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OVERLY DETERMINED AGENTS PREVENT CONSENSUS IN A GENERALIZED DEFFUANT MODEL ON $\mathbb{Z}$ WITH DISPERSED OPINIONS

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Abstract

During the last decades, quite a number of interacting particle systems have been introduced and studied in the crossover area of mathematics and statistical physics. Some of these can be seen as simplistic models for opinion formation processes in groups of interacting people. In the model introduced by Deffuant *et al.* (2000), agents that are neighbors on a given network graph, randomly meet in pairs and approach a compromise if their current opinions do not differ by more than a given threshold value θ. We consider the two-sided infinite path $\mathbb{Z}$ as the underlying graph and extend existing models to a setting in which opinions are given by probability distributions. Similar to what has been shown for finite-dimensional opinions, we observe a dichotomy in the long-term behavior of the model, but only if the initial narrow mindedness of the agents is restricted.

Keywords: Deffuant model; opinion dynamics; random measure; total variation distance

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Secondary 60G57; 82C22

1. Introduction

The research field that became known as *opinion dynamics* originated from simple models for interacting elementary particles established in statistical physics. These models were introduced in order to determine how microscopic interaction rules lead to macroscopic properties of the whole system. Due to the strong link between statistical mechanics and spatial stochastic processes, there was interest among mathematicians, and in the course of a few decades an abundance of new models with similar but qualitatively different interaction schemes were introduced and analyzed, primarily by computer-based stochastic simulations. The survey article [1] gives a broad overview of the different models and their analyses and applications.

Despite their radical limitations in terms of complexity, these models increasingly attracted the attention of the social sciences and were used to describe group behavior on an elementary level and to explain real-life phenomena. Deffuant *et al.* [3] suggested a simple model that features a bounded confidence restriction: neighbors talk to each other in pairs and their opinions are updated towards a compromise only if the opinions they hold just before they meet are not further apart than a given threshold. This is meant to capture the realistic phenomenon that people tend to modify their attitude on a specific topic when talking to others, but not if they consider the views of their discussion partner as so alien as to seem like complete nonsense.

In mathematical terms, the *Deffuant model* is structured as follows. First, we are given a simple connected graph $G = (V, E)$ that shapes the underlying network. The vertices are understood to represent agents holding individual opinions on a certain topic. The edges of the

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graph are supposed to represent the connections between these individuals and entail a possible mutual influence among neighbors. The vertex set V can be either finite or countably infinite. In the latter case, the maximal degree in G is commonly assumed to be bounded.

Then there are two model parameters: the already mentioned confidence bound $\theta > 0$ and the convergence parameter $\mu \in (0, \frac{1}{2}]$, shaping the step-size towards a compromise when two opinions are updated. Opinions usually take values in $\mathbb{R}$. A higher-dimensional analog was considered in [9] and here we will extend the model further. For these generalizations, we need to specify a metric d that is used to measure the distance of two opinions and replaces the absolute value used in the original model.

The first source of randomness is the configuration of initial opinions. Even though there have been attempts to look at settings with dependent initial opinions (see, e.g. Section 2.2 in [7]) the usual setting is to take independent and identically distributed (i.i.d.) initial opinions which then evolve dependencies by interacting. The opinion value at vertex $v \in V$ and time $t \geq 0$ will be denoted by $\eta_t(v)$.

The second source of randomness in the model is the succession of pairwise encounters. On finite graphs, the next pair of neighbors to meet is picked uniformly at random. On infinite graphs, the corresponding equivalent is to assign i.i.d. Poisson processes to all edges of the graph. Whenever a *Poisson event* occurs on the edge $e = \langle u, v \rangle$, i.e. a jump in the Poisson process associated with e, the agents u and v interact in the following way: assume the event happens at time t and the opinions of u and v just before are given by $\eta_{t-}(u) = \lim_{s \uparrow t} \eta_s(u) =: a$ and $\eta_{t-}(v) = \lim_{s \uparrow t} \eta_s(v) =: b$, respectively. Then, depending on the distance of a and b, there might be an update according to the following rule:

$$\eta_t(u) = \begin{cases} a + \mu(b-a) & \text{if } d(a,b) \leq \theta, \\ a & \text{otherwise,} \end{cases}$$

and, similarly, (1.1)

$$\eta_t(v) = \begin{cases} b + \mu(a-b) & \text{if } d(a,b) \leq \theta, \\ b & \text{otherwise.} \end{cases}$$

Given our assumptions, E is countable, so there will almost surely be neither two simultaneous Poisson events nor a limit point in time for the Poisson events on edges incident to one fixed vertex. This guarantees the well definedness of the process by (1.1) for finite G. The extension to infinite graphs with bounded degree is not immediately obvious but a standard argument; see [13, Theorem 3.9].

When it comes to the long-term behavior of the system, it is quite natural to ask whether or not the agents will form a consensus to which all the opinions converge. Let us properly define and distinguish the following two opposing asymptotics of the Deffuant model as time tends to ∞.

Definition 1.1. (i) *Disagreement*. There will be finally blocked edges, i.e. edges $e = \langle u, v \rangle \in E$ such that

$$d(\eta_t(u), \eta_t(v)) > \theta$$

for all times t large enough. Hence, the vertices fall into different opinion groups that are incompatible with neighboring ones.

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(ii) *Consensus*. The value at every vertex converges, as $t \to \infty$, to a common limit l, where

$$l = \begin{cases} \dfrac{1}{|V|} \displaystyle\sum_{v \in V} \eta_0(v) & \text{if } G \text{ is finite,} \\ \mathbb{E}\eta_0 & \text{if } G \text{ is infinite,} \end{cases}$$

and $\mathcal{L}(\eta_0)$ denotes the common marginal distribution of the initial opinion values.

Even though these two regimes intuitively seem to be complementary, for infinite graphs it is far from obvious that the asymptotics of the model is necessarily given by one or the other; see [7, Definition 1.1] and also the remark at the end of Section 5.

Initially, the studies concerning the long-term behavior of the model were all based on computer simulation experiments performed for diverse finite networks (such as complete graphs, square lattices, and even random graphs; see, e.g. [3] and [4]). Later, this approach was extended to the generalization of the Deffuant model to higher-dimensional opinion values; see, e.g. [14].

In this paper we consider the two-sided infinite path $\mathbb{Z}$ as an underlying network topology, i.e. $V = \mathbb{Z}$ and $E = \{\langle v, v+1\rangle,\ v \in \mathbb{Z}\}$. The first result for the model on $\mathbb{Z}$ is due to Lanchier [12], who showed a sharp phase transition from almost sure disagreement to almost sure consensus at $\theta = \frac{1}{2}$, given initial opinions that are independent and uniformly distributed on the unit interval, i.e. $\mathcal{L}(\eta_0) = \mathrm{unif}([0,1])$. This critical value was, in fact, originally conjectured for general networks by Fortunato [4]. Shortly after Lanchier, Häggström [6] reproved the rigorous result for $\mathbb{Z}$, using a quite different approach; this was then extended to general univariate distributions for $\mathcal{L}(\eta_0)$ by Häggström and Hirscher [7] as well as Shang [17] independently. In [9], the Deffuant model on $\mathbb{Z}$ with vector-valued opinions and more general distance measures was investigated.

Note, however, that the extension of the original model to higher-dimensional opinions (as in [9] and [14]) does not coincide with the vectorial model proposed in the original paper [3]: apart from the above, they introduced another model in which opinions are given by binary vectors of fixed length and interactions proceed as follows. When two neighbors meet and their opinions' Hamming distance (number of differing bits) does not exceed a given threshold, one of the two agents is randomly chosen to pick a bit on which they disagree (uniformly at random) and flip it in order to adopt the other's opinion with probability $\mu \in [0,1]$. Even though of similar type, the finite opinion space forces an adaptive behavior (not average preserving), which makes this model qualitatively quite different from the original model described in (1.1).

One aspect that can be considered unrealistic in all these models is the fact that even though opinions are random, for a fixed realization they are given by numbers or vectors, in other words entirely determined by the initial configuration and the succession of interactions—not doing justice to the common phenomenon of inconclusiveness in people's opinions. In this regard, it seems to be a more reasonable choice to shape individual opinions by probability distributions, thus accounting for uncertainty. In what follows, we are thus going to introduce and analyze a variant of the Deffuant model on $\mathbb{Z}$, where the opinions are given by random absolutely continuous measures on $[0,1]$. The support of these measure-valued opinions can be seen to represent the degree of indecision: the more concentrated the measure, the more determined the agent. An extension in this direction could pave the way to more realistic simulation models that might be useful in applications, as we will briefly discuss in the final section.

As a general preparation for the extension to measure-valued opinions, in Section 2 we will introduce the total variation distance (which will be used as the distance measure) and recall a

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strong law of large numbers (SLLN) for continuous densities, due to Rubin [16], replacing the common SLLN which was a crucial ingredient in the case of finite-dimensional opinions.

The Deffuant model with measure-valued opinions, outlined already at the beginning of Section 2, is properly described at the beginning of Section 3. We consider random symmetric triangular distributions as initial opinions and find that for this setting overly determined agents (i.e. agents whose initial opinion is concentrated on sufficiently short intervals) prevent consensus for all $\theta \in [0, 1)$; see Theorem 3.1.

