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VOLATILITY ANALYSIS OF REGIME-SWITCHING MODELS

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This paper investigates the volatility in regime-switching models formulated based on the geometric Brownian motion with its drift and volatility factors randomized with Markov chains. By developing explicit formulas about occupation time of Markov chains, we analysis the difference between global volatility of this model and the volatility caused by Brownian randomness, in order to measure the volatility caused by regime-switching after justifying its existence. Utilizing this structure of volatility, we optimize the methods of volatility parameters estimation.

Keywords: markov chains, regime-switching model, volatility analysis

1. INTRODUCTION

Since introduced by Hamilton in [3], regime-switching models are reckoned practical especially for modeling those risky assets. The switched regimes are applied to simulate the changeable states of its general backgrounds or underlying environment. In most cases, regime-switching is accompanied with some existed famous models by randomizing some parameters, which previous are assumed as constants. Particularly, geometric Brownian motion is a common choice for this combination since its drift and volatility parameters can be made switchable and governed by a Markov chain denoting the regimes.

Consider the well-known Black–Scholes model, whose two parameter are assumed as constants, consequently, these constant return rate and volatility may fail to capture the dynamics from reality. Hence, many researches extend its model to include drift and volatility variables driven by Markov chains denoting the different regimes (see [6,8,13]). Besides the obvious advantage of practical adaption, another merit is the presentation of a clear volatility smile, as shown in [7,13]. Volatility analysis plays an essential role on risk management, derivative pricing, portfolio selection, etc. To meet the demands of real market, many researches spare no effort to model and forecast volatility on some regime-switching models, as we can seen in [2,9,15].

This paper investigates the volatility of regime-switching models, mainly about two aspects. One is the structure of volatility, that is, to detect the sources contributing to the model volatility and another is the application of this finding, in particular, clarification of volatility sources improves the estimation of volatility parameters. The volatility analysis is

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proceeded by developing a closed-form formula for the suggestive volatility, it pioneers a new technique for occupation time computation of Markov chains. Thereby the numerical testing for volatilities estimation is based on explicit formulas instead of Monte Carlo simulation. For the existed advances of computational techniques and results, we refer to [1,4,11].

The remaining part is arranged as follows. Section 2 formulates the model and defines the suggestive volatility and averaged volatility, based on which we raise the questions about sources of volatility in Section 3 and state the main hypothesis. For further analysis and justify this hypothesis, Section 4 develops formulas of expectation of integration in regime-switching models. Applying these formulas, we proceed to check the existence and influences of volatility caused by regime-switching in Section 5 and conclude to the hypothesis. Finally, we apply this findings to estimate the volatility parameters in regime-switching models in Section 6.

2. SUGGESTIVE VOLTALITY AND AVERAGED VOLTALITY

Given a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ independent of the Markov chain $(\alpha_t)_{t\in\mathbb{R}_+}$ and the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(\alpha_t)_{t\in\mathbb{R}_+}$, we consider an asset price $(Y_t)_{t\in\mathbb{R}_+}$ modeled by a geometric Brownian motion

$$dY_t = \mu(\alpha_t)Y_t dt + \sigma(\alpha_t)Y_t dB_t, \quad 0 \le t \le T, \tag{2.1}$$

with regime-switching driven by a finite-state continuous-time Markov chain $(\alpha_t)_{t\in\mathbb{R}_+}$ on $\mathcal{M} = \{e_1, e_2, \ldots, e_m\}$. Here, $\mu : \mathcal{M} \longrightarrow \mathbb{R}$ and $\sigma : \mathcal{M} \longrightarrow (0,\infty)$ are deterministic functions. For sake of volatility analysis, we defined an equivalent process $(\hat{Y}_t)_{t\in\mathbb{R}_+}$ of $(Y_t)_{t\in\mathbb{R}_+}$:

$$d\hat{Y}_t = \mu\hat{Y}_t dt + \sigma\hat{Y}_t dB_t, \quad 0 \le t \le T, \tag{2.2}$$

