} \\ \xi_{j,N} &= \sup\{d_{j,t',N'} : t' \in T_j \text{ and } N' \in \mathrm{Wd}(N) \text{ with } \min N' \geqslant \ell_{T_j}(t')\}. \end{aligned} \tag{3.6}$$

Observe that for every $j \in [q]$, $t \in T_j$ and $N, N' \in \mathrm{Wd}(L)$ with $\min N, \min N' \geqslant \ell_{T_j}(t)$ and $N' \subseteq N$, we have that

$$d_{j,t,N} \leqslant d_{j,t,N'}. \tag{3.7}$$

Also, observe that for every $j \in [q]$, and $N, N' \in \mathrm{Wd}(L)$ with $N' \subseteq N$, we have that $\xi_{j,N'} \leqslant \xi_{j,N}$, and therefore

$$\xi_{j,N'} - \xi^2_{j,N'}/18 \leqslant \xi_{j,N} - \xi^2_{j,N}/18. \tag{3.8}$$

Claim 1. There exist $t_1 \in T_1, \ldots, t_q \in T_q$ with $\ell_{T_1}(t_1) = \cdots = \ell_{T_q}(t_q)$ and $L_1 \in \mathrm{Wd}(L)$ with $\ell_{T_1}(t_1) \leqslant \min L_1$ such that $d_{j,t_j,L_1} \geqslant \xi_{j,L_1} - \xi^2_{j,L_1}/18$ for all $j \in [q]$.

Proof of Claim 1. We set $N'_0 = L$. We inductively choose a decreasing sequence $(N'_j)_{j=1}^q$ in $\mathrm{Wd}(L)$ and $t'_1 \in T_1, \ldots, t'_q \in T_q$ satisfying for every $j \in [q]$ the following:

(a) $\ell_{T_j}(t'_j) \leqslant \min N'_j$, and

(b) $d_{j,t'_j,N'_j} \geqslant \xi_{j,N'_{j-1}} - \xi^2_{j,N'_{j-1}}/18$.

The construction is immediate by (3.6). We set $m = \max\{\ell_{T_j}(t'_j) : j \in [q]\}$ and $N_0 = \{n \in N'_q : n \geqslant m\}$. Applying Lemma 3.3, we inductively choose a decreasing sequence $(N_j)_{j=1}^q$ in $\mathrm{Wd}(N_0)$ and

$$t_1 \in \mathrm{Succ}_{T_1}(t'_1) \cap T_1(m), \ldots, t_q \in \mathrm{Succ}_{T_q}(t'_q) \cap T_1(m)$$

such that

$$d_{j,t_j,N_j} \geqslant \xi_{j,N'_{j-1}} - \xi^2_{j,N'_{j-1}}/18. \tag{3.9}$$

Let $L_1 = N_q$ and observe that for every $j \in [q]$ we have that $L_1 \subseteq N_j \subseteq N'_j \subseteq N'_{j-1}$ and therefore

$$d_{j,t_j,L_1} \overset{(3.7)}{\geqslant} d_{j,t_j,N_j} \overset{(3.9)}{\geqslant} \xi_{j,N'_{j-1}} - \xi^2_{j,N'_{j-1}}/18 \overset{(3.8)}{\geqslant} \xi_{j,L_1} - \xi^2_{j,L_1}/18. \tag{3.10}$$

Finally, observe that $\min L_1 \geqslant \min N_0 \geqslant m = \ell_{T_1}(t_1)$. The proof of Claim 1 is complete. □

Claim 2. For every $p \in L_1$ and for every $L'_1 \in \mathrm{Wd}(L_1)$ with $p \leqslant \min L'_1$, there exist $L''_1 \in \mathrm{Wd}(L'_1)$ and $B_1, \ldots, B_q$ subsets of

$$A_1 \cap \mathrm{Succ}_{T_1}(t_1) \cap T_1(p), \ldots, A_q \cap \mathrm{Succ}_{T_q}(t_q) \cap T_q(p)$$

respectively, such that for every $j \in [q]$ we have the following:

(1) $\mathrm{dens}_{T_j}(B_j|t_j,p) \geqslant \eta/2$, and

(2) $d_{j,t,L''_1} \geqslant \eta/2$ for all $t \in B_j$.

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Proof of Claim 2. Let $p \in L_1$ and $L_1' \in \mathrm{Wd}(L_1)$ with $p \leqslant \min L_1'$. We set $N_0 = L_1'$. We inductively construct a decreasing sequence $(N_j)_{j=1}^q$ in $\mathrm{Wd}(N_0)$ and $B_1, \ldots, B_q$ subsets of

$$A_1 \cap \mathrm{Succ}_{T_1}(t_1) \cap T_1(p), \ldots, A_q \cap \mathrm{Succ}_{T_q}(t_q) \cap T_q(p)$$

respectively, satisfying for every $j \in [q]$ the following:

(α) $\mathrm{dens}_{T_j}(B_j | t_j, p) \geqslant \eta/2$, and

(β) $d_{j,t,N_j} \geqslant \eta/2$ for all $t \in B_j$.

Claim 2 follows easily by the above construction and inequality (3.7), setting $L_1'' = N_q$. Let us assume that for some $j \in [q]$ we have chosen $(N_i)_{i=0}^{j-1}$ and if $j > 1$, $(B_i)_{i=1}^{j-1}$ satisfying (α) and (β) above. We describe the construction of B_j and N_j.

We set $\Gamma = \mathrm{Succ}_{T_j}(t_j) \cap T_j(p)$. For every $n \in N_{j-1}$ we have the following. By the homogeneity of the tree T_j, we have that

$$\mathbb{E}_{t \in \Gamma} \mathrm{dens}_{T_j}(A_j | t, n) = \mathrm{dens}_{T_j}(A_j | t_j, n) \geqslant d_{j,t_j,L_1} \geqslant \xi_{j,L_1} - \xi_{j,L_1}^2/18. \tag{3.11}$$

The first inequality holds, since n belongs to L_1 too, while the second one holds by Claim 1. Then either

(a) there exists $t_n' \in \Gamma$ such that $\mathrm{dens}_{T_j}(A_j | t_n', n) \geqslant \xi_{j,L_1} + \xi_{j,L_1}^2/18$, or

(b) for every $t \in \Gamma$ we have that $\mathrm{dens}_{T_j}(A_j | t, n) < \xi_{j,L_1} + \xi_{j,L_1}^2/18$.

By Fact 3.2 there exists $N_j' \in \mathrm{Wd}(N_{j-1})$ such that either (a) occurs for every $n \in N_j'$ or (b) occurs for every $n \in N_j'$. We will show that (b) occurs for every $n \in N_j'$ by showing that the first alternative leads to a contradiction. Indeed, assume that (a) occurs for every $n \in N_j'$. By Fact 3.2 there exist $\widetilde{N}_j \in \mathrm{Wd}(N_j')$ and $t \in \Gamma$ such that $t_n' = t$ for all $n \in \widetilde{N}_j$. Hence $d_{j,t,\widetilde{N}_j} \geqslant \xi_{j,L_1} + \xi_{j,L_1}^2/18 > \xi_{j,L_1} \geqslant \xi_{j,\widetilde{N}_j}$, which is a contradiction by the definition of ξ_{j,L_1} in (3.6). Hence, for every $t \in \Gamma$ we have that

$$\mathrm{dens}_{T_j}(A_j | t, n) < \xi_{j,L_1} + \xi_{j,L_1}^2/18 \tag{3.12}$$

for all $n \in N_j'$.