The results for finite-dimensional opinion spaces listed above will be sketched in more detail in Section 4. Central ideas from [6] will be presented as they prove to be useful in our setting as well.

In Section 5 the main result for the setting with unrestricted symmetric triangular distributions, Theorem 3.1, is proved. We show that the behavior of the model is trivial in this case as extremely determined agents will have and keep a total variation distance close to 1 to their neighbors' opinions.

However, if we put a restriction on the (initial) determination of the agents by disallowing triangular distributions that have a support of length less than a fixed value γ, the familiar phenomenon of a phase transition in θ from almost sure (a.s.) disagreement to a.s. consensus reappears. This case, as well as the precise dependency of the threshold value θ_c on the parameter γ, is elaborated in Section 6. In the final section, we briefly discuss possible other frameworks for the Deffuant model with infinite-dimensional opinion space and earlier attempts to incorporate inhomogeneous open mindedness of the agents.

2. Distance and convergence of absolutely continuous random measures

As indicated, we want to generalize the Deffuant model on $\mathbb{Z}$ further by looking at opinions that are no longer numbers or vectors but probability measures instead. These random distributions can be seen to shape indeterminacy in the agents: even with initial opinion profile and sequence of encounters fixed, the opinion of an individual at a given time is not a fixed value but a probability measure. Initially, the agents are independently assigned random measures from a common distribution. When they meet and their current opinion measures do not differ by more than θ with respect to a fixed metric on the space of probability measures, the new opinions will be given by convex combinations of the old ones, as described in (1.1).

In order to quantify the difference between two distributions there are quite a few metrics to choose from. The so-called total variation distance is among the most commonly used.

Definition 2.1. Let μ and ν be two probability distributions on a set S. The *total variation distance* between the two measures is then defined as

$$\|\mu - \nu\|_{\mathrm{TV}} := \sup_{A \subseteq S} |\mu(A) - \nu(A)|.$$

As the total variation distance of two probability distributions is a number in $[0, 1]$, the nontrivial values for θ lie in $(0, 1)$ for this model. Further, note that the total variation distance of two probability distributions μ and ν, that are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and have densities f and g respectively, is given by

$$\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2} \int_{\mathbb{R}} |f(x) - g(x)| \, \mathrm{d}x.$$

In addition, if μ and ν are distributions on $[0, 1]$, it can immediately be concluded that $\|\mu - \nu\|_{\mathrm{TV}} \leq \frac{1}{2}\|f - g\|_\infty$, where $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$ denotes the supremum norm on $[0, 1]^{\mathbb{R}}$.

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To be able to transfer the findings from the Deffuant model on $\mathbb{Z}$ featuring real- or vector-valued opinions, we further need an equivalent for the SLLN geared towards the densities of random measures. The following result of Rubin [16] serves our purposes. Let $U, U_1, U_2, \ldots$ denote a sequence of i.i.d. random variables with values in an arbitrary space Y and let their common distribution be denoted by $\mu := \mathcal{L}(U)$. Given a compact topological space X, consider a map $f\colon X \times Y \to \mathbb{R}$ that is measurable in the second argument for each $x \in X$.

Theorem 2.1. (SLLN for continuous densities.) *If there exists an integrable function g on Y such that $|f_y(x)| \leq g(y)$ for all $x \in X$ and $y \in Y$, as well as a sequence of measurable sets $(S_i)_{i\in\mathbb{N}}$ with*

$$\mathbb{P}\Big(U \in \bigcap_{i\in\mathbb{N}} S_i^{\mathrm{c}}\Big) = 0,$$

and the property that $\{f_y(\cdot),\ y \in S_i\}$ is equicontinuous on X for all $i \in \mathbb{N}$, then with probability 1*,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f_{U_i}(x) = \mathbb{E} f_U(x) = \int_Y f_y(x)\mu(\mathrm{d}y)$$

uniformly in $x \in X$ and the limit function is continuous.

The measure on X with density $\mathbb{E}f_U$ is commonly called the *intensity* or *first moment measure*; see, e.g. [10, Section 1.2] for a more general introduction. The idea of an intensity measure comes from the domain of point processes, but also in this form with density functions, there are natural interpretations from physics. For instance, consider a particle that can be in different states $y \in Y$ and assume that f_y describes the distribution of its energy level when in state y. If the random state of the particle is given by the variable U then $\mathbb{E}f_U$ describes the weighted distribution of energy values. In our context, the intensity measure will correspond to the relative popularity of single opinion values when averaged over the opinion measures of many independent agents.

3. Initial opinions given by random triangular distributions

For concreteness, let us pick the initial opinions from a specific class of absolutely continuous distributions. A rather natural choice, departing from real-valued opinions (which can be seen as Dirac delta measures), are symmetric triangular distributions on random subintervals of [0, 1], the endpoints of which are chosen uniformly from [0, 1].

More precisely, let us consider the initial opinions $\{\eta_0(v),\ v \in \mathbb{Z}\}$ to be picked in the following way. Consider $\{U(v),\ v \in \mathbb{Z}\}$ to be an i.i.d. sequence of unif$([0,1]^2)$ random vectors. The node v will be assigned an initial opinion given by the random absolutely continuous probability measure with density

$$\begin{aligned} f_0^{(v)}(x) &= \begin{cases} 0, & x \notin (m, M), \\ \left(\dfrac{2}{M-m}\right)^2 (x-m), & x \in (m, (m+M)/2], \\ -\left(\dfrac{2}{M-m}\right)^2 (x-M), & x \in ((m+M)/2, M), \end{cases} \\ &= \frac{2}{|y-z|}\left(1 - \frac{2}{|y-z|}\left|x - \frac{y+z}{2}\right|\right)^+, \quad x \in [0,1], \end{aligned} \tag{3.1}$$

where $U(v) = (y, z)$, $m := \min\{y, z\}$, and $M := \max\{y, z\}$; see Figure 1.

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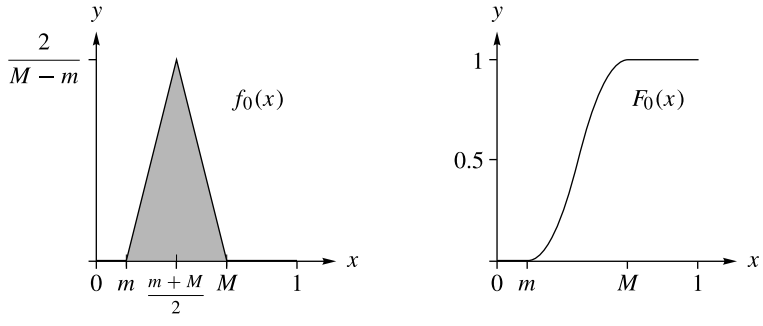


FIGURE 1: The density and distribution function of a symmetric triangular distribution on $[m, M]$.

Seen from a different angle, to determine the initial opinion of a fixed agent, we first choose a central opinion value C uniformly from $[0, 1]$ and then a spread for the support of the distribution, uniformly among $[0, \min\{C, 1 - C\}]$. The fact that this procedure is equivalent to the one described above is an immediate consequence of the change of variable formula; see the proof of Lemma 5.1 (especially Figure 3) for more details.

Note that this model features two qualitatively different forms of extreme initial opinions. On the one hand—as in the original model—the agents can have opinions lying at the edges of the spectrum (i.e. concentrated close to 0 or 1 in this case). On the other hand, an individual opinion can be very determined in the sense that $U(v)$ is close to the diagonal (i.e. $|y - z|$ very small), which provokes a highly concentrated density and necessarily a large distance to a vast majority of possible initial opinions.

This effect is quite realistic: irrespectively of their opinion being exceptional or mainstream, people that are extremely narrow minded or determined are usually neither willing to consider the opinion nor to accept the arguments of others, let alone to compromise. In this sense, even though the mathematics is closely related to the case of finite-dimensional opinions, the extension of the model to measure-valued opinions introduces an additional real-life phenomenon.

For symmetric triangular distributions on $[0, 1]$ without any restriction on the minimal length of their support, we show that the model exhibits a trivial behavior.

Theorem 3.1. *Consider the Deffuant model on $\mathbb{Z}$, where the initial opinions are given by independently assigned random triangular distributions as described in (3.1). Then, for all $\theta \in [0, 1)$, the system almost surely approaches disagreement in the long run if the total variation distance is used to measure the distance between two opinions.*

For the proof of this result, we refer the reader to Section 5. Since this setting allows us to reuse results from the finite-dimensional case, we will first give a brief overview of these in the following section.

4. Background

As mentioned in the introduction, the first analytic result concerning consensus formation in the Deffuant model on $\mathbb{Z}$ was established by Lanchier [12] and dealt with opinion profiles that are initially given by an i.i.d. sequence of unif$([0, 1])$ random variables. The distance between two opinions was taken to be the absolute value of their difference. Häggström [6] used different techniques to reprove and slightly sharpen this result. His arguments were later adapted to accommodate other univariate initial distributions as well, leading to an analog

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covering all marginal distributions that have a first moment $\mathbb{E}\eta_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$; see [7, Theorem 2.2].