where $\hat{Y}_0 = Y_0$, $\mu \in \mathbb{R}$ and $\sigma > 0$ are real constants, namely the drift factor and volatility factor, respectively. Hence, we try to determine this σ, whose value may indicate the general volatility of Y_t during the time period $[0, T]$. Formally, we defined this σ as the *suggestive volatility* of $(Y_t)_{t\in[0,T]}$ when the equivalent process $(\hat{Y}_t)_{t\in\mathbb{R}_+}$ satisfies the condition below:

$$E[Y_T] = E[\hat{Y}_T] \quad \text{and} \quad E[Y_T^2] = E[\hat{Y}_T^2], \tag{2.3}$$

namely Y_T and $\hat{Y}_T$ are consistent in terms of first and second moments. Similar simple arguments as in [6,10] yield that equations in (2.3) will be equivalently written as follows:

$$E[e^{\int_0^T \mu(\alpha_t)dt}] = e^{\mu T} \quad \text{and} \quad E[e^{\int_0^T \sigma^2(\alpha_t)dt}] = e^{\sigma^2 T}. \tag{2.4}$$

Equivalently, we write

$$\sigma = \left(\frac{1}{T}\log(E[e^{\int_0^T \sigma^2(\alpha_t)dt}])\right)^{1/2}. \tag{2.5}$$

On the other hand, we consider the *averaged volatility* $\bar{\sigma}$ defined as follows:

$$\bar{\sigma} = \left(\frac{1}{T}E\left[\int_0^T \sigma^2(\alpha_t)dt\right]\right)^{1/2}, \tag{2.6}$$

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where the "average" is implied by the following equation:

$$E\left[\int_0^T \sigma^2(\alpha_t)dt\right] = \bar{\sigma}^2 T. \tag{2.7}$$

By Jensen's inequality, we see that

$$\sigma \geq \bar{\sigma}.$$

Notice that the equality holds only if $\sigma^2(\alpha_t)$ is degraded to be deterministic. One easy way to achieve this is to let $\sigma(\cdot)$ be a constant. Otherwise, in most cases, $\sigma > \bar{\sigma}$ strictly. The positive difference between σ and $\bar{\sigma}$ plays a key role on the volatility analysis of regime-switching models, we will further explore its underlying meaning and practical uses in later sections.

3. SOURCES OF VOLTALITY

Inspired by the positive difference between σ and $\bar{\sigma}$ and practices of using regime-switching models especially of setting the volatility parameters, it seems that the volatility of the entire model is, in many situations, larger than that caused by the randomness of Brownian motion. Therefore, we raise this hypothesis:

HYPOTHESIS 3.1: *Switching of regimes generates extra volatility, hence the model volatility actually involves variations from both sources: Brownian randomness and regime-switchings.*

Illustrated by (2.7), the averaged volatility $\bar{\sigma}$ records the variation caused by Brownian randomness. Meanwhile, the suggestive volatility σ defined by (2.2) and (2.3) reflects the all variations of the entire system. We use σ to measure the model volatility and use $\sigma - \bar{\sigma}$ to count for extra volatility mentioned in Hypothesis 3.1.

Next, we partially justify Hypothesis 3.1 by two propositions below:

PROPOSITION 3.1: *The suggestive volatility σ of $(Y_t)_{t\in[0,T]}$ (under consideration (2.3) of the first two moments) is larger than averaged volatility $\bar{\sigma}$ (2.6).*

PROPOSITION 3.2: *Given fixed time horizon $[0,T]$ and volatility $\sigma(e_i)$, the difference $\sigma - \bar{\sigma}$ is mainly determined by the transition rate of Markov chains.*

The first proposition is proved in Section 2. It remains to prove or numerically justify Proposition 3.3. Before this approach, some formulas for later computation and analysis are prepared.

4. COMPUTATION OF INTEGRATION IN THE REGIME-SWITCHING MODEL

Although we have shown in Section 2 that $\sigma \geq \bar{\sigma}$, it is also attempted to calculate $\sigma - \bar{\sigma}$ measuring the additional volatility besides those caused by Brownian randomness. Based on simplified expression, it might be easier to test whether $\sigma - \bar{\sigma}$ is caused by regime-switchings. Therefore, for convenience and possibilities of further analysis, we proceed to establish some computations results for regime-switching models.