By Fact 3.4 applied for '$\delta = \xi_{j,L_1} - \xi_{j,L_1}^2/18$' and '$\varepsilon = \xi_{j,L_1}/3$' and inequalities (3.11) and (3.12), for every $n \in N_j'$, setting

$$D_n = \{t \in \Gamma : \mathrm{dens}_{T_j}(A_j | t, n) \geqslant \xi_{j,L_1} - \xi_{j,L_1}^2/18 - \xi_{j,L_1}/3\}, \tag{3.13}$$

we have that $|D_n|/|\Gamma| \geqslant 1 - \xi_{j,L_1}/3$. Applying Fact 3.2, we obtain $N_j \in \mathrm{Wd}(N_j')$ and $D \subseteq \Gamma$ such that $D_n = D$ for all $n \in N_j$. Clearly, $\mathrm{dens}_{T_j}(D | t_j, p) = |D|/|\Gamma| \geqslant 1 - \xi_{j,L_1}/3$ and since $\mathrm{dens}_{T_j}(A_j | t_j, p) \geqslant d_{j,t_j,L_1} \geqslant \xi_{j,L_1} - \xi_{j,L_1}^2/18$, setting $B_j = A_j \cap D$, we have that

$$\mathrm{dens}_{T_j}(B_j | t_j, p) \geqslant \xi_{j,L_1} - \xi_{j,L_1}^2/18 - \xi_{j,L_1}/3 \geqslant \xi_{j,L_1}/2 \geqslant \eta/2. \tag{3.14}$$

Moreover, by (3.13), for every $n \in N_j$, we have that

$$\mathrm{dens}_{T_j}(A_j | t, n) \geqslant \xi_{j,L_1} - \xi_{j,L_1}^2/18 - \xi_{j,L_1}/3 \geqslant \xi_{j,L_1}/2 \geqslant \eta/2. \tag{3.15}$$

The proof of the inductive step and of the inductive construction is complete. As we have already noticed, Claim 2 follows easily by the above construction and inequality (3.7), setting $L_1'' = N_q$. □

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The lemma follows by an iterated use of Claim 2 and an application of Lemma 3.1. Indeed, let P be an arithmetic progression of length $f_q(b,k,\eta/2)$ contained in L_1. Set $L_2 = \{n \in L_1 : n \geqslant \max P\}$. Clearly, L_2 is a van der Waerden set. Making use of Claim 2, we inductively construct a decreasing sequence $(L^p)_{p\in P}$ in $\mathrm{Wd}(L_2)$ and sequences $(B_1^p)_{p\in P},\dots,(B_q^p)_{p\in P}$ of subsets of $A_1 \cap \mathrm{Succ}_{T_1}(t_1),\dots,A_q \cap \mathrm{Succ}_{T_q}(t_q)$ respectively, such that for every $j \in [q]$ and $p \in P$ we have the following:

(1) $B_j^p \subseteq T_j(p)$,

(2) $\mathrm{dens}_{T_j}(B_j^p|t_j,p) \geqslant \eta/2$, and

(3) $d_{j,t,L^p} \geqslant \eta/2$ for all $t \in B_j^p$.

For every $j \in [q]$ we set $B_j = \cup_{p\in P} B_j^p$. By (2) above, we have that $\mathrm{dens}_{T_j}(B_j|p) \geqslant \eta/2$ for all $j \in [q]$ and $p \in P$. By Lemma 3.1, there exist an arithmetic progression Q of length k contained in P and strong subtrees $S_1,\dots,S_q$ of $T_1,\dots,T_q$ respectively, such that S_j is contained in B_j and $L_{T_j}(S_j) = Q$ for all $j \in [q]$. Setting $L' = L^q$, the proof is complete. □

Before we proceed to the proof of Theorem 1.4 let us introduce some additional notation. Let T be a tree and m a positive integer with $m \leqslant h(T)$. We set

$$T|m = \bigcup_{n=0}^{m-1} T(n). \tag{3.16}$$

Clearly $T|m$ is strong subtree of T.

Proof of Theorem 1.4. Let b the branching number of T, *i.e.*, the positive integer such that T is a b-homogeneous tree. Also, let $h_r = 1 + 2 + \cdots + r$ for every positive integer r. We inductively construct an increasing sequence $(W_r)_{r=1}^{\infty}$ of finite strong subtrees of T, a sequence $(Q_r)_{r=1}^{\infty}$ of arithmetic progressions contained in L and a decreasing sequence $(L_r)_{r=1}^{\infty}$ in $\mathrm{Wd}(L)$ satisfying for every positive integer r the following conditions:

(a) the arithmetic progression Q_r is of length r,

(b) $\max Q_r < \min Q_{r+1}$,

(c) $h(W_r) = h_r + 1$ and $L_T(W_r|h_r) = \cup_{j=1}^{r} Q_r$,

(d) $W_r|h_r = W_{r+1}|h_r$,

(e) W_r is contained in A,

(f) $\max L_T(W_r) \leqslant L_r$,

(g) for every $t \in W_r$ and $n \in L_r$, we have that $\mathrm{dens}_T(A|t,n) \geqslant \delta/2^r$.

By a straightforward application of Lemma 3.5 for '$q = 1$', '$k = 2$', '$\eta = \delta$', '$T_1 = T$' and '$A_1 = A$', we obtain Q_1, W_1 and L_1 as desired. Assume that for some positive integer r the sequences $(W_j)_{j=1}^{r}$, $(Q_j)_{j=1}^{r}$ and $(L_j)_{j=1}^{r}$ have been chosen satisfying (a)–(e) above. We describe the construction of W_{r+1}, Q_{r+1} and L_{r+1}.

Let $q = |W(h_r)| = b^{h_r+1}$, and let $(t_j)_{j=1}^{q}$ be an enumeration of the set $W(h_r)$. For every $j \in [q]$ we set $T_j = \mathrm{Succ}_{T_j}(t_j)$ and $A_j = A \cap T_j$. Also, let $L'_{r+1} = \{n - h_r : n \in L_r\}$. Applying Lemma 3.5 for '$k = r + 2$' and '$\eta = \delta/2^r$', we obtain an arithmetic progression

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Q'_{r+1} of length $r+2$ contained in L'_{r+1}, a van der Waerden subset L''_{r+1} of L'_{r+1} with $\max Q'_{r+1} \leqslant \min L''_{r+1}$ and strong subtrees $S_1, \ldots, S_q$ of $T_1, \ldots, T_q$ respectively, such that for every $j \in [q]$ the following are satisfied:

(i) S_j is contained in A_j,

(ii) $L_{T_j}(S_j) = Q'_{r+1}$, and

(iii) $\mathrm{dens}_{T_j}(A_j|t,n) \geqslant \delta/2^{r+1}$ for all $t \in S_j$ and $n \in L''_{r+1}$.

We set

$$\begin{aligned} L_{r+1} &= \{n + h_t : n \in L''_{r+1}\}, \\ Q_{r+1} &= \{n + h_t : n \in Q'_{r+1} \setminus \{\max Q'_{r+1}\}\} \quad \text{and} \\ W_{r+1} &= (W_r|h_r) \cup \cup_{j=1}^{q} S_j. \end{aligned}$$

It follows readily that W_{r+1}, Q_{r+1} and L_{r+1} are as desired and the proof of the inductive construction is complete. Finally, setting $S = \cup_{r=1}^{\infty}(W_r|h_r)$, it is immediate that S is a strong subtree of T contained in A having level set $L' = \cup_{r=1}^{\infty} Q_r$, which is a van der Waerden subset of L. □

4. Correlation of measurable events on arithmetic progressions

This section contains some preliminary work for the proof of Theorem 1.6. We proceed to define some numerical invariants. For every $0 < \eta \leqslant 1$ and every integer $k \geqslant 2$, we set

$$\theta_1(k,\eta) = (\eta/2) \cdot \binom{\mathrm{Sz}(k,\eta/2)}{2}^{-1}, \tag{4.1}$$

while for every $0 < \eta \leqslant 1$ we set

$$\theta_1(1,\eta) = \eta. \tag{4.2}$$

We will need the following lemma.