Theorem 4.1. *Consider the Deffuant model on the graph $(\mathbb{Z}, E)$, where the edge set is given by $E = \{\langle v, v+1\rangle,\ v \in \mathbb{Z}\}$ with fixed parameter $\mu \in (0, \frac{1}{2}]$. Let the initial configuration be given by an i.i.d. sequence of real-valued random variables having the common distribution $\mathcal{L}(\eta_0)$, and the distance of two opinions by the absolute value of their difference.*

(i) *Given a bounded distribution $\mathcal{L}(\eta_0)$ with expected value $\mathbb{E}\eta_0$, let $[a, b]$ denote the smallest closed interval containing its support. If $\mathbb{E}\eta_0$ does not lie in the support, let $I \subset [a, b]$ denote the maximal, open interval with $\mathbb{E}\eta_0 \in I$ and $\mathbb{P}(\eta_0 \in I) = 0$. In this case, set h to be the length of I, otherwise set $h = 0$.*
Then the critical value for θ, marking the phase transition from a.s. disagreement to a.s. consensus, becomes $\theta_c = \max\{\mathbb{E}\eta_0 - a, b - \mathbb{E}\eta_0, h\}$. The common limit value in the supercritical regime is $\mathbb{E}\eta_0$.

(ii) *Suppose that the distribution $\mathcal{L}(\eta_0)$ is unbounded but its expected value exists, i.e. $\mathbb{E}\eta_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$. Then the Deffuant model with arbitrary fixed parameter $\theta \in (0, \infty)$ will almost surely behave subcritically, meaning that disagreement will be approached in the long run.*

With an appropriate adaptation to the more involved geometry of vector-valued opinions, the main ideas in [6] further served to establish similar results for the Deffuant model on $\mathbb{Z}$ with opinion space $\mathbb{R}^d$, $d > 2$, and more general distance measures; see [9, Theorems 3.15 and 4.11].

Since the same line of reasoning was used in [6], [7], and [9] to derive the results for the finite-dimensional opinion spaces we have just mentioned, let us now take a closer look at the key concepts and crucial auxiliary results. They form the basis for most of the conclusions we will be able to draw in the case of infinite-dimensional opinions.

In [6], Häggström presented two central ideas, whose effect turns out to be highly limited to paths, but combined they prove to be quite powerful in the analysis of the Deffuant model on the infinite path $\mathbb{Z}$. The first one is the notion of *flat points*.

Definition 4.1. Consider the initial i.i.d. configuration $\{\eta_0(u)\}_{u\in\mathbb{Z}}$ with common marginal distribution $\mathcal{L}(\eta_0)$ on an opinion space X (e.g. $\mathbb{R}$ or $\mathbb{R}^d$), which we consider to be equipped with the metric ρ. Under the premise that the mean $\mathbb{E}\eta_0$ of the initial distribution exists and is finite, given $\varepsilon > 0$, a vertex $v \in \mathbb{Z}$ is called *ε-flat to the right* (with respect to the initial configuration), if, for all $n \geq 0$,

$$\frac{1}{n+1}\sum_{u=v}^{v+n} \eta_0(u) \in B_\varepsilon(\mathbb{E}\eta_0),$$

where $B_r(x) := \{y \in X,\ \rho(x, y) \leq r\}$ denotes the (closed) ρ-ball around $x \in X$ with radius $r > 0$. A vertex v is called *ε-flat to the left* if the above condition is met, with the sum running from $v - n$ to v instead. Finally, v is called *two-sidedly ε-flat*, if, for all $m, n \geq 0$,

$$\frac{1}{m+n+1}\sum_{u=v-m}^{v+n} \eta_0(u) \in B_\varepsilon(\mathbb{E}\eta_0).$$

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The crucial role that vertices, which are one- or two-sidedly ε-flat with respect to the initial configuration can play in the further evolution of the configuration becomes more obvious in the light of the second key idea, the nonrandom pairwise averaging procedure Häggström [6] proposed to call *sharing a drink* (SAD) on $\mathbb{Z}$.

Glasses are put along the infinite path at all integers; the one at site 0 is full, all others are empty. Similarly to the Deffuant model, neighbors interact and share, but now we skip randomness and confidence bound: the procedure starts with the initial profile $\{\xi_0(v)\}_{v\in\mathbb{Z}}$, given by $\xi_0(0) = 1$ and $\xi_0(v) = 0$ for all $v \neq 0$. In each step, we choose an edge, along which an update of the form (1.1) is executed; more precisely, if we are given the profile $\{\xi_n(v)\}_{v\in\mathbb{Z}}$ after step n and choose $\langle u, u+1\rangle$ for the next round, we obtain

$$\xi_{n+1}(u) = (1-\mu)\xi_n(u) + \mu\xi_n(u+1), \qquad \xi_{n+1}(u+1) = \mu\xi_n(u) + (1-\mu)\xi_n(u+1),$$
$$\xi_{n+1}(v) = \xi_n(v) \quad \text{for all } v \notin \{u, u+1\}.$$

The resulting profiles, after we have performed this procedure a finite number of rounds, will be called *SAD-profiles*. Besides the facts that they feature only finitely many nonzero elements, and the elements are all positive and sum to 1, there are less obvious properties that these profiles share which we will collect in the following lemma (for proofs, see [6, Lemmas 2.1 and 2.2, and Theorem 2.3]).

Lemma 4.1. *Consider the SAD-procedure on the infinite path $\mathbb{Z}$, started in vertex v, i.e. with $\xi_0(u) = \delta_v(u)$, $u \in V$. Then we obtain the following.*

(i) *All achievable SAD-profiles are unimodal.*

(ii) *If the vertex v shares the drink only to one side, it will remain a mode of the SAD-profile.*

(iii) *The supremum over all achievable SAD-profiles started with δ_v at another vertex w is equal to $1/(d+1)$, where d is the graph distance between v and w.*

The connection to the Deffuant model was established in [6, Lemma 3.1]: the opinion value $\eta_t(0)$ at any given time $t > 0$ can be written as a weighted average of the initial opinions, where the weights are given by the (random) SAD-profile which is dual to the dynamics in the Deffuant model in the sense that the order of updates has to be reversed.

Combining this link with the concept of ε-flatness makes it possible to derive the following crucial auxiliary results (which are obvious generalizations of intermediate results, established in the proofs of [6, Proposition 5.1 and Lemma 6.3]).

Lemma 4.2. *Consider the Deffuant model on $\mathbb{Z}$ with the initial configuration given by an i.i.d. sequence of random variables having the common marginal distribution $\mathcal{L}(\eta_0)$.*

(i) *If vertex v is ε-flat to the right with respect to the initial configuration and does not interact with vertex $v-1$, its opinion stays inside $B_\varepsilon(\mathbb{E}\eta_0)$. The same holds for ε-flatness to the left and with $v+1$ in place of $v-1$.*

(ii) *If vertex v is two-sidedly ε-flat with respect to the initial configuration, its opinion value will stay inside $B_{6\varepsilon}(\mathbb{E}\eta_0)$, irrespectively of the dynamics.*

Without much further work, these findings can be used to analyze the behavior of the model featuring unrestricted symmetric triangular distributions, as we will see in the following section.

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5. Overly determined agents prevent consensus

As the expectation of the initial distribution played a central role in the model featuring real- or vector-valued opinions, we first have to obtain its counterpart in the context of random measures, the intensity, before we can set about proving Theorem 3.1.

Lemma 5.1. *Consider the absolutely continuous random measure η to be given by the density*

$$f_U(x) = \frac{2}{|y-z|}\left(1 - \frac{2}{|y-z|}\left|x - \frac{y+z}{2}\right|\right)^+, \qquad x \in [0,1], \tag{5.1}$$

where $U = (y, z)$ is taken uniformly from the unit square $[0,1]^2$, as introduced in (3.1). Then its intensity measure (commonly denoted $\mathbb{E}\eta$) is given by the density

$$\varphi(x) = \begin{cases} -8[(1-x)\ln(1-x) + x(1-\ln(2))], & x \in \left[0, \frac{1}{2}\right], \\ -8[x\ln(x) + (1-x)(1-\ln(2))], & x \in \left[\frac{1}{2}, 1\right]; \end{cases}$$

see Figure 2.

Proof. Fix $x \in (0,1)$. First, by symmetry, we can take U to be uniform on the set $A := \{(y,z) \in \mathbb{R}^2,\ 0 \le z \le y \le 1\}$. To further simplify the calculations, let us consider the simple linear transform $T((y,z)) = \frac{1}{2}(y+z, y-z)$, depicted in Figure 3. From the change of variable formula, we know that $T(U)$ is uniform on the set $B := \{(u,v),\ v \in [0, \frac{1}{2}],\ u \in [v, 1-v]\}$.