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4.1. Expectation of Exponential of Integration Over Markov Chain

As we find in the expression (2.4), expectation of the exponential of Markov chain is frequently used in the computation based on the Markovian regime-switching models. Thereby in this paper, we develop some closed-form expressions which can be easily implemented in the computation of regime-switching models. To consider a general case, we define the integration of Markov process by

$$X_t := \int_0^t u(\alpha_s)ds, \quad t \in \mathbb{R}_+, \tag{4.1}$$

where $u(\cdot)$ is a bounded real-valued function on $\mathcal{M}$. Two steps are taken to compute $E[\exp(X_t)]$.

4.1.1. Step 1: Determine infinitesimal generator of (X_t, α_t). As for the first step, it is necessary to figure out the infinitesimal generator of $(X_t, \alpha_t)_{t\in\mathbb{R}_+}$ as a preparation to compute the expectation. Direct computation of the infinitesimal generator of X_t only is impossible because X_t alone is not a Markov process. Indeed, (X_t, α_t) follows the Markovian property by checking the fact that

$$E\left[\int_0^t u(\alpha_r)dr \,\middle|\, \mathcal{F}_s^m\right] = E\left[\int_0^t u(\alpha_r)dr \,\middle|\, \int_0^s u(\alpha_r)dr, \alpha_s\right], \quad 0 \le s \le t,$$

where the filtration $(\mathcal{F}_t^m)_{t\in\mathbb{R}_+}$ is generated by $(\alpha_t)_{t\in\mathbb{R}_+}$. As we also note that [13] provides the infinitesimal generator $\mathcal{A}_3$ of (t, Y_t, α_t) that for any smooth function $F(t, y, e_i)$, we have

$$\mathcal{A}_3F = \frac{\partial}{\partial t}F(t,y,e_i) + \frac{1}{2}y^2\sigma^2(e_i)\frac{\partial^2}{\partial y^2}F(t,y,e_i) + ry\frac{\partial}{\partial y}F(t,y,e_i) + QF(t,y,\cdot)(i).$$

Our infinitesimal generator of (X_t, α_t) should be similar to the above expression, we process to establish ours applying standard approaches as follows. Formally, consider an arbitrary function $f(x, e_i)$ on $\mathbb{R} \times \mathcal{M}$ which allows at least second derivative with respect to $x \in \mathbb{R}$, we define the infinitesimal generator $\mathcal{A}_2$ as follows:

$$\mathcal{A}_2 f(x, e_i) := \lim_{\varepsilon\to 0} \frac{E[f(X_{s+\varepsilon}, \alpha_{s+\varepsilon}) \,|\, X_s = x, \alpha_s = e_i] - f(x, e_i)}{\varepsilon}, \tag{4.2}$$

for any $x \in \mathbb{R}_+$, $e_i \in \mathcal{M}$ and arbitrary $s \in \mathbb{R}_+$. Here, we recall and define some notations, let $Q = \{q_{i,j}\}_{m\times m}$ denote the transition rate matrix of the Markov chain $(\alpha_t)_{t\in\mathbb{R}_+}$ and let $P(t) := \{p_{i,j}(t)\}_{m\times m}$ denote the transition probability matrix of $(\alpha_t)_{t\in\mathbb{R}_+}$, namely for each entry of this matrix $P(t)$, $p_{i,j}(t) := P(\alpha_t = j \,|\, \alpha_t = e_i)$, and satisfies the formula that $P(t) = \exp(tQ)$ for any $t \in \mathbb{R}_+$. Rewrite (4.2) in form of a summation as follows:

$$E[f(X_{s+\varepsilon}, \alpha_{s+\varepsilon} \,|\, X_s = x, \alpha_s = e_i] = \sum_{j=1}^m E[f(X_{s+\varepsilon}, \alpha_{s+\varepsilon}) \,|\, X_s = x, \alpha_s = e_i, \alpha_{s+\varepsilon} = e_j]p_{i,j}(\varepsilon)$$

and by splitting $f(X_{s+\varepsilon}, \alpha_{s+\varepsilon}) - f(x, e_i)$ into

$$f(X_{s+\varepsilon}, \alpha_{s+\varepsilon}) - f(x, e_j)) + (f(x, e_j) - f(x, e_i)),$$