Lemma 4.1. *Let k be a positive integer and η a real with $0 < \eta \leqslant 1$. Then, for every integer n with $n \geqslant \mathrm{Sz}(k,\eta/2)$, and every family $(A_i)_{i=1}^n$ of measurable events in a probability space (Ω, Σ, μ) such that $\mu(A_i) \geqslant \eta$ for all $i \in [n]$, there exists an arithmetic progression P of length k contained in $[n]$ such that*

$$\mu\left(\bigcap_{i \in P} A_i\right) \geqslant \theta_1(k,\eta). \tag{4.3}$$

Proof. For $k = 1$ the result is immediate. Thus, let as assume that $k \geqslant 2$. Fix $n \geqslant \mathrm{Sz}(k,\eta/2)$ and a family $(A_i)_{i=1}^n$ of measurable events in a probability space (Ω, Σ, μ) satisfying $\mu(A_i) \geqslant \eta$ for all $i \in [n]$. We set $n_0 = \mathrm{Sz}(k,\eta/2)$ and

$$A = \{(i,x) \in [n_0] \times \Omega : x \in A_i\}. \tag{4.4}$$

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Clearly, the product probability measure $\mathrm{dens}_{[n_0]} \otimes \mu$ of A is at least η. For every $x \in \Omega$ let

$$A^x = \{i \in [n_0] : (i, x) \in A\}. \tag{4.5}$$

By Fubini's theorem, setting

$$C = \left\{x \in \Omega : \mathrm{dens}_{[n_0]}(A^x) \geqslant \frac{\eta}{2}\right\}, \tag{4.6}$$

we have that $C \in \Sigma$ and $\mu(C) \geqslant \eta/2$. By Theorem 1.1 and the choice of n_0, for every $x \in C$ there exists an arithmetic progression P_x of length k contained in A^x. Observe that for every $x \in C$ we have

$$x \in \bigcap_{i \in P_x} A_i. \tag{4.7}$$

There are at most $\binom{n_0}{2}$ arithmetic progressions of length k contained in $[n_0]$. Therefore, there exist an arithmetic progression P of length k contained in $[n_0]$ and a measurable subset C' of C with

$$\mu(C') \geqslant \mu(C) \cdot \binom{\mathrm{Sz}(k, \eta/2)}{2}^{-1} \overset{(4.1)}{\geqslant} \theta_1(k, \eta) \tag{4.8}$$

and such that $P_x = P$ for all $x \in C'$. Invoking (4.7) we see that $C' \subseteq \cap_{i \in P} A_i$. Hence

$$\mu(\cap_{i \in P} A_i) \geqslant \mu(C') \overset{(4.8)}{\geqslant} \theta_1(k, \eta),$$

and the proof is complete. □

We will also need a variant of Lemma 4.1 which is stated in the more general context of uniformly density regular families. To state it we need, first, to introduce some further invariants. Specifically, for every $0 < \eta \leqslant 1$ and every uniformly density regular family $\mathcal{F}$, we set

$$M(\mathcal{F}, \eta) = \max\{|\{F \in \mathcal{F} : F \subseteq I\}| : I \text{ is an interval of length } B(\mathcal{F}, \eta)\} \tag{4.9}$$

(where $B(\mathcal{F}, \eta)$ is defined in Definition 1.5), and we define

$$\theta_2(\mathcal{F}, \eta) = \frac{\eta}{4 \cdot M(\mathcal{F}, \eta/4)}. \tag{4.10}$$

Lemma 4.2. *Let $0 < \eta \leqslant 1$ and let $\mathcal{F}$ be a uniformly density regular family. Also, let n be an integer with $n \geqslant (2/\varepsilon) \cdot B(\mathcal{F}, \varepsilon/4)$ and let (Ω, Σ, μ) be a probability space. Finally, let A be a subset of $[n] \times \Omega$ with $(\mathrm{dens}_{[n]} \otimes \mu)(A) \geqslant \eta$. Then there exists an element F of $\mathcal{F}$ such that, setting*

$$\widetilde{A} = \{x \in \Omega : (i, x) \in A \text{ for all } i \in F\},$$

we have

$$\mu(\widetilde{A}) \geqslant \theta_2(\mathcal{F}, \eta). \tag{4.11}$$

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Proof. We set $n_0 = B(\mathcal{F}, \eta/4)$. First we pick a subinterval I of $[n]$ of length n_0 such that

$$(\mathrm{dens}_I \otimes \mu)\big(A \cap (I \times \Omega)\big) \geqslant \eta/2, \tag{4.12}$$

as follows. We set $\ell = \lfloor n/n_0 \rfloor$ and we pick $I_1, \ldots, I_\ell$ disjoint subintervals of $[n]$ each of length n_0. We set $J = \cup_{j=1}^{\ell} I_j$. By the assumptions on n, we have that $\mathrm{dens}_{[n]}([n] \setminus J) < \eta/2$. Consequently, since $I_1, \ldots, I_\ell$ are of the same length, we have that

$$\frac{\eta}{2} \leqslant (\mathrm{dens}_J \otimes \mu)\big(A \cap (J \times \Omega)\big) = \frac{1}{\ell} \sum_{j=1}^{\ell} (\mathrm{dens}_{I_j} \otimes \mu)\big(A \cap (I_j \times \Omega)\big). \tag{4.13}$$

Hence, for some $j_0 \in [\ell]$ we have that

$$(\mathrm{dens}_{I_{j_0}} \otimes \mu)\big(A \cap (I_{j_0} \times \Omega)\big) \geqslant \eta/2.$$

Let $I = I_{j_0}$.

For every $i \in I$ we set

$$A_i = \{x \in \Omega : (i, x) \in A\} \tag{4.14}$$

and for every $x \in \Omega$ we set

$$A^x = \{i \in I : (i, x) \in A\}. \tag{4.15}$$

By (4.12) and Fubini's theorem, we have that the set

$$C = \{x \in \Omega : \mathrm{dens}_I(A^x) \geqslant \eta/4\} \tag{4.16}$$

is a measurable event of probability at least $\eta/4$. Since I is of length $B(\mathcal{F}, \eta/4)$, for every $x \in C$ we have that there exists an element F_x of $\mathcal{F}$ contained in A^x. Let us observe that for every $x \in C$, by the definition of the set A^x, we have that

$$x \in \bigcap_{i \in F_x} A_i. \tag{4.17}$$

Since I contains at most $M(\mathcal{F}, \eta/4)$ elements of $\mathcal{F}$, we have that there exist an element F of $\mathcal{F}$ contained in I and a measurable subset C' of C such that

$$\mu(C') \geqslant \frac{\mu(C)}{M(\mathcal{F}, \eta/4)} \overset{(4.10)}{\geqslant} \theta_2(\mathcal{F}, \eta) \tag{4.18}$$

and $F_x = F$ for all $x \in C'$. Invoking (4.17), we have that $C' \subset \cap_{i \in F} A_i$. Setting $\widetilde{A}$ as in the statement, we clearly have that the intersection $\cap_{i \in F} A_i$ is a subset of $\widetilde{A}$. Thus

$$\mu(\widetilde{A}) \geqslant \mu(\cap_{i \in F} A_i) \geqslant \mu(C') \overset{(4.17)}{\geqslant} \theta_2(\mathcal{F}, \eta),$$

as desired. □

Before we proceed, let us introduce some additional notation. Let (Ω, Σ, μ) be a probability space and let B be a measurable event of positive probability. For every $A \in \Sigma$ we set

$$\mu_B(A) = \frac{\mu(A \cap B)}{\mu(B)}. \tag{4.19}$$

We have the following elementary fact.