Given the random density f_U as in (5.1), we can write

$$f_{T(U)}(x) = \frac{1}{v}\left(1 - \frac{1}{v}|x-u|\right)^+, \qquad x \in [0,1]$$

and conclude that $f_{(u,v)}(x)$ is nonzero for (u,v) in $B_x = B_1 \cup B_2 \cup B_3$, where

$$B_1 := \left\{(u,v),\ v \in \left[0, \frac{x}{2}\right],\ u \in [x-v, x+v]\right\},$$
$$B_2 := \left\{(u,v),\ v \in \left[\frac{x}{2}, \frac{1-x}{2}\right],\ u \in [v, x+v]\right\},$$
$$B_3 := \left\{(u,v),\ v \in \left[\frac{1-x}{2}, \frac{1}{2}\right],\ u \in [v, 1-v]\right\}.$$

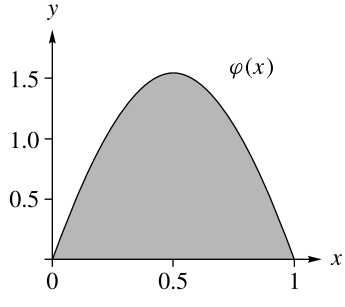


FIGURE 2: Density of the intensity measure corresponding to random symmetric triangular distributions on [0, 1].

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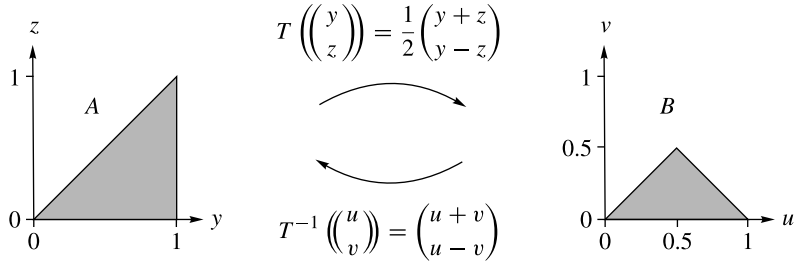


FIGURE 3: Using transform T to consider the arithmetic mean and half the distance of two independent unif$([0, 1])$ random variables instead.

Hence, for $x \in [0, \frac{1}{2}]$, tedious but elementary calculations lead to

$$\varphi(x) := \mathbb{E}[f_U(x)] = 4 \iint_{B_x} \frac{1}{v} - \frac{|x-u|}{v^2} \, \mathrm{d}u \, \mathrm{d}v = -8(1-x)\ln(1-x) - 8x(1-\ln(2)).$$

By symmetry around $x = \frac{1}{2}$, the claim follows. □

Proof of Theorem 3.1. As usual, let $\eta_t(v)$ denote the opinion of individual $v \in \mathbb{Z}$ at time $t > 0$ and, further, let $f_t^{(v)}$ be the density corresponding to this random measure. For any fixed $\delta > 0$, let us define the random variables

$$F_t^{(v)}(\delta) := \eta_t(v)([0, \delta]) = \int_0^\delta f_t^{(v)}(x) \, \mathrm{d}x \quad \text{for all } t > 0, \ v \in \mathbb{Z}.$$

Their values lie in the interval $[0, 1]$, which actually coincides with the support of their distributions. Further, we know that $\{F_0^{(v)}(\delta), \ v \in \mathbb{Z}\}$ are i.i.d. random variables and Fubini's theorem yields

$$\mathbb{E}[F_0^{(v)}(\delta)] = \int_0^\delta \varphi(x) \, \mathrm{d}x. \tag{5.2}$$

We can disregard the $\theta = 0$ case, since there will not be any dynamics and, hence, a.s. disagreement. Given $\theta \in (0, 1)$, define $\varepsilon := \frac{1}{2}(1-\theta) > 0$ and choose $\delta > 0$ such that $\int_0^\delta \varphi(x) \, \mathrm{d}x < \varepsilon$.

As mentioned above, the support of the distribution of $F_0^{(v)}(\delta)$ is $[0, 1]$ (without gaps); hence, we can conclude as in [6, Lemma 4.2] that any vertex is (one-sidedly) ε-flat with positive probability with respect to the sequence $\{F_0^{(v)}(\delta), \ v \in \mathbb{Z}\}$. Due to $\mathbb{P}(F_0^{(v)}(\delta) = 1) = \delta^2 > 0$ and independence, the coincidence of the following two events occurs with positive probability for any $v \in \mathbb{Z}$:

- vertex $v - 1$ is ε-flat to the left and vertex $v + 1$ is ε-flat to the right with respect to $\{F_0^{(v)}(\delta), \ v \in \mathbb{Z}\}$;
- $F_0^{(v)}(\delta) = 1$.

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Using Lemma 4.2(i) and the same line of reasoning as in the proof of [6, Proposition 5.1], we conclude that the edge $\langle v-1, v\rangle$—and, similarly, $\langle v, v+1\rangle$—will be blocked forever, as

$$\begin{aligned}\|\eta_t(v) - \eta_t(v-1)\|_{\mathrm{TV}} &\geq |F_t^{(v)}(\delta) - F_t^{(v-1)}(\delta)| \\ &= F_0^{(v)}(\delta) - F_t^{(v-1)}(\delta) \\ &\geq 1 - (\mathbb{E}F_0^{(v-1)}(\delta) + \varepsilon) \\ &> 1 - 2\varepsilon \\ &= \theta.\end{aligned}$$

From the fact that approaching disagreement is shift-invariant, hence a 0-1-event, we conclude that for $\theta \in [0, 1)$ there will be a.s. disagreement. □

Remark 5.1. The trivial case $\theta = 1$ in Theorem 3.1 will surely not lead to blocked edges, so disagreement can be ruled out. However, this does not necessarily imply a consensus formation. The standard energy argument (which we will also use in Lemma 6.4) fails, since the random symmetric triangular distribution does not have a finite second moment, i.e. $\mathbb{E}[f_U(x)]^2 = \infty$ for all $x \in (0, 1)$.

Using the results for univariate opinions once more, we can, however, conclude consensus for $\theta = 1$ if we change to a different distance measure: the so-called Lévy distance. Consider two probability distributions μ and ν on $[0, 1]$. Their *Lévy distance* $\rho(\mu, \nu)$ is defined as the infimum of the set

$$\{\varepsilon > 0 \text{ such that } \mu([0, x-\varepsilon]) - \varepsilon \leq \nu([0, x]) \leq \mu([0, x+\varepsilon]) + \varepsilon \text{ for all } x \in [0, 1]\}.$$

To settle the case with $\theta = 1$ and ρ as a distance measure, let us consider the univariate case, where $\{F_t^{(v)}(\delta) = \eta_t(v)([0, \delta]),\ t > 0,\ v \in \mathbb{Z}\}$ are the opinions assigned to the agents: as there is no bounded confidence restriction, any encounter leads to an update and the update rule (1.1) applies to both $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ and $\{F_t^{(v)}(\delta)\}_{v\in\mathbb{Z}}$.

Hence, for any fixed $\delta \in [0, 1]$, from Theorem 4.1 and (5.2), we know that $F_t^{(v)}(\delta)$ converges to $\Phi(\delta) := \int_0^\delta \varphi(x)\,\mathrm{d}x$ almost surely. Consequently, with probability 1, it holds that

$$\lim_{t\to\infty} F_t^{(v)}(\delta) = \Phi(\delta) \quad \text{for all } v \in \mathbb{Z},\ \delta \in [0, 1] \cap \mathbb{Q}.$$

Since all $F_t^{(v)}$ and Φ are continuous and increasing, this implies a.s. pointwise (in fact, even uniform) convergence. In other words, for any $v \in \mathbb{Z}$, the opinion measure $\eta_t(v)$ converges with probability 1 *vaguely* to the intensity measure $\mathbb{E}\eta$, having density φ. As vague convergence of measures on a compact interval is metrized by the corresponding Lévy metric (see, e.g. [5, Lemma 2]), this implies that $\lim_{t\to\infty} \rho(\eta_t(v), \mathbb{E}\eta) = 0$ almost surely for all $v \in \mathbb{Z}$.

Note that a.s. consensus for $\theta = 1$ and the Lévy metric does not immediately imply a result for the total variation case, as $\rho(\mu, \nu) \leq \|\mu - \nu\|_{\mathrm{TV}}$ for two probability measures μ and ν.

6. Agents with bounded determination

In order to obtain a nontrivial phase transition in the parameter θ, let us now consider a situation in which all the agents feature at least a certain minimum of open mindedness. This will be incorporated in our model by disallowing the initial random measure to be concentrated on a subinterval of length less than γ, for a fixed constant $\gamma \in (0, 1)$. We will refer to these as random *restricted triangular distributions*.

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Before we can show the main result, Theorem 6.1 (which states that there is a phase transition and gives the precise threshold value for the parameter θ), we need to study the altered intensity measure and verify a few auxiliary results that are needed to guarantee the existence of ε-flat vertices; see Lemma 6.6.