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we see that

$$\mathcal{A}_2 f(x,e_i) = \sum_{j=1}^{m} \left(\frac{\partial f(x,e_j)}{\partial x} u(e_i) \mathbf{1}_{\{i=j\}} + q_{i,j} f(x,e_j) \right)$$
$$= \frac{\partial f(x,e_i)}{\partial x} u(e_i) + \sum_{j=1}^{m} q_{i,j} f(x,e_j), \tag{4.3}$$

where we applied an obvious result that

$$\lim_{\varepsilon \to 0} E[f(X_{s+\varepsilon}, \alpha_{s+\varepsilon}) - f(X_s, \alpha_s) \,|\, \alpha_s = e_i, \alpha_{s+\varepsilon} = e_j] \frac{p_{i,j}(\varepsilon)}{\varepsilon} = 0.$$

Expressing (4.3) in another way, we rewrite the infinitesimal generator $\mathcal{A}_2$ in the following proposition.

PROPOSITION 4.1: *Denoted as $\mathcal{A}_2$ in (4.2), the infinitesimal generator of (X_t, α_t) is specified according to the expression below with an arbitrary partial differentiable (w.r.t. x) function $f(x,e_i)$ on $\mathbb{R} \times \mathcal{M}_m$:*

$$\mathcal{A} f(x,e_i) = \left[\left(\mathrm{Diag}(u(e_1), \ldots, u(e_n)) \frac{\partial}{\partial x} + Q \right) \overline{f(x,\cdot)} \right]_{(i)},$$

where $\mathrm{Diag}(u(e_1), \ldots, u(e_n))$ denotes an $m \times m$ diagonal matrix with its diagonal entries being $u(e_1), \ldots, u(e_m)$ orderly, $\overline{f(x,\cdot)}$ denotes a vertical vector whose elements are $f(x,e_1), \ldots, f(x,e_m)$ orderly, and we use the notation $[\eta]_{(i)}$ to denote the ith element of any vector η and $f(x,\cdot)$ has infinite order of derivatives with respect to x.

4.1.2. Step 2: Calculate the expectation of $\exp(X_t)$. To apply Proposition 4.1 for the computation of moment generating function, we specify this function f in Proposition 4.1 as $f(x,e_i) := \exp(x)$ for any $x \in \mathbb{R}_+$, $e_i \in \mathcal{M}$. It follows that $\partial f(x,e_i)/\partial x = \exp(x)$ and

$$\mathcal{A} f(x,e_i) = \left[\left(\begin{pmatrix} u(e_1) & & \\ & \ddots & \\ & & u(e_m) \end{pmatrix} + Q \right) \begin{pmatrix} f(x,e_1) \\ \vdots \\ f(x,e_m) \end{pmatrix} \right]_{(i)}.$$

Therefore, we define an $m \times m$ matrix A that

$$A := \mathrm{Diag}(u(e_1), \ldots, u(e_m)) + Q, \tag{4.4}$$

where $\mathrm{Diag}(u(e_1), \ldots, u(e_m))$ denotes an $m \times m$ diagonal matrix with its diagonal entries being $u(e_1), \ldots, u(e_m)$ orderly. Hence, we have

$$\mathcal{A} f(x,e_i) = [A \overline{f(x,\cdot)}]_{(i)}, \tag{4.5}$$

where $\overline{f(x,\cdot)}$ denotes a vertical vector whose elements are $f(x,e_1), \ldots, f(x,e_m)$ orderly, and we use the notation $[\eta]_{(i)}$ to denote the ith element of any vector η. Applying the properties