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Fact 4.3. *Let η, θ be reals with $0 < \eta, \theta \leqslant 1$. Also, let (Ω, Σ, μ) be a probability space and let A, B be measurable events such that $\mu(A) \geqslant \eta$ and $\mu(B) \geqslant \theta$. If $\mu_B(A) \leqslant \eta/2$, then $\mu(\Omega \setminus B) \geqslant \eta/2$ and $\mu_{\Omega\setminus B}(A) \geqslant \eta + \eta\theta/2$.*

Proof. Assuming that $\mu_B(A) \leqslant \eta/2$, we have the following. First we observe that

$$\mu(\Omega \setminus B) \geqslant \mu(A \setminus B) = \mu(A) - \mu(A \cap B) \geqslant \mu(A) - \mu_B(A) \geqslant \eta/2. \tag{4.20}$$

Since

$$\begin{aligned} \eta \leqslant \mu(A) &= \mu(\Omega \setminus B) \cdot \mu_{\Omega\setminus B}(A) + \mu(B) \cdot \mu_B(A) \\ &\leqslant \big(1 - \mu(B)\big) \cdot \mu_{\Omega\setminus B}(A) + (\eta/2) \cdot \mu(B), \end{aligned} \tag{4.21}$$

we have that

$$\begin{aligned} \mu_{\Omega\setminus B}(A) &\geqslant \eta \cdot \frac{1 - \mu(B)/2}{1 - \mu(B)} = \eta \cdot \left(1 + \frac{\mu(B)/2}{1 - \mu(B)}\right) \\ &\geqslant \eta + \eta \cdot \mu(B)/2 \geqslant \eta + \eta\theta/2, \end{aligned} \tag{4.22}$$

as desired. □

Lemma 4.4. *Let k be a positive integer and η a real with $0 < \eta \leqslant 1$. Also, let L be a van der Waerden set and let $(A_\ell)_{\ell\in L}$ be a family of measurable events in a probability space (Ω, Σ, μ) such that $\mu(A_\ell) \geqslant \eta$ for all $\ell \in L$. Then there exist an arithmetic progression P of length k contained in L, a van der Waerden subset L' of L and a subset B of $\cap_{i\in P} A_i$ such that*

$$\mu(B) \geqslant \left(\frac{\eta}{2}\right)^{\lfloor 2/(\eta\cdot\theta_1(k,\eta))\rfloor - 1} \theta_1(k, \eta) \tag{4.23}$$

and $\mu_B(A_\ell) \geqslant \eta/2$, for all $\ell \in L'$. Moreover, the set B belongs to the algebra generated by the family $\{A_\ell : \ell \in L \text{ and } \ell \leqslant \max P\}$.

Proof. We set $L_0 = L$ and $\Omega_0 = \Omega$. We pick a positive integer s_0 with $s_0 \leqslant \lfloor 2/(\eta \cdot \theta_1(k, \eta))\rfloor$, and we construct by induction $L_1, \ldots, L_{s_0}$ and $P_1, \ldots, P_{s_0}$ such that setting inductively $B_t = (\cap_{i\in P_t} A_i) \cap \Omega_{t-1}$ and $\Omega_t = \Omega_{t-1} \setminus B_t$ for all $t = 1, \ldots, s_0$, we have that the following conditions are satisfied.

(i) For every $t = 1, \ldots, s_0$, we have that L_t is a van der Waerden subset of L_{t-1}.

(ii) For every $t = 1, \ldots, s_0$, we have that P_t is an arithmetic progression of length k contained in L_{t-1}.

(iii) For every $t = 0, \ldots, s_0 - 1$, we have $\mu(\Omega_t) \geqslant (\eta/2)^t$.

(iv) For every $t = 1, \ldots, s_0$ we have $\mu_{\Omega_{t-1}}(B_t) \geqslant \theta_1(k, \eta)$.

(v) For every $t = 0, \ldots, s_0 - 1$, we have $\mu_{\Omega_t}(A_\ell) \geqslant \eta + t \cdot \eta \cdot \theta_1(k, \eta)/2$ for all $\ell \in L_t$.

(vi) For every $t = 1, \ldots, s_0 - 1$, we have that $\mu_{B_t}(A_\ell) < \eta/2$ for all $\ell \in L_t$.

(vii) $\mu_{B_{s_0}}(A_\ell) \geqslant \eta/2$ for all $\ell \in L_{s_0}$.

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Assume that for some $s < \lfloor 2/(\eta \cdot \theta_1(k,\eta))\rfloor$ we have constructed $(L_t)_{t=0}^s$ and if $s \geqslant 1$, $(P_t)_{t=1}^s$, satisfying (i)–(vi) above. Let $(\Omega_t)_{t=0}^s$ be as defined above. By the inductive assumption (i), we have that L_s is a van der Waerden set and therefore we may pick an arithmetic progression P of length $\mathrm{Sz}(k,\eta/2)$ contained in L_s. By the inductive assumption (v) and Lemma 4.1, there exists an arithmetic progression P_{s+1} of length k contained in P such that

$$\mu_{\Omega_s}(B_{s+1}) \geqslant \theta_1(k,\eta), \tag{4.24}$$

where $B_{s+1} = (\cap_{i\in P_{s+1}} A_i) \cap \Omega_s$. By Fact 3.2 we pass to a van der Waerden subset L_{s+1} of L_s such that either

(a) $\mu_{B_{s+1}}(A_\ell) \geqslant \eta/2$, for all $\ell \in L_{s+1}$, or

(b) $\mu_{B_{s+1}}(A_\ell) < \eta/2$, for all $\ell \in L_{s+1}$.

If (a) occurs, then we set $s_0 = s+1$ and the inductive construction is complete. Let us assume that (b) holds. Then invoking (4.24) and (v) of the inductive assumptions, by Fact 4.3, we have that

$$\mu_{\Omega_{s+1}}(A_\ell) \geqslant \eta + (s+1)\cdot\eta\cdot\theta_1(k,\eta)/2 \tag{4.25}$$

for all $\ell \in L_{s+1}$, where $\Omega_{s+1} = \Omega_s \setminus B_{s+1}$. Moreover, by Fact 4.3 and the inductive assumptions (iii) and (v), we have that $\mu(\Omega_{s+1}) \geqslant (\eta/2)\cdot\mu(\Omega_s) \geqslant (\eta/2)^{s+1}$. The inductive step of the construction is complete. Finally, let as point out that if $s = \lfloor 2/(\eta\cdot\theta_1(k,\eta))\rfloor - 1$, then (a) has to occur. Indeed, assuming that (b) occurs then by (4.25) we would have that the relative probability of A_ℓ inside Ω_{s+1} exceeds 1.