Lemma 6.1. *For fixed $\gamma \in (0,1)$, consider the absolutely continuous random measure $\eta^{(\gamma)}$ to be given by the density*

$$f_U(x) = \frac{2}{|y-z|}\left(1 - \frac{2}{|y-z|}\left|x - \frac{y+z}{2}\right|\right)^+, \qquad x \in [0,1], \tag{6.1}$$

where $U = (y,z)$ is taken uniformly from the set $\{y, z \in [0,1],\ |y-z| \geq \gamma\}$ (and note that this corresponds to the expression in (3.1), conditional on the support being an interval of length at least γ). Then the density of its intensity measure $\mathbb{E}\eta^{(\gamma)}$ is given by the following expressions (assuming $0 \leq x \leq \frac{1}{2}$):

(i) *for $x \geq \gamma$,*

$$\varphi_\gamma(x) = -\frac{8}{(1-\gamma)^2}\left[(1-x)\ln(1-x) + x(1-\ln(2)) + \frac{\gamma}{4}\right];$$

(ii) *for $x \geq 1-\gamma$,*

$$\varphi_\gamma(x) = \begin{cases} -\dfrac{8}{(1-\gamma)^2}\left[(1-x)\ln\gamma + x + \dfrac{1-2x}{2\gamma} - \dfrac{\gamma}{2}\right], & x \leq \gamma/2, \\ -\dfrac{8}{(1-\gamma)^2}\left[-x\ln(2x) + \ln\gamma + x + \dfrac{(1-x)^2+x^2}{2\gamma} - \dfrac{3}{4}\gamma\right], & x \geq \gamma/2; \end{cases}$$

(iii) *for $x \leq \gamma$, $x \leq 1-\gamma$,*

$$\varphi_\gamma(x) = \begin{cases} -\dfrac{8}{(1-\gamma)^2}\left[(1-x)\ln(1-x) + x - \dfrac{x^2}{2\gamma}\right], & x \leq \gamma/2, \\ -\dfrac{8}{(1-\gamma)^2}\left[(1-x)\ln(1-x) - x\ln\left(\dfrac{2x}{\gamma}\right) + x + \dfrac{x^2}{2\gamma} - \dfrac{\gamma}{4}\right], & x \geq \gamma/2. \end{cases}$$

The corresponding expressions for $x \in [\frac{1}{2}, 1]$ are obtained by replacing x by $1-x$.

Proof. As in the proof of Lemma 5.1, we can take U to be uniform on the set $A_\gamma := \{(y,z) \in \mathbb{R}^2,\ \gamma \leq y \leq 1,\ 0 \leq z \leq y-\gamma\}$ and consider the very same linear transform T; see Figure 4.

Then $T(U)$ is uniform on the set $B_\gamma := \{(u,v),\ v \in [\gamma/2, \frac{1}{2}],\ u \in [v, 1-v]\}$ and the corresponding random density is still

$$f_{T(U)}(x) = \frac{1}{v}\left(1 - \frac{1}{v}|x-u|\right)^+, \qquad x \in [0,1].$$

Depending on the values of $x \in [0, \frac{1}{2}]$ and $\gamma \in (0,1)$ (see Figure 5 for an illustration) quite cumbersome but nevertheless elementary calculations in the same vein as in the proof of Lemma 5.1 (which we will leave to the reader to perform) lead to the formulae stated above.

The last claim follows again by the symmetry in x. □

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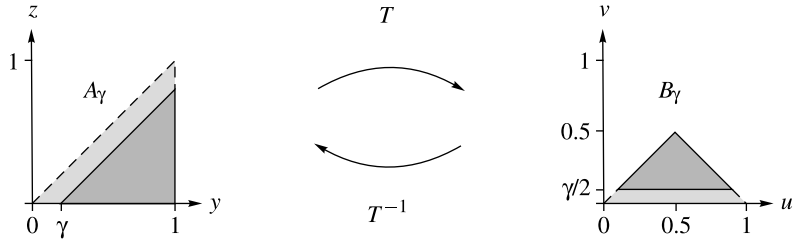


FIGURE 4: The restricted set A_γ forces a minimum amount of open mindedness.

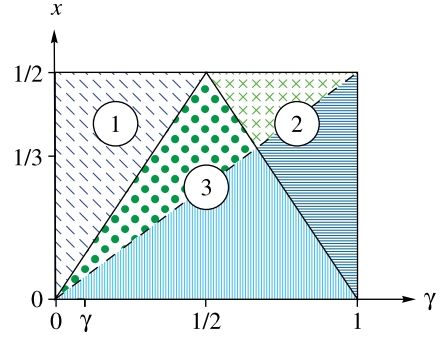


FIGURE 5: Different regimes for the form of $\varphi_\gamma(x)$.

Lemma 6.2. *Consider $\mathbb{E}\eta^{(\gamma)}$ as in the previous lemma. Irrespectively of the value of $\gamma \in (0, 1)$, the density function $\varphi_\gamma(x), x \in [0, 1]$, corresponding to the intensity measure, is (strictly) increasing on $[0, \frac{1}{2}]$ and (strictly) decreasing on $[\frac{1}{2}, 1]$.*

Proof. In principle, one could simply check the expressions for φ_γ given in Lemma 6.1. However, this simple fact can also be seen directly from the construction. Let us consider the random density in (6.1) to be generated by a vector $T = (U, V)$ that is chosen uniformly from B_γ, as described in the proof of Lemma 6.1. After having picked $V \sim \text{unif}([\gamma/2, \frac{1}{2}])$, we take U to be uniform on $[V, 1 - V]$.

Consider $x_1, x_2 \in [0, \frac{1}{2}]$, such that $x_1 < x_2$, and $V = v$ to be already fixed. First note that $f_T(x_1) < f_T(x_2)$ for $U > (x_1 + x_2)/2 \in (0, \frac{1}{2})$. If $v \le (x_1 + x_2)/2$, symmetry around $(x_1+x_2)/2$ shows that $f_{(U,v)}(x_1)$ and $f_{(U,v)}(x_2)$ have the same distribution for U conditioned on $[v, x_1+x_2-v]$. In conclusion, we obtain the first-order stochastic dominance $f_T(x_1) \prec f_T(x_2)$ and, in particular,

$$\varphi_\gamma(x_1) = \mathbb{E} f_T(x_1) < \mathbb{E} f_T(x_2) = \varphi_\gamma(x_2).$$

Symmetry of φ_γ around $x = \frac{1}{2}$ implies the second part of the claim. □

Note that φ_γ cannot be arbitrarily well approximated by the density of a restricted triangular distribution. Consequently, for $\varepsilon > 0$ sufficiently small, there cannot be any ε-flat vertices with respect to the initial configuration, as all triangular distributions have a positive distance to the intensity measure $\mathbb{E}\eta^{(\gamma)}$ bounded away from 0. For this reason, we have to make the same detour as in the proof of [7, Theorem 2.2(ii)]. We need to verify that the density of the intensity measure can actually appear at a later time, more precisely be arbitrarily well approximated

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by the opinions that form when agents have interacted. This happens, in fact, for all positive values of the model parameter θ.

Lemma 6.3. *Consider the Deffuant model on $\mathbb{Z}$ with arbitrary parameter $\theta \in (0, 1]$ in which the initial opinions are i.i.d. absolutely continuous measures given by the random densities described in (6.1). Then, at any time $t > 0$ and for any $\varepsilon > 0$, a (sufficiently long) fixed finite section of the infinite path will hold opinions that are less than ε away from $\mathbb{E}\eta^{(\gamma)}$ in total variation distance (and will be bounded by edges on which no Poisson events occurred up to time t) with positive probability.*

Proof. Fix $\theta \in (0, 1]$, $\varepsilon > 0$, and $t > 0$. The idea is to show that a set of agents with suitably assigned initial opinions can interact in such a way that at time t opinions close to $\mathbb{E}\eta^{(\gamma)}$ are formed.

Let us consider an i.i.d. sequence $(U_n)_{n\in\mathbb{N}}$ of random variables uniformly distributed on $A_\gamma = \{(y, z) \in \mathbb{R}^2,\ \gamma \le y \le 1,\ 0 \le z \le y - \gamma\}$; see Figure 4. Then we obtain the density corresponding to the initial opinion for agent $v \in \mathbb{N}$ by

$$f_0^{(v)} = f_{U_v}, \tag{6.2}$$

where f_{U_v} is taken to be as in (6.1) and $f_t^{(v)}$ denotes the random density corresponding to the opinion of agent v at time t.

From Theorem 2.1, we know that, with probability 1,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_0^{(v)}(x) = \varphi_\gamma(x) \quad \text{uniformly in } x \in [0, 1]. \tag{6.3}$$

It is not difficult to check that $\max\{|y_1 - y_2|, |z_1 - z_2|\} \le \delta$ entails

$$\frac{1}{2} \int_0^1 |f_{(y_1,z_1)}(x) - f_{(y_2,z_2)}(x)|\, \mathrm{d}x \le \frac{2\delta}{\gamma}. \tag{6.4}$$

In other words, if the coordinates of two vectors, (y_1, z_1) and (y_2, z_2), shaping restricted triangular distributions in the sense of (6.1) do not differ by more than δ, the total variation distance between the corresponding measures is at most $2\delta/\gamma$.

Fix an integer $m \ge 16/(\gamma\theta)$ and subdivide the unit square $[0, 1]^2$ into m^2 squares. The standard SLLN implies that the fraction of $(U_n)_{n\in\mathbb{N}}$ landing in a square completely contained in A_γ almost surely tends to $2/(m^2(1 - \gamma)^2) > 0$ as $n \to \infty$. We can therefore choose $N \in \mathbb{N}$ large enough such that, with positive probability, every square that is a subset of A_γ contains at least one of $(U_n)_{n=1}^N$ and

$$\left\| \frac{1}{N} \sum_{v=1}^{N} f_0^{(v)} - \varphi_\gamma \right\|_\infty \le \varepsilon.$$

Note that by symmetry under permutations, there is at least a chance of $1/N!$ that the agents $\{1, \ldots, N\}$ are assigned these values from A_γ in such a way that those of neighboring agents do not differ much in both coordinates; more precisely, matching the values in a serpentine fashion as depicted in Figure 6 will keep discrepancies in the y-coordinate below $4/m$ and in the z-coordinate below $3/m$.

