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106.21 A class of interesting integrals

In this Note we shall show that:

$$\int_0^{+\infty} \frac{\log x \arctan x}{x(1+x)}\,dx = \frac{\pi^3}{96}. \tag{1}$$

The calculation uses a trick, which reduces the integral to two easy ones. The same trick can be applied to show that: (k, m, n positive integers)

$$\int_0^{+\infty} \frac{\log^{2k-1} x \arctan x^m}{x(1+x^n)}\,dx = Q_{k,m,n}\,\pi^{2k+1},$$

$$\int_0^{+\infty} \frac{\log^{2k-2} x \log(1+x^m)}{x(1+x^n)}\,dx = R_{k,m,n}\,\pi^{2k}$$

with $Q_{k,m,n}$, $R_{k,m,n}$ rational numbers depending on k, m and n. These integrals cannot be found in tables of integrals such as Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* [1].

Proof: To prove (1), we denote the integral by I and split up the interval at $x = 1$. We then make the substitution $x = \frac{1}{t}$ in the resultant second integral. So we have

$$\begin{aligned} I &= \int_0^1 \frac{\log x \arctan x}{x(1+x)}dx + \int_1^0 \left(\log\frac{1}{t}\right)\left(\arctan\frac{1}{t}\right)\frac{t^2}{t+1}\left(-\frac{1}{t^2}\right)dt \\ &= \int_0^1 \frac{\log x \arctan x}{x(1+x)}dx - \int_0^1 \log t\left\{\frac{\pi}{2} - \arctan t\right\}\frac{1}{t+1}dt \\ &= \int_0^1 \log x \arctan x\left\{\frac{1}{x} - \frac{1}{1+x}\right\}dx - \int_0^1 \log x\left\{\frac{\pi}{2} - \arctan x\right\}\frac{1}{1+x}dx \\ &= \int_0^1 \frac{1}{x}\log x \arctan x\,dx - \frac{\pi}{2}\int_0^1 \frac{\log x}{1+x}dx. \end{aligned}$$

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We will now evaluate these two integrals, treating the second and simpler one first.

The second integral is a special case of [1, p. 550, 4.271.2]

$$\int_0^1 \frac{\log^{2k-1} x}{1+x}\,dx = -(2k-1)!\left(1-2^{1-2k}\right)\zeta(2k) \tag{2}$$

$$= \frac{(-1)^k}{2k}\left(2^{2k-1}-1\right)\pi^{2k}\,\mathrm{B}_{2k} \qquad (k = 1, 2, 3, \ldots)$$

(where B_j denotes the jth Bernoulli number). For $k = 1$ we get the value $-\frac{\pi^2}{12}$. The first integral can be rewritten using integration by parts

$$\int_0^1 \frac{1}{x}\log x \arctan x\,dx = -\frac{1}{2}\int_0^1 \frac{\log^2 x}{1+x^2}\,dx.$$

The right-hand side is a special case of [1, p. 550, 4.271.6]

$$\int_0^1 \frac{\log^{2k} x}{1+x^2}\,dx = \frac{(-1)^k}{2^{2k+2}}\pi^{2k+1}\,\mathrm{E}_{2k}\,. \tag{3}$$

Here E_j denotes the jth Euler number. The Euler numbers are defined by

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty}\frac{\mathrm{E}_n}{n!}x^n.$$

Note that $\mathrm{E}_2 = -1$, which follows by taking the product of this series with the series for $\cosh x$. Hence, for $k = 1$ we have

$$\int_0^1 \frac{\log^2 x}{1+x^2}\,dx = \frac{\pi^3}{16.}$$

Combining these two results leads to $I = -\frac{\pi^3}{32} - \frac{\pi}{2}\left(-\frac{\pi^2}{12}\right) = \frac{\pi^3}{96}$.

Note that both integrals (2) and (3) can be evaluated without the use of tables, using the Maclaurin series for $\frac{1}{1+x}$ and $\frac{1}{1+x^2}$, and the easy-to-prove result

$$\int_0^1 x^i \log^m x\,dx = \frac{(-1)^m m!}{(i+1)^{m+1}}.$$

The same trick can be employed to prove the following general formula

$$\int_0^{+\infty} \frac{\log^{2k-1} x \arctan x^m}{x\left(1+x^n\right)}\,dx = \frac{(-1)^{k-1}}{2k}\left(\frac{\mathrm{E}_{2k}}{2^{2k+2}m^{2k}} + \left(2^{2k-1}-1\right)\frac{\mathrm{B}_{2k}}{2n^{2k}}\right)\pi^{2k+1} \tag{4}$$

(k, m, n positive integers). This time we use

$$\frac{1}{x\left(1+x^n\right)} = \frac{1}{x} - \frac{x^{n-1}}{1+x^n}$$

giving the two integrals

$$\int_0^1 \frac{\log^{2k-1} x \arctan x^m}{x}\,dx \text{ and } \int_0^1 \left(\log^{2k-1} x\right)\frac{x^{n-1}}{1+x^n}\,dx$$

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which can be reduced beautifully to (3) and (2) by the substitutions $t = x^m$ and $u = x^n$ respectively. And in a similar way we obtain

$$\int_0^{+\infty} \frac{\log^{2k-2} x \log(1+x^m)}{x(1+x^n)}\,dx = (-1)^{k-1}\frac{m(2^{2k-1}-1)}{2k(2k-1)}\left(\frac{1}{m^{2k}}+\frac{2k-1}{n^{2k}}\right)\mathrm{B}_{2k}\pi^{2k} \quad (5)$$

(k, m, n positive integers).

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106.22 The golden ratio represented by a tangent

It is known that the golden ratio $\Phi = (\sqrt{5}+1)/2$ can be represented by using the trigonometric functions sin and cos as follows:

$$\Phi = 2\sin\frac{3\pi}{10} = 2\cos\frac{\pi}{5},$$

$$\Phi = \left[2\sin\frac{\pi}{10}\right]^{-1} = \left[2\cos\frac{2\pi}{5}\right]^{-1}$$

(see, e.g. [1]). The proofs are based on Ptolemy's theorem and the consideration of two triangles consisting of sides and diagonals of a regular pentagon.

To the best of our knowledge there appears to be no known result for tangent representation of the golden ratio Φ as above. Our purpose is to investigate the relationship between the golden ratio Φ and the trigonometric function tan, and to represent Φ by tan in two ways. We use the notation $\varphi = (\sqrt{5}-1)/2$.

We mention the paper [2] by MacHale which shows that $\sin\theta = \varphi = 1/\Phi$ if $\tan\theta = \cos\theta$. In Glaister [3] the following theorem was established by using the techniques of elementary geometry.

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