Hence, setting $L' = L_{s_0}$, $P = P_{s_0}$ and $B = B_{s_0}$, we have that L' is a van der Waerden subset of L and P is an arithmetic progression of length k contained in L. Moreover, we have that $B = (\cap_{i\in P} A_i)\cap\Omega_{s_0-1} \subseteq \cap_{i\in P} A_i$ and

$$\begin{aligned}\mu(B) = \mu(B_{s_0}) &= \mu_{\Omega_{s_0-1}}(B_{s_0})\cdot\mu(\Omega_{s_0-1}) \overset{\text{(iv),(iii)}}{\geqslant} \theta_1(k,\eta)\cdot(\eta/2)^{s_0-1}\\ &\geqslant \theta_1(k,\eta)\cdot(\eta/2)^{\lfloor 2/(\eta\cdot\theta_1(k,\eta))\rfloor - 1}.\end{aligned} \tag{4.26}$$

By (vii), we have that $\mu_B(A_\ell) \geqslant \eta/2$, for all $\ell \in L'$. Finally, it is immediate that the set B, by its definition, belongs to the algebra generated by the family $\{A_\ell : \ell \in L \text{ and } \ell \leqslant \max P\}$ as desired. □

5. The auxiliary map T

As we have already mentioned, the definition of the map V_δ makes use of an auxiliary map T. Recall that by $\mathcal{R}$ we denote the set of all uniformly density regular families (see Definition 1.5). We define the map $T : \mathcal{R}^{<\omega}\times\mathbb{R}_+ \to \mathbb{N}$, where $\mathbb{R}_+$ denotes the set of all positive reals, as follows. Let q be a non-negative integer and $((\mathcal{F}_p)_{p=0}^q, \varepsilon)$ an element of $\mathcal{R}^{<\omega}\times\mathbb{R}_+$. We inductively define $(\varepsilon_p)_{p=0}^q$ by setting

$$\varepsilon_0 = \varepsilon \quad \text{and} \quad \varepsilon_{p+1} = \theta_2(\mathcal{F}_p, \varepsilon_p) \tag{5.1}$$

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for all $p = 0, \dots, q-1$. Finally we set

$$T\big((\mathcal{F}_p)_{p=0}^{q}, \varepsilon\big) = \left\lceil \frac{2}{\varepsilon_q} \cdot B(\mathcal{F}_q, \varepsilon_q/4) \right\rceil. \tag{5.2}$$

We then extend T to $\mathcal{R}^{<\omega} \times \mathbb{R}_+$ arbitrarily. Let us observe, for later use, that if q is positive then

$$T\big((\mathcal{F}_p)_{p=0}^{q}, \varepsilon\big) = T\big((\mathcal{F}_p)_{p=1}^{q}, \theta_2(\mathcal{F}_0, \varepsilon)\big). \tag{5.3}$$

Although the following notation is quite standard in the literature, we include it below for clarity.

Notation. Let $q_0 < q_1 < q_2$ be non-negative integers and let $(n_q)_q$ be a sequence of positive integers. Also, let

$$x \in \prod_{p=q_0}^{q_1-1} [n_p] \quad \text{and} \quad y \in \prod_{p=q_1}^{q_2-1} [n_p].$$

By $x^\frown y$ we denote the concatenation of the sequences x, y, *i.e.*, the sequence $z \in \prod_{p=q_0}^{q_2-1}[n_p]$ satisfying $z(p) = x(p)$ for all $p = q_0, \dots, q_1 - 1$ and $z(p) = y(p)$ for all $p = q_1, \dots, q_2 - 1$. Moreover, for

$$A \subseteq \prod_{p=q_0}^{q_1-1} [n_p] \quad \text{and} \quad B \subseteq \prod_{p=q_1}^{q_2-1} [n_p]$$

we set

$$x^\frown B = \{x^\frown y : y \in B\} \tag{5.4}$$

and

$$A^\frown B = \bigcup_{x \in A} x^\frown B. \tag{5.5}$$

The main property of the map T that we are interested in is described by the following lemma. Similar results to this one have already been considered (see [5, 6, 10]).

Lemma 5.1. *Let $0 < \varepsilon \leqslant 1$, let q be a non-negative integer, let $\mathcal{F}_0, \dots, \mathcal{F}_q$ be uniformly density regular families, and let $n_0, \dots, n_q$ be integers such that $n_p \geqslant T\big((\mathcal{F}_s)_{s=0}^{p}, \varepsilon\big)$ for all $p = 0, \dots, q$. Finally, let D be a subset of $\prod_{p=0}^{q}[n_p]$ of density at least ε. Then there exists a sequence $(I_p)_{p=0}^{q}$ such that*

(i) *I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$ for all $p = 0, \dots, q$, and*

(ii) *$\prod_{p=0}^{q} I_p$ is a subset of D.*

Proof. We proceed by induction on q. First let us observe that for $q = 0$ we have that

$$T\big((\mathcal{F}_0), \varepsilon\big) \geqslant B(\mathcal{F}_0, \varepsilon)$$

and therefore the result follows immediately by the definition of the number $B(\mathcal{F}_0, \varepsilon)$.

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Assume that the statement holds for some q. Fix a real ε with $0 < \varepsilon \leqslant 1$, uniformly density regular families $\mathcal{F}_0, \ldots, \mathcal{F}_{q+1}$ and integers $n_0, \ldots, n_{q+1}$ satisfying $n_p \geqslant T\big((\mathcal{F}_s)_{s=0}^{p}, \varepsilon\big)$ for all $p = 0, \ldots, q+1$. Finally, let D be a subset of $\prod_{p=0}^{q+1}[n_p]$ of density at least ε. We set $\Omega = \prod_{p=1}^{q+1}[n_p]$. Observe that $\prod_{p=0}^{q+1}[n_p] = [n_0] \times \Omega$ and that the probability measures $\mathrm{dens}_{\prod_{p=0}^{q+1}[n_p]}$ and $\mathrm{dens}_{n_0} \otimes \mathrm{dens}_{\Omega}$ are equal. Thus $(\mathrm{dens}_{n_0} \otimes \mathrm{dens}_{\Omega})(D) \geqslant \varepsilon$. Since

$$n_0 \geqslant T\big((\mathcal{F}_0), \varepsilon\big) \overset{(5.2)}{=} (2/\varepsilon) \cdot B(\mathcal{F}_0, \varepsilon/4),$$

by Lemma 4.2, there exists an element I_0 of $\mathcal{F}_0$ such that setting

$$\widetilde{D} = \{x \in \Omega : (i, x) \in D \text{ for all } i \in I_0\},$$

we have that

$$\mu(\widetilde{D}) \geqslant \theta_2(\mathcal{F}_0, \varepsilon). \tag{5.6}$$

By the definition of $\widetilde{D}$, it is immediate that

$$I_0^{\frown} \widetilde{D} \subseteq D. \tag{5.7}$$

Also, notice that for every $p = 1, \ldots, q+1$,

$$n_p \geqslant T\big((\mathcal{F}_s)_{s=0}^{p}, \varepsilon\big) \overset{(5.3)}{=} T\big((\mathcal{F}_s)_{s=1}^{p}, \theta_2(\mathcal{F}_0, \varepsilon)\big). \tag{5.8}$$

By (5.6), (5.8) and the inductive assumption we have that there exists a sequence $(I_p)_{p=1}^{q+1}$ such that

(a) I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$ for all $p = 1, \ldots, q+1$, and
(b) $\prod_{p=1}^{q+1} I_p$ is a subset of $\widetilde{D}$.