Gathering our results, we find that, for large enough N with nonzero probability, the agents 1 through N have an initial configuration with the following two properties. First, the total

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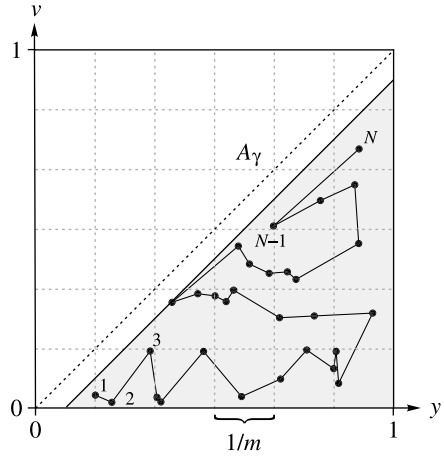


FIGURE 6: Finding suitable values for the initial opinions that can generate opinions close to the intensity measure $\mathbb{E}\eta^{(\gamma)}$ on a finite section.

variation distance of neighboring opinions is at most $8/(\gamma m) \leq \theta/2$, second, their arithmetic mean has a total variation distance of at most $\epsilon/2$ to $\mathbb{E}\eta^{(\gamma)}$.

Assume that there are no updates on the edges $\langle 0, 1\rangle$ and $\langle N, N+1\rangle$ up to time t. It is easy to check (by induction) that in this case, updates on the considered section in sweeps from left to right, i.e. first on $\langle 1, 2\rangle$, then $\langle 2, 3\rangle$ and so on until $\langle N-1, N\rangle$ repetitively, will keep the total variation distance of neighbors inside the section always below θ. Note that in this regard, the exact value of t is irrelevant: if we fix a finite collection of edges, the set of possible interaction patterns on these edges, happening until some time $t > 0$ with positive probability, is the same for every $t > 0$; see the proof of [9, Lemma 3.12].

The following lemma finally verifies that a sufficiently large number of such sweeps will eventually bring the considered opinions within a total variation distance $\varepsilon/2$ of their mean, due to the fact that the mean is preserved, given that there are no updates on neither $\langle 0, 1\rangle$ nor $\langle N, N+1\rangle$.

Since the Poisson clocks and the initial configuration are independent, these two events coincide with positive probability and the claim is verified. □

Lemma 6.4. *If there are infinitely many (performed) updates along an edge, the total variation distance of the corresponding neighbors' opinions almost surely converges to 0.*

Proof. This statement follows immediately using the energy idea used in the proofs of [6, Theorems 2.3 and 5.3]. Consider the energy of vertex v at time t to be given by

$$W_t(v) := \int_0^1 [f_t^{(v)}(x)]^2 \,\mathrm{d}x.$$

By an update along the edge $\langle u, v\rangle$, i.e. when the opinion values $(\eta_{t-}(u), \eta_{t-}(v))$ are replaced by $(\eta_t(u), \eta_t(v)) = ((1-\mu)\eta_{t-}(u) + \mu\eta_{t-}(v), (1-\mu)\eta_{t-}(v) + \mu\eta_{t-}(u))$, energy is lost to the amount of

$$\begin{aligned} 2\mu(1-\mu)\int_0^1 [f_{t-}^{(u)}(x) - f_{t-}^{(v)}(x)]^2 \,\mathrm{d}x &\geq 2\mu(1-\mu)\left(\int_0^1 |f_{t-}^{(u)}(x) - f_{t-}^{(v)}(x)| \,\mathrm{d}x\right)^2 \\ &= 8\mu(1-\mu)\left\|\eta_{t-}(u) - \eta_{t-}(v)\right\|_{\mathrm{TV}}^2. \end{aligned}$$

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As in [6, Lemma 6.2], define the total energy $W_t^{\mathrm{tot}}(v)$ at v to be $W_t(v)$ plus the energy lost on $\langle v, v+1\rangle$ until time t. If we let $X(v)$ denote the random variable consisting of $\eta_0(v)$ and the Poisson process associated with the edge $\langle v, v+1\rangle$, $\{X(v),\ v \in \mathbb{Z}\}$ is an i.i.d. sequence. Then $W_t^{\mathrm{tot}}(0)$ is a measurable function of the random variables $\{X(v),\ v \in \mathbb{Z}\}$ and $\{W_t^{\mathrm{tot}}(v),\ v \in \mathbb{Z}\}$ its corresponding shifted equivalents. The well-known pointwise ergodic theorem due to Birkhoff and Khinchin thus implies

$$\mathbb{E}[W_t^{\mathrm{tot}}(0)] = \lim_{y\to-\infty,\, z\to\infty} \frac{1}{z-y+1} \sum_{u=y}^{z} W_t^{\mathrm{tot}}(u) \quad \text{almost surely.}$$

Note that there are almost surely infinitely many edges on which no Poisson event has occurred up to time t and that on a section between two such edges, the sum of total energies is preserved until t. Gathering our results, we find

$$\begin{aligned}
\mathbb{E}[W_t^{\mathrm{tot}}(v)] &= \lim_{y\to-\infty,\, z\to\infty} \frac{1}{z-y+1} \sum_{u=y}^{z} W_t^{\mathrm{tot}}(u) \quad \text{almost surely} \\
&= \lim_{y\to-\infty,\, z\to\infty} \frac{1}{z-y+1} \sum_{u=y}^{z} W_0^{\mathrm{tot}}(u) \quad \text{almost surely} \\
&= \mathbb{E}[W_0^{\mathrm{tot}}(v)] \\
&= \mathbb{E}[W_0(v)] \\
&= -\frac{8}{3(1-\gamma)}\left(1 + \frac{\ln(\gamma)}{1-\gamma}\right).
\end{aligned}$$

If we assume for contradiction that with positive probability for some $\delta > 0$ the total variation distance $\|\eta_t(v) - \eta_t(v+1)\|_{\mathrm{TV}}$ lies in $[\delta, \theta]$ for arbitrarily large t, the conditional Borel–Cantelli lemma (see, e.g. [11, Corollary 6.20]) forces infinitely many performed updates with the total variation distance being at least δ. As the total energy is always nonnegative, this implies that $\lim_{t\to\infty} \mathbb{E}[W_t^{\mathrm{tot}}(v)] = \infty$, a contradiction. □

Before we can use Lemma 6.3 to guarantee the existence of flat vertices at time $t > 0$, we need to check that (6.3) also holds for time $t > 0$.

Lemma 6.5. *Given the Deffuant model as described in Lemma 6.3, for all $t \geq 0$ it holds that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_t^{(v)} = \varphi_\gamma$$

almost surely with respect to the supremum norm on $[0, 1]$.

Proof. Fix $t > 0$. From Theorem 2.1, we know that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_0^{(v)} = \varphi_\gamma \quad \text{almost surely}$$

with respect to the supremum norm. Using the fact that the densities are uniformly bounded by $2/\gamma$, we can conclude that this convergence holds even for $t > 0$, by the same token as in [7].

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To the right of site 1, almost surely there is an infinite increasing sequence of nodes $(v_k)_{k\in\mathbb{N}}$, such that there was no Poisson event up to time t on the collection of edges $\{\langle v_k, v_k+1\rangle,\ k \in \mathbb{N}\}$. We denote the random lengths of the intervals in between by $L_k := v_{k+1} - v_k$ for $k \in \mathbb{N}$. In addition, let $L_0 := v_1 - v_0$ be the length of the interval including agent 1, where $\langle v_0, v_0 + 1\rangle$ is the first edge to the left of site 1 without a Poisson event until time t. Independence of the involved Poisson processes entails that $(L_k)_{k\in\mathbb{N}_0}$ is an i.i.d. sequence of random variables having geometric distribution on $\mathbb{N}$ with parameter e^{-t}.

Fix $\delta > 0$. Using the Borel–Cantelli lemma, we find that the event

$$E_\delta := \{L_0 = \infty\} \cup \limsup_{k\to\infty} \left\{ L_k \geq k\frac{\delta\gamma}{2} \right\}$$

has probability 0.

The Deffuant model is mass preserving in the sense that the sum of opinions of two interacting agents is always preserved. Therefore, it holds that, for all $k \in \mathbb{N}$,

$$\sum_{u=v_0+1}^{v_k} f_0^{(v)} = \sum_{u=v_0+1}^{v_k} f_t^{(u)}.$$

For the average at time $t > 0$ running from $v_0 + 1$ to some $v \in \{v_k + 1, \dots, v_{k+1}\}$ to differ by more than δ from the one at time 0, i.e.

$$\left\| \frac{1}{v - v_0} \sum_{u=v_0+1}^{v} f_0^{(u)} - \frac{1}{v - v_0} \sum_{u=v_0+1}^{v} f_t^{(u)} \right\|_\infty \geq \delta,$$

the interval length L_k has to be at least $k(\delta\gamma/2)$, since $v_k \geq k$, $v_0 \leq 0$, and the density $f_s^{(u)}$ is nonnegative and uniformly bounded by $2/\gamma$ for all $u \in \mathbb{Z}$ and times $s \geq 0$.