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of Markov semigroup (see [12]) and formula (4.5), we see that for any $t \in \mathbb{R}_+$ and $e_i \in \mathcal{M}$,

$$E[\exp(X_t) \,|\, \alpha_0 = e_i] = e^{\mathcal{A}t} f(0, e_i) = \sum_{k=0}^{\infty} \frac{(\mathcal{A}t)^k}{k!} f(0, e_i)$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} [(At)^k \overline{f(0,\cdot)}]_{(i)} = [e^{At} \mathbf{1}_{m,1}]_{(i)}, \tag{4.6}$$

where the matrix A is given by (4.4) and $\mathbf{1}_{m,1}$ denotes a vertical vector of m elements all being 1. In most cases, m eigenvectors of A defined in (4.4) are linearly independent, listed as $\eta_1, \ldots, \eta_m$, and their correspondent eigenvalues are $r_1, \ldots, r_m$. Denote the matrix $(\eta_1, \ldots, \eta_m)$ as M and let Λ denotes an $m \times m$ diagonal matrix with its diagonal entries being $r_1, \ldots, r_m$ orderly, then we have

$$\begin{aligned} e^{At} &= e^{M\Lambda M^{-1} t} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (M\Lambda M^{-1} t)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} M(\Lambda t)^k M^{-1} \\ &= M e^{\Lambda t} M^{-1}. \end{aligned} \tag{4.7}$$

Denoted by Λ_t, the $m \times m$ diagonal matrix with its diagonal entries being $e^{r_1 t}, \ldots, e^{r_m t}$ orderly and combine (4.6) and (4.7) to see that

$$E[\exp(X_t) \,|\, \alpha_0 = e_i] = [M \Lambda_t M^{-1} \mathbf{1}_{m,1}]_{(i)}. \tag{4.8}$$

Applying (4.6), and with equation (2.4) of σ, the suggestive volatility σ is given by the following proposition after defining the matrix

$$\mathrm{Diag}(\sigma^2(e_1), \ldots, \sigma^2(e_m)) + Q$$

as A_σ.

PROPOSITION 4.2: *Defined by (2.5), the suggestive volatility σ of Y_t in (2.1) during $t \in [0, T]$ is expressed by*

$$\sigma = \left(\frac{1}{T} \log([e^{A_\sigma T} \mathbf{1}_{m,1}]_{(i)}) \right)^{1/2}, \tag{4.9}$$

given that $\alpha_0 = e_i$. Should e^{A_σ} be similarity diagonalized as that in (4.7), (4.9) will be further simplified as that in (4.8), hence numerically realizable directly.

4.2. Expectation of Integration Over Markov Chain

To express and further numerical implement the averaged volatility $\bar{\sigma}$ in (2.6), we compute $E[X_t]$, where $(X_t)_{t \in \mathbb{R}_+}$ is given by (4.1).

Suppose that m eigenvectors of the transition rate matrix Q are linearly independent, listed as $\eta_1, \ldots, \eta_m$, and their correspondent eigenvalues are $r_1, \ldots, r_m$, where $r_1 = 0$ and $\eta_1 = (1, \ldots, 1)^T$. Denote the diagonal matrix $\mathrm{Diag}(e^{r_1 t}, \ldots, e^{r_m t})$ as $D_\alpha(t)$ for $t \in \mathbb{R}_+$ and

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denote the matrix $(\eta_1, \ldots, \eta_m)$ as M_α, hence we have the transition probability matrix $(p_{i,j})_{m\times m}$:

$$P(t) = e^{tQ} = M_\alpha D_\alpha(t) M_\alpha^{-1}. \tag{4.10}$$

Applying the Fubini's theorem to exchange the order of integration and expectation, we have