By (b) and (5.7), we have that $\prod_{p=0}^{q+1} I_p \subset I_0^{\frown} \widetilde{D} \subseteq D$ and the proof is complete. □

Definition 5.2. Let $0 < \varepsilon \leqslant 1$ and let r be a non-negative integer. Also, let L be a van der Waerden subset of $\mathbb{N}_+$ and let $(n_q)_{q \in \mathbb{N}}$ be a sequence of positive integers. We will say that a sequence $(D_\ell)_{\ell \in L}$ is $(r, \varepsilon, (n_q)_{q \in \mathbb{N}})$-dense if, for every $\ell \in L$ with $\ell > r$, we have that D_ℓ is a subset of $\prod_{p=r}^{\ell-1}[n_p]$ of density at least ε.

For every $0 < \varepsilon \leqslant 1$ and every positive integer k we define

$$\theta_3(k, \varepsilon) = \frac{1}{2} \cdot \left(\frac{\varepsilon}{2}\right)^{\lfloor 2/(\varepsilon \cdot \theta_1(k, \varepsilon))\rfloor} \cdot \theta_1(k, \varepsilon). \tag{5.9}$$

Finally, we recall that for every non-negative integer n and every sequence x of length at least n (finite or infinite), by $R_n(x)$ we denote the initial segment of x of length n.

Lemma 5.3. *Let $0 < \varepsilon \leqslant 1$, let r be a non-negative integer and let k be a positive integer. Also, let $(\mathcal{F}_q)_{q \in \mathbb{N}}$ be a sequence of uniformly density regular families and let $(n_q)_{q \in \mathbb{N}}$ be a sequence of positive integers such that $n_q \geqslant T\big((\mathcal{F}_p)_{p=r}^{q}, \theta_3(k, \varepsilon)\big)$, for all $q \geqslant r$. Finally, let L be a van der Waerden subset of $\mathbb{N}_+$ and $(D_\ell)_{\ell \in L}$ a sequence which is $(r, \varepsilon, (n_q)_{q \in \mathbb{N}})$-dense. Then there exist an arithmetic progression P of length k inside L, a van der Waerden subset L' of*

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L, a finite sequence $(I_p)_{p=r}^{r'-1}$ and an $(r', \varepsilon', (n_q)_{q\in\mathbb{N}})$-dense sequence $(\widetilde{D}_\ell)_{\ell\in L'}$, where $r' = \max P$ and $\varepsilon' = \varepsilon \cdot 2^{-(\prod_{p=r}^{r'-1} n_p + 2)}$, satisfying the following.

(i) *For every $p = r, \dots, r'-1$, the set I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$.*

(ii) $r < \min P$.

(iii) *For every $q \in P$, the set $\prod_{p=r}^{q-1} I_p$ is a subset of D_q.*

(iv) *For every $\ell \in L'$, the set $(\prod_{p=r}^{r'-1} I_p)^\frown \widetilde{D}_\ell$ is a subset of D_ℓ.*

Proof. Passing to a final segment of L, if necessary, we may assume that $r < \min L$. Let $\Omega = \prod_{p=r}^{\infty}[n_p]$ and let μ be the Lebesgue (probability) measure on Ω. Also, for every $\ell \in L$, let $A_\ell = \{x \in \Omega : R_{\ell - r}(x) \in D_\ell\}$. By Lemma 4.4, applied for '$\eta = \varepsilon$', there exist an arithmetic progression P of length k contained in L, a van der Waerden subset L'' of L and a subset $\widehat{B}$ of $\cap_{i\in P} A_i$ such that

$$\mu(\widehat{B}) \geqslant \left(\frac{\varepsilon}{2}\right)^{\lfloor 2/(\varepsilon\cdot\theta_1(k,\varepsilon))\rfloor - 1} \theta_1(k,\varepsilon), \tag{5.10}$$

for every $\ell \in L''$ we have $\mu_{\widehat{B}}(A_\ell) \geqslant \varepsilon/2$ and the set $\widehat{B}$ belongs to the algebra generated by the family $\{A_\ell : \ell \in L \text{ and } \ell \leqslant \max P\}$. Thus, setting $r' = \max P$, there exists a subset B of $\prod_{p=r}^{r'-1}[n_p]$ such that

$$\operatorname{dens}_{\prod_{p=r}^{r'-1}[n_p]}(B) = \mu(\widehat{B}) \overset{(5.10)}{\geqslant} \left(\frac{\varepsilon}{2}\right)^{\lfloor 2/(\varepsilon\cdot\theta_1(k,\varepsilon))\rfloor - 1} \theta_1(k,\varepsilon), \tag{5.11}$$

for every $\ell \in L''$. Setting

$$B_\ell = \left\{ x \in \prod_{p=r}^{\ell-1}[n_p] : R_{r'-r}(x) \in B \right\},$$

we have

$$\operatorname{dens}_{B_\ell}(D_\ell) \geqslant \varepsilon/2 \tag{5.12}$$

and

$$R_{q-r}(x) \in D_q \tag{5.13}$$

for all $q \in P$ and $x \in B$. Passing to a final subset L'', if necessary, we may assume $\min L'' > r'$.

For every ℓ in L'' we have the following. For each element y in $\prod_{p=r'}^{\ell-1}[n_p]$ we define

$$\Gamma_y = \{z \in B : z^\frown y \in D_\ell\}.$$

By (5.12) and Fubini's theorem, we have that the set

$$D'_\ell = \left\{ y \in \prod_{p=r'}^{\ell-1}[n_p] : \operatorname{dens}_B(\Gamma_y) \geqslant \varepsilon/4 \right\}$$

is of density at least $\varepsilon/4$ inside $\prod_{p=r'}^{\ell-1}[n_p]$. Since Γ_y is a subset of B and therefore a subset of $\prod_{p=r}^{r'-1}[n_p]$, we have that there exist a subset Γ_ℓ of B and a subset $\widetilde{D}_\ell$ of D'_ℓ of density

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at least $(\varepsilon/4)\cdot 2^{-\prod_{p=r}^{r'-1} n_p}$ inside $\prod_{p=r}^{r'-1}[n_p]$ such that $\Gamma_y = \Gamma_\ell$ for all y in $\widetilde{D}_\ell$. Let us observe that by the choice of Γ_ℓ and D'_ℓ we have that

$$\Gamma_\ell^\frown \widetilde{D}_\ell \subseteq D_\ell \quad \text{and} \quad \mathrm{dens}_B(\Gamma_\ell) \geqslant \varepsilon/4. \tag{5.14}$$

By Fact 3.2 there exist a subset Γ of B and a van der Waerden subset L' of L'', such that $\Gamma_\ell = \Gamma$ for all $\ell \in L'$. Clearly, $(\widetilde{D}_\ell)_{\ell\in L'}$ is $(r', \varepsilon', (n_q)_q)$-dense, where ε' is defined in the statement of the lemma. By (5.9), (5.11) and (5.14) we have

$$\mathrm{dens}_{\prod_{p=r}^{r'-1}[n_p]}(\Gamma) = \mathrm{dens}_B(\Gamma)\cdot \mathrm{dens}_{\prod_{p=r}^{r'-1}[n_p]}(B) \geqslant \theta_3(k,\varepsilon). \tag{5.15}$$

Moreover, by (5.14), we have that

$$\Gamma^\frown \widetilde{D}_\ell \subseteq D_\ell \tag{5.16}$$

for all $\ell \in L'$. Since for every $q = r, \dots, r'-1$ we have that

$$n_q \geqslant T\big((\mathcal{F}_p)_{p=r}^q, \theta_3(k,\varepsilon)\big),$$

by (5.15) and Lemma 5.1 there exists a sequence $(I_p)_{p=r}^{r'-1}$ such that

(a) I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$ for all $p = r, \dots, r'-1$, and

(b) $\prod_{p=r}^{r'-1} I_p$ is a subset of Γ.