In conclusion, given E_δ^{c}, it holds that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_t^{(v)}(x) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{v=v_0+1}^{n} f_t^{(v)}(x) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \sum_{v=v_0+1}^{n} f_0^{(v)}(x) + \delta \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_0^{(v)}(x) + \delta \\
&= \varphi_\gamma(x) + \delta \quad \text{uniformly in } x \in [0, 1].
\end{aligned}$$

In the same way, we obtain $\varphi_\gamma(x) - \delta$ as a lower bound on the limits inferior and letting δ go to 0 finally verifies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{v=1}^{n} f_t^{(v)} = \varphi_\gamma \quad \text{almost surely}$$

with respect to the supremum norm. □

Lemma 6.6. *Given the Deffuant model as described in Lemma 6.3 and $\varepsilon > 0$, the following holds for all $t > 0$:*

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(i) *with nonzero probability, there has been no Poisson event on the edge* $\langle 0, 1\rangle$ *until time* t *and site 1 is* ε*-flat to the right with respect to the configuration* $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ *and distance measure* $\|.\|_{\mathrm{TV}}$*;*

(ii) *with nonzero probability, site 0 is two-sidedly* ε*-flat with respect to the configuration* $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ *and distance measure* $\|\cdot\|_{\mathrm{TV}}$.

Proof. In order to verify these claims, we have to put together only the ingredients established in Lemmas 6.3 and 6.5. As in the proof of [7, Theorem 2.2], we do this by using a conditional variant of the coupling technique introduced in [15], that became known as *local modification* in percolation theory.

Fix $\varepsilon > 0$. Recall that for absolutely continuous measures μ and ν on $[0, 1]$ with densities f and g, respectively, we have $\|\mu - \nu\|_{\mathrm{TV}} \le \frac{1}{2}\|f - g\|_\infty$, and let B denote the following event:

$$\left\{ \left\| \frac{1}{n}\sum_{v=1}^{n} \eta_t(v) - \mathbb{E}\eta^{(\gamma)} \right\|_{\mathrm{TV}} \le \frac{\varepsilon}{3} \text{ and } \left\| \frac{1}{n}\sum_{v=-n}^{-1} \eta_t(v) - \mathbb{E}\eta^{(\gamma)} \right\|_{\mathrm{TV}} \le \frac{\varepsilon}{3}, \text{ for all } n \ge N \right\}.$$

From Lemmas 6.3 and 6.5, we know that $N \in \mathbb{N}$ can be chosen sufficiently large such that $\mathbb{P}(B) > 1-\mathrm{e}^{-2t}$ and $\mathbb{P}(C \cap D) > 0$, where C denotes the event that there have been no Poisson events on the two edges $\langle 0, 1\rangle$ and $\langle N, N + 1\rangle$ up to time t and

$$D := \left\{ \|\eta_t(v) - \mathbb{E}\eta^{(\gamma)}\|_{\mathrm{TV}} \le \frac{\varepsilon}{3} \text{ for all } 1 \le v \le N \right\}.$$

Additionally, since $\mathbb{P}(C) = \mathrm{e}^{-2t}$, we must have $\mathbb{P}(B \cap C) > 0$.

Now let $\boldsymbol{\eta}_t := \{\eta_t(v)\}_{v\in\mathbb{Z}}$ and $\boldsymbol{\eta}'_t := \{\eta'_t(v)\}_{v\in\mathbb{Z}}$ be the configurations at time t coming from two independent copies of the considered model. There is a strictly positive probability that $B \cap C$ occurs for $\boldsymbol{\eta}_t$ as well as $C \cap D$ occurring for $\boldsymbol{\eta}'_t$. Given C, the hybrid process $\tilde{\boldsymbol{\eta}}_t$ defined by

$$\tilde{\eta}_t(v) = \begin{cases} \eta_t(v) & \text{if } v \notin \{1, \dots, N\}, \\ \eta'_t(v) & \text{if } v \in \{1, \dots, N\}, \end{cases}$$

is a perfectly fine copy of the model as well, showing that the event $B \cap D$ has nonzero probability. It is an easy exercise to check that $B \cap D$ actually implies the ε-flatness to the right of site 1.

In fact, the same argument applies to the second setting. Here, however, we choose C to be the event that there were no Poisson events on $\langle -N - 1, -N\rangle$ and $\langle N, N + 1\rangle$ as well as $D := \{\|\eta_t(v) - \mathbb{E}\eta^{(\gamma)}\|_{\mathrm{TV}} \le \varepsilon/3 \text{ for all } -N \le v \le N\}$. Then the two-sidedly ε-flatness of site 0 follows from $B \cap D$. □

Let us now use Lemmas 6.4 and 6.6 to prove the main statement about the model featuring restricted random triangular distributions.

Theorem 6.1. *Consider the Deffuant model on* $\mathbb{Z}$*, in which the total variation distance is used to measure the difference between two opinions and in which the initial opinions are given by independently assigned random restricted triangular distributions with fixed* $\gamma \in (0, 1)$ *as described in (6.1). Then there is a sharp phase transition in the following sense: for* $\theta \in [0, \theta_c)$*, the system almost surely approaches disagreement in the long run; for* $\theta \in (\theta_c, 1]$*, it almost surely approaches consensus. The threshold* $\theta_c \in (0, 1)$ *is given by*

$$\theta_c = \frac{1}{2}\int_0^1 |f_{(0,\gamma)}(x) - \varphi_\gamma(x)|\,\mathrm{d}x = \frac{1}{2}\int_0^1 \left| \frac{2}{\gamma}\left(1 - \frac{2}{\gamma}\left|x - \frac{\gamma}{2}\right|\right)^+ - \varphi_\gamma(x) \right| \mathrm{d}x.$$

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Proof. As mentioned in Section 4, we will closely follow the ideas in [6] to establish this result, just now the opinions are given by (random) absolutely continuous measures, or rather their density functions. In fact, most of the work has already been done by showing Lemma 6.6. Let us define

$$\theta_{\mathrm{c}} := \frac{1}{2}\int_0^1 |f_{(0,\gamma)}(x) - \varphi_\gamma(x)|\,\mathrm{d}x \quad \text{and} \quad \varepsilon := \frac{1}{7}|\theta_{\mathrm{c}} - \theta|.$$

In what follows, we will consider the two regimes: the subcritical one ($\theta < \theta_{\mathrm{c}}$) and the supercritical one ($\theta > \theta_{\mathrm{c}}$).

In the subcritical regime, fix $t > 0$ and let B denote the event that there are no Poisson events either on $\langle -1, 0\rangle$ or on $\langle 0, 1\rangle$ during $[0, t]$ and site -1 is ε-flat to the left, site 1 is ε-flat to the right with respect to the configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$. By Lemma 6.6(i), the obvious symmetry and conditional independence, we know that B occurs with positive probability. Let the initial opinion of agent 0 be given by f_{U_0} in the sense of (6.2) and C be the event that the first coordinate of U_0 is less than $\gamma + \varepsilon\gamma/2$, which by the shape of A_γ (see Figure 4) and (6.4) entails

$$\|\eta_0(0) - \mathbb{E}\eta^{(\gamma)}\|_{\mathrm{TV}} \geq \theta_{\mathrm{c}} - \varepsilon.$$

Given that there are no Poisson events on edges incident to site 0, we can (again by local modification) conclude that $B \cap C$ has positive probability. From Lemma 4.2, we know that the total variation distance between $\eta_s(1)$ and the intensity measure $\mathbb{E}\eta^{(\gamma)}$ will not exceed ε for $s \geq t$ due to its one-sided ε-flatness if there is no interaction with site 0 (same for $\eta_s(-1)$). However, given $B \cap C$, the opinions $\eta_t(1)$ and $\eta_t(0) = \eta_0(0)$ are at distance larger than $\theta_{\mathrm{c}} - 2\varepsilon > \theta$ and, hence, they will never be close enough to interact, since the same holds for $\eta_t(-1)$, which leaves the opinion at site 0 unchanged for all time. In other words, with nonzero probability, the edge $\langle 0, 1\rangle$ will be finally blocked (and the same for $\langle -1, 0\rangle$).

To conclude the claimed a.s. behavior, we apply the ergodicity argument used in the proof of Lemma 6.4 once again. Whether the configuration approaches disagreement or not can be checked, given the initial configuration plus all Poisson processes associated to the edges. The sequence $\{X(v),\ v \in \mathbb{Z}\}$ (as defined in the proof of Lemma 6.4) is i.i.d. and, hence, ergodic with respect to shifts. Thus, the translation-invariant event 'disagreement' necessarily has to be trivial, i.e. must have probability either 0 or 1. Since we have already shown that its probability is nonzero, the event has to be an a.s. one in the subcritical regime.