Since $\prod_{p=r}^{r'-1} I_p \subseteq \Gamma \subseteq B$, by (5.13) we have that $\prod_{p=r}^{q-1} I_p$ is a subset of D_q for all $q \in P$. By (b) and (5.16) we have that $\prod_{p=r}^{r'-1} I_p^\frown \widetilde{D}_\ell$ is a subset of D_ℓ for all $\ell \in L'$ and the proof is complete. □

6. Definition of the map V_δ and the proof of Theorem 1.6

Let us recall that $\mathcal{R}$ denotes the set of all uniformly density regular families. In what follows, let us adopt the following convention. For a sequence of positive integers $(n_q)_{q\in\mathbb{N}}$ and a non-negative integer r , we consider $(n_p)_{p=r}^{r-1}$ to be the empty sequence and $\prod_{p=r}^{r-1} n_p$ to be equal to zero. Fix some real δ with $0 < \delta \leqslant 1$. We define the map

$$V_\delta : \mathcal{R}^{<\omega} \times \mathbb{N}_+^{<\omega} \to \mathbb{N}$$

as follows. For every non-negative integer q, every finite sequence $(n_p)_{p=0}^{q-1}$ of positive integers and every finite sequence $(\mathcal{F}_p)_{p=0}^q$ of uniformly density regular families, we set

$$V_\delta\big((\mathcal{F}_p)_{p=0}^q, (n_p)_{p=0}^{q-1}\big) = \max_{0\leqslant r\leqslant q} T\big((\mathcal{F}_p)_{p=r}^q, \theta_3(r+1, \delta\cdot 2^{-(\prod_{p=0}^{r-1} n_p + 2r)})\big). \tag{6.1}$$

Proof of Theorem 1.6. Let $(n_q)_{q\in\mathbb{N}}$ be a sequence of positive integers, let $(\mathcal{F}_q)_{q\in\mathbb{N}}$ be a sequence of uniformly density regular families, let L be a van der Waerden subset of $\mathbb{N}_+$ and let $(D_\ell)_{\ell\in L}$ be $(0, \delta, (n_q)_{q\in\mathbb{N}})$-dense such that

$$n_q \geqslant V_\delta\big((\mathcal{F}_p)_{p=0}^q, (n_p)_{p=0}^{q-1}\big) \tag{6.2}$$

for every non-negative integer q. We set $L_0 = L$, $r_0 = 0$ and $(D_\ell^0)_{\ell\in L_0} = (D_\ell)_{\ell\in L}$. We inductively construct a sequence of arithmetic progressions $(P_n)_{n=1}^\infty$ contained in L, a

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decreasing sequence of van der Waerden sets $(L_n)_{n\in\mathbb{N}}$, a sequence $\big((I_q)_{q=r_{n-1}}^{r_n-1}\big)_{n=1}^{\infty}$, where $r_n = \max P_n$ for all positive integers n, and a sequence $\big((D_\ell^n)_{\ell\in L_n}\big)_{n\in\mathbb{N}}$ such that for every non-negative integer n we have the following.

(i) $r_n < \min P_{n+1}$.

(ii) $(D_\ell^n)_{\ell\in L_n}$ is $(r_n, \delta\cdot 2^{-(\prod_{p=0}^{r_n-1} n_p + 2r_n)}, (n_q)_{q\in\mathbb{N}})$-dense.

(iii) If n is positive, then I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$ for every $p = r_{n-1},\dots,r_n-1$.

(iv) If n is positive, then P_n is an arithmetic progression of length $r_{n-1}+1$ contained in L_{n-1} such that $\prod_{p=0}^{q-1} I_q \subseteq D_q$ for every $q\in P_n$.

(v) $(\prod_{p=0}^{r_n-1} I_p)^\frown D_\ell^n \subseteq D_\ell$ for all $\ell\in L_n$, under the convention $\prod_{p=0}^{-1} I_p = \{\varnothing\}$.

Notice first that for $n=0$ the properties (i)–(v) are satisfied. Assume that for some non-negative integer n we have constructed $(L_m)_{m=0}^n$, $\big((D_\ell^m)_{\ell\in L_m}\big)_{m=0}^n$, and if $n\geqslant 1$ we have constructed $(P_m)_{m=1}^n$ and $\big((I_q)_{q=r_{m-1}}^{r_m-1}\big)_{m=1}^n$ satisfying (i)–(v). Then, for every integer q with $q\geqslant r_n$, we have that

$$n_q \overset{(6.2)}{\geqslant} V_\delta\big((\mathcal{F}_p)_{p=0}^q, (n_p)_{p=0}^{q-1}\big) \overset{(6.1)}{\geqslant} T\big((\mathcal{F}_p)_{p=r_n}^q, \theta_3(r_n+1, \delta\cdot 2^{-(\prod_{p=0}^{r_n-1} n_p+2r_n)})\big). \tag{6.3}$$

By (6.3) and the inductive assumption (ii), we have that the assumptions of Lemma 5.3 for '$\varepsilon = \delta\cdot 2^{-(\prod_{p=0}^{r_n-1} n_p+2r_n)}$', '$k = r_n+1$', '$r = r_n$', '$L = L_n$' and '$(D_\ell)_{\ell\in L} = (D_\ell^n)_{\ell\in L_n}$' are satisfied. Hence there exist an arithmetic progression P_{n+1} of length r_n+1 inside L_n, a van der Waerden subset L_{n+1} of L_n, a finite sequence $(I_p)_{p=r_n}^{r_{n+1}-1}$ and an $(r_{n+1}, \varepsilon', (n_q)_{q\in\mathbb{N}})$-dense sequence $(D_\ell^{n+1})_{\ell\in L_{n+1}}$, where $r_{n+1} = \max P_{n+1}$ and

$$\varepsilon' = \delta\cdot 2^{-(\prod_{p=0}^{r_n-1} n_p+2r_n)}\cdot 2^{-(\prod_{p=r_n}^{r_{n+1}-1} n_p+2)} \geqslant \delta\cdot 2^{-(\prod_{p=0}^{r_{n+1}-1} n_p+2r_{n+1})}, \tag{6.4}$$

satisfying the following.

(a) For all $p = r_n,\dots,r_{n+1}-1$, the set I_p is an element of $\mathcal{F}_p$ contained in $[n_p]$.

(b) $r_n < \min P_{n+1}$.

(c) For every $q\in P_{n+1}$, the set $\prod_{p=r_n}^{q-1} I_p$ is a subset of D_q^n.

(d) For every $\ell\in L_{n+1}$, the set $(\prod_{p=r_n}^{r_{n+1}-1} I_p)^\frown D_\ell^{n+1}$ is a subset of D_ℓ^n.

By (c) and the inductive assumption (v), we have for every $q\in P_{n+1}$ that the set $\prod_{p=0}^{q-1} I_p$ is a subset of D_q. By (d) and the inductive assumption (v), we have for every $\ell\in L_{n+1}$ that the set $(\prod_{p=r_n}^{r_{n+1}-1} I_p)^\frown D_\ell^{n+1}$ is a subset of D_ℓ. The proof of the inductive step is complete.

We set $L' = \bigcup_{n=1}^{\infty} P_n$. Moreover, observe that r_n tends to infinity as n tends to infinity. Thus L' is a van der Waerden set. It is straightforward that L' and $(I_q)_{q\in\mathbb{N}}$ satisfy the conclusion of the theorem. □

7. Bounds for the map V_δ

In this section we are interested in bounds for the map V_δ. For every positive integer m, we let $\mathcal{F}_{[m]}$ denote the family of all subsets of the positive integers with m elements. It is

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immediate that

$$B(\mathcal{F}_{[m]}, \varepsilon) = \lceil m/\varepsilon \rceil. \tag{7.1}$$

We set

$$\mathcal{R}_c = \{\mathcal{F}_{[m]} : m \text{ is a positive integer}\}. \tag{7.2}$$

We will also need the following remark.

Remark 7.1. Let $\mathcal{R}'$ be a subfamily of $\mathcal{R}$ and let $T' : \mathcal{R}'^{<\omega} \times \mathbb{R}_+ \to \mathbb{N}$ be a map satisfying the conclusion of Lemma 5.1 for $\mathcal{F}_0, \ldots, \mathcal{F}_q$ from $\mathcal{R}'$. Also, let $V'_\delta : \mathcal{R}^{<\omega} \times \mathbb{N}_+^{<\omega} \to \mathbb{N}$ be the map defined as in (6.1) using T' instead of T. Then one can check that V'_δ satisfies the conclusion of Theorem 1.6 under the restriction that each $\mathcal{F}_q$ is chosen from $\mathcal{R}'$.

We define $T' : \mathcal{R}_c^{<\omega} \times \mathbb{R}_+ \to \mathbb{N}$, as follows. For every non-negative integer q, every positive integer $m_0, \ldots, m_q$ and every real ε with $0 < \varepsilon \leqslant 1$, we set

$$T'\big((\mathcal{F}_{[m_p]})_{p=0}^q, \varepsilon\big) = \frac{2}{\tilde{\varepsilon}} \left\lceil \frac{m_q(m_q-1)}{\tilde{\varepsilon}^q 2^{1-q}(1-2^{-q})} \right\rceil, \tag{7.3}$$

where $\tilde{\varepsilon} = \varepsilon$ if $q = 0$ and

$$\tilde{\varepsilon} = \varepsilon^{\prod_{p=0}^{q-1} m_p} 2^{-2\sum_{j=0}^{q-1} \prod_{p=j}^{q-1} m_p}$$

otherwise. Actually,

$$T'\big((\mathcal{F}_{[m_p]})_{p=0}^q, \varepsilon\big) = T_\varepsilon\big((m_p)_{p=0}^q\big)$$

for every choice of non-negative integer q, positive integers $m_0, \ldots, m_q$ and real ε with $0 < \varepsilon \leqslant 1$, where T_ε is defined in equation (3) from [18]. We also define $V'_\delta : \mathcal{R}^{<\omega} \times \mathbb{N}_+^{<\omega} \to \mathbb{N}$ as in (6.1) using T' instead of T. Lemma 3 from [18] yields that T' satisfies the conclusion of Lemma 5.1. Hence by Remark 7.1 we have that V'_δ satisfies the conclusion of Theorem 1.6. In what follows we fix $0 < \delta \leqslant 1$ and a sequence $(m_q)_{q\in\mathbb{N}}$ of integers such that $m_q \geqslant 2$ for every non-negative integer q.

We define a map $f_c : \mathbb{N} \to \mathbb{N}$ inductively as follows. We set

$$f_c(0) = V'_\delta\big((\mathcal{F}_{[m_0]}), \varnothing\big) \quad \text{and} \quad f_c(q+1) = V'_\delta\big((\mathcal{F}_{[m_p]})_{p=0}^{q+1}, (f_c(p))_{p=0}^q\big) \tag{7.4}$$

for all $q = 0, 1, \ldots$. In particular, we are interested in the rate of growth of the map f_c. We will need the following inequalities. By Lemma 14 of [18] we have that

$$T'\big((\mathcal{F}_{[m_p]})_{p=0}^q, \varepsilon\big) \leqslant A_2\left(5\log_2(1/\varepsilon)\prod_{p=0}^q m_p\right) \tag{7.5}$$

for every non-negative integer q and every $0 < \varepsilon \leqslant \delta/2$, where $A_2(x) = 2^x$ for every real x. Moreover, by Theorem 18.2 of [9] we have that

$$\mathrm{Sz}(k, \varepsilon) \leqslant A_2^{(3)}(\log_2(1/\varepsilon) A_2^{(2)}(k+9)), \tag{7.6}$$

for all positive integers k and all reals ε with $0 < \varepsilon \leqslant 1$.

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Proposition 7.2. *We have that*

$$f_c(q) \leqslant A_2^{(1+6q)}\Big(5\big((2/\delta^2)+1\big)\log_2(2/\delta)m_0 + \sum_{p=1}^{q} m_p\Big),$$

for every non-negative integer q. □

Proof. By (4.1) and (7.6), for every positive integer k and every $0 < \varepsilon \leqslant 1$ we have

$$\begin{aligned}\theta_1(k,\varepsilon) &\geqslant \varepsilon \cdot [A_2^{(3)}(\log_2(2/\varepsilon)A_2^{(2)}(k+9))]^{-2}\\ &\geqslant A_2\big(\log_2(1/\varepsilon)\big)^{-1} \cdot A_2^{(2)}\big(1 + A_2\big(\log_2(2/\varepsilon)\cdot A_2^{(2)}(k+9)\big)\big)^{-1}\\ &\geqslant A_2^{(2)}\big(1 + A_2\big(\log_2(2/\varepsilon)\cdot A_2^{(2)}(k+9)\big)\big)^{-1}, \end{aligned} \tag{7.7}$$

and therefore invoking (5.9) we have that

$$\begin{aligned}\theta_3(k,\varepsilon) &\geqslant A_2\big(1+(2/\varepsilon)\log_2(2/\varepsilon)A_2^{(2)}\big(1+A_2\big(\log_2(2/\varepsilon)\cdot A_2^{(2)}(k+9)\big)\big)\big)^{-1}\\ &\quad\times A_2^{(2)}\big(1+A_2\big(\log_2(2/\varepsilon)\cdot A_2^{(2)}(k+9)\big)\big)^{-1}\\ &\geqslant A_2^{(3)}\big(3+A_2\big(\log_2(2/\varepsilon)A_2^{(2)}(k+9)\big)\big)^{-1}. \end{aligned} \tag{7.8}$$

By (6.1) and (7.4), for every positive integer q, we have that

$$\begin{aligned} f_c(q) &\leqslant T'\big((\mathcal{F}_{[m_p]})_{p=0}^{q}, \theta_3(q+1, \delta\cdot 2^{-(\prod_{p=0}^{q-1} f_c(p)+2q)})\big)\\ &\overset{(7.5),(7.8)}{\leqslant} A_2\Big(5A_2^{(2)}\big(3+A_2\big(\log_2(2/\varepsilon)A_2^{(2)}(k+9)\big)\big)\prod_{p=0}^{q} m_p\Big)\\ &\leqslant A_2^{(6)}\Big(\log_2(2/\delta) + \prod_{p=0}^{q-1} f_c(p) + 2q + m_q\Big). \end{aligned} \tag{7.9}$$

By (4.2) and (5.9), we have that $\theta_3(1,\varepsilon) \geqslant (\varepsilon/2)^{(2/\varepsilon^2)+1}$, for all $0 < \varepsilon \leqslant 1$. Thus, by (6.1), (7.4) and (7.5), we have that

$$f_c(0) = T'\big((\mathcal{F}_{[m_0]}), \theta_3(1,\delta)\big) \leqslant A_2\big(5\log_2(2/\delta)(2/\delta^2+1)m_0\big). \tag{7.10}$$

By inequalities (7.9), (7.10) and using induction on q, the result follows. □

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