In the supercritical case, we know that at time $t > 0$ any fixed site is two-sidedly ε-flat with positive probability (Lemma 6.6(ii)). Lemma 6.2 implies that the largest total variation distance of a restricted triangular distribution as defined in (6.1) to the intensity measure $\mathbb{E}\eta^{(\gamma)}$ is given by what we defined as θ_{c}. Since all opinions at a later time are convex combinations of the initial ones, this distance cannot be exceeded. If site 0 is two-sidedly ε-flat with respect to the configuration $\{\eta_t(v)\}_{v\in\mathbb{Z}}$, the opinion $\eta_s(0)$ will be at total variation distance at most 6ε to $\mathbb{E}\eta^{(\gamma)}$ for all $s \geq t$ (Lemma 4.2(ii)). This implies that its neighboring opinions differ by not more than $\theta_{\mathrm{c}} + 6\varepsilon < \theta$; hence, Lemma 6.4 forces the differences to converge to 0. By induction, this is actually true for any pair of neighbors. Again, ergodicity ensures that this consensus behavior occurs with probability 1.

It further implies the a.s. existence of two-sidedly ε-flat vertices for any strictly positive value of ε. For this reason, the measure the opinions converge to must be the intensity measure, which concludes the proof. □

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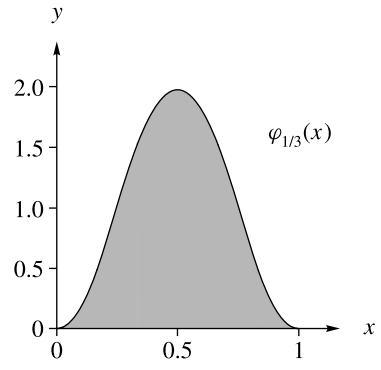


FIGURE 7: The intensity measure is concentrated more towards the center if the random triangular distributions are restricted to a minimum width.

Example 6.1. Let us consider the Deffuant model on $\mathbb{Z}$ with opinions being absolutely continuous probability distributions and the initial ones given by random restricted triangular distributions with parameter $\gamma = \frac{1}{3}$. From Lemma 6.1, we know that the corresponding intensity measure has the somewhat cumbersome density (see Figure 7):

$$\varphi_{1/3}(x) = \begin{cases} -18\big[(1-x)\ln(1-x) - \frac{3}{2}x^2 + x\big], & 0 \le x \le \frac{1}{6}, \\ -18\big[(1-x)\ln(1-x) - x\ln(6x) + x + \frac{3}{2}x^2 - \frac{1}{12}\big], & \frac{1}{6} \le x \le \frac{1}{3}, \\ -18\big[(1-x)\ln(1-x) + x(1-\ln(2)) + \frac{1}{12}\big], & \frac{1}{3} \le x \le \frac{1}{2}, \\ -18\big[x\ln(x) + (1-x)(1-\ln(2)) + \frac{1}{12}\big], & \frac{1}{2} \le x \le \frac{2}{3}, \\ -18\big[x\ln(x) - (1-x)\ln(6-6x) + \frac{3}{2}x^2 - 4x + \frac{29}{12}\big], & \frac{2}{3} \le x \le \frac{5}{6}, \\ -18\big[x\ln(x) - \frac{3}{2}x^2 + 2x - \frac{1}{2}\big], & \frac{5}{6} \le x \le 1. \end{cases}$$

If the total variation distance is used to measure the disparity of two opinions, we can conclude from Theorem 6.1 that the threshold θ_c for this model takes on the value

$$\theta_c = \frac{1}{2}\int_0^1 |f_{(0,1/3)}(x) - \varphi_{1/3}(x)|\,\mathrm{d}x \approx 0.831\,72.$$

7. Alternative choices (concerning the initial configuration, distance measure, and network graph) and inhomogeneous open mindedness

No doubt that the extension of the Deffuant model on $\mathbb{Z}$ to measure-valued opinions leaves a wide range of possible laws for the initial configuration to be examined. We saw trivial behavior for triangular distributions and a nontrivial phase transition for restricted triangular distributions.

If we want to stick to absolutely continuous measures on a compact support S and $\|\cdot\|_{\mathrm{TV}}$ as distance measure, the line of argument from Section 6 will, in principle, carry over on condition that

$$\mathbb{E}\int_S [f_\omega(x)]^2\,\mathrm{d}x < \infty,$$

where $f_\omega, \omega \in \Omega$, denotes the random initial density shaped by a probability space $(\Omega, \mathbb{P})$, and that the total variation distance of an initial opinion to the intensity measure is almost surely bounded away from 1. The first condition is needed to prove Lemma 6.4; the latter will,

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in fact, give the threshold θ_c, as essential supremum of the total variation distance of initial opinions to the intensity measure. There is, however, one issue that must not be overlooked: in order to establish Lemma 6.3, we need the absence of major gaps in the support of the initial opinions—just as in the case of finite-dimensional opinions. To examine this problem more closely, elaborate geometric considerations as in [9] seem to be necessary and we will thus leave this for future studies.

If one wants to include point processes as opinions, in many situations the total variation distance will not work as a meaningful measure for the discrepancy of two opinions, at least if the support of two such processes is disjoint with positive probability. Using the Lévy distance instead could, however, lead to interesting models in such a setting. In fact, even in the case of triangular distributions, it seems to be unrealistic that two determined agents are at maximal distance, whether the intervals on which their opinions are concentrated are in close proximity or at different ends of the spectrum. From this point of view, albeit more difficult to handle, the Lévy distance appears to be a more suitable choice.

In terms of network topology, the graph $\mathbb{Z}$ appears to be a rather restrictive and unnatural choice. The reason for this choice is twofold. On the one hand, the concept of a phase transition in the strict sense exists only in (the limit of) an infinite network. On the other, the arguments and techniques from [6] that were also heavily used here, do not apply beyond dimension one. A handful of much weaker results could be obtained for higher-dimensional grids; see [7, Section 3]. One reason why the methods in [6] are limited to dimension one is elaborated in the last section of [8]. Another attempt was made by Shang [20], who considered multiplex networks. However, this idea is essentially a generalization of $\mathbb{Z}$ in which multiple edges (of a different kind) between neighbors are considered. The case of more general infinite networks is wide open and the Deffuant model with univariate opinions is definitely easier to handle when trying to advance in this direction.

Finally, it might be worth accentuating the new feature of the presented model (in contrast to finite-dimensional opinions), which was already mentioned in the beginning of Section 5, and to put it into a broader context. In the Deffuant model with triangular distributions, besides the common tolerance parameter θ and willingness to compromise μ, we have a diversified scale of open mindedness of the agents shaped by the random support of their initial opinion.

There have been attempts, e.g. by Deffuant *et al.* in [21], to simulate the long-term behavior of a variant of the original model in which the agents have different θ-values. From an analytical point of view, this is quite a crucial change since it brings along situations in which the opinion of only one of two interacting neighbors is updated. Then the sum of opinions is no longer preserved, which renders void many of our central arguments. Another advance in the same direction is the so-called *relative agreement model*, introduced in [2]. There, the bounded confidence rule is dropped and replaced by a continuous counterpart: agents feature both a real-valued opinion and a separate value corresponding to their individual uncertainty, which taken together shape a dispersed opinion in the form of a symmetric interval of length twice the uncertainty around the opinion value. If an agent is influenced by another, the impact depends on the overlap of the two opinion intervals relative to the length of the interval corresponding to the influenced agent. Again, the asymmetric way of updating opinion values makes the relative agreement model, although based on the same principles and ideas, qualitatively quite different.

Closer to the original Deffuant model with univariate opinions, Shang considered two more variants on $\mathbb{Z}$: one where the individual confidence thresholds can take on two different values [19] and one where they are random [18]. In the former, agents still update their opinions in pairs but the effective θ-value is now given by the average of the two involved confidence

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thresholds; in the latter, the thresholds are assigned to the edges directly. In this way, the interactions are still average preserving and the methods from [6] apply without further ado.

The way in which inhomogeneous open mindedness or uncertainty is incorporated in our extension of the model not only avoids the issue of asymmetric updates, but also leads to the realistic property that agents themselves become more open minded by interacting with open-minded neighbors (in contrast to the variants of the Deffuant model with individual confidence bounds that have been studied so far). Owing to this additional feature, the model with measure-valued opinions may be relevant also in applications, in spite of its increased complexity. Unquestionably, the additional computational cost caused by infinite-dimensional opinions is nonnegligible. However, on the one hand, it can be alleviated by restricting the initial opinions to a certain class of distributions (as proposed in Section 3). On the other hand, it is not difficult to envision circumstances in which vectors are not sufficient to capture the stance of a single agent: take quantitative risk analysis in investment banking for example. When the payoff of a given investment is evaluated by different methods of probabilistic risk assessment, the results will be given by probability distributions. An implementation of the Deffuant model with measure-valued opinions could be used to get an idea of whether or not the members of a group of experts are likely to come to a consensus about the evaluation in question.

Moreover, when implementing the extension of the Deffuant model presented here in order to simulate the opinion formation process in a social network, one could alter the interaction rule to become a double-stage process capturing the changeability of individual views: agents still hold opinions that are given by probability measures. Every time two agents meet, they independently draw a numerical value from their current distribution and proceed as in the case of finite-dimensional opinions. Note that, with this change, the distance of two opinions is no longer deterministic but random, which is why it is qualitatively different from both the original Deffuant model and the extension proposed here.

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