$$\begin{aligned} E[X_t \,|\, \alpha_0 = e_i] &= E\left[\sum_{j=1}^m \int_0^t u(e_j)\mathbf{1}_{\{\alpha_s = e_j\}} ds \,\middle|\, \alpha_0 = e_i\right] \\ &= \sum_{j=1}^m u(e_j) \int_0^t E[\mathbf{1}_{\{\alpha_s = e_j\}} \,|\, \alpha_0 = e_i] ds \\ &= \sum_{j=1}^m u(e_j) \int_0^t P(\alpha_s = e_j \,|\, \alpha_0 = e_i) ds \\ &= \int_0^t [M_\alpha D_\alpha(s) M_\alpha^{-1} \bar{u}]_{(i)} ds \\ &= [M_\alpha D_e(t) M_\alpha^{-1} \bar{u}]_{(i)}, \end{aligned}$$

where we use the notation $[\eta]_{(i)}$ to denote the ith element of any vector η, $\bar{u} := (u(e_1), \ldots, u(e_m))'$ is a vertical vector, and $D_e(t)$ is given by:

$$D_e(t) = \begin{pmatrix} \frac{1}{r_1}(e^{r_1 t} - 1) & & 0 \\ & \ddots & \\ 0 & & \frac{1}{r_m}(e^{r_m t} - 1) \end{pmatrix}, \quad t > 0,$$

where $(1/r_i)(e^{r_i t} - 1)$ is replaced by t when $r_i = 0$ for some $i \in \{1, \ldots, m\}$. Similarly, we use the notation $\widehat{\sigma^2}$ to denote $(\sigma^2(e_1), \ldots, \sigma^2(e_m))'$.

PROPOSITION 4.3: *Assume that $(e^{Qt})_{t>0}$ can be similarity diagonalized as that in (4.10). Defined by (2.6), the suggestive volatility $\bar{\sigma}$ of Y_t in (2.1) during $t \in [0, T]$ is expressed by*

$$\bar{\sigma} = \left(\frac{1}{T}\int_0^T [e^{Qt}\widehat{\sigma^2}]_{(i)} dt\right)^{1/2} = \left(\frac{1}{T}\left[M_\alpha D_e(T) M_\alpha^{-1} \widehat{\sigma^2}\right]_{(i)}\right)^{1/2}$$

given that $\alpha_0 = e_i$.

5. VOLTALITY CAUSED BY REGIME-SWITCHING

Combining Propositions 4.2 and 4.3 yields that

$$\sigma - \bar{\sigma} = \left(\frac{1}{T}\log([e^{A_\sigma T}\mathbf{1}_{m,1}]_{(i)})\right)^{1/2} - \left(\frac{1}{T}\int_0^T [e^{Qt}\widehat{\sigma^2}]_{(i)} dt\right)^{1/2} \tag{5.1}$$

given that $\alpha_0 = e_i$. To the best of our knowledge, the existing technique can not explicitly express $\sigma - \bar{\sigma} > 0$. However, on the two-state space, it is possible to further simplify (5.1) and obtain its closed-form expression.

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5.1. For Two-State Markov Chain with $\lambda > 0$, $\nu > 0$

In this section, we consider the Markov chain $(\alpha_t)_{t\in\mathbb{R}}$ on two-state space $\mathcal{M}_2 := \{e_1, e_2\}$ with its transition rate matrix that

$$Q_2 := \begin{pmatrix} -\lambda & \lambda \\ \nu & -\nu \end{pmatrix},$$

where $\lambda > 0, \nu > 0$. Hence, by similarity diagonalizing as shown in (4.7) and (4.10), we rewrite (5.1) as follows:

$$\sigma - \bar{\sigma} = \left(\frac{1}{T}\log\left([M\Lambda_T M^{-1}\mathbf{1}_{m,1}]_{(i)}\right)\right)^{1/2} - \left(\frac{1}{T}\left[M_\alpha D_e(T) M_\alpha^{-1}\widehat{\sigma^2}\right]_{(i)}\right)^{1/2}, \tag{5.2}$$

where

$$\begin{aligned}
M &= \begin{pmatrix} \dfrac{\nu - \lambda + \sigma^2(e_1) - \sigma^2(e_2) - \pi_1}{2\nu} & \dfrac{\nu - \lambda + \sigma^2(e_1) - \sigma^2(e_2) + \pi_1}{2\nu} \\ 1 & 1 \end{pmatrix}, \\
\Lambda_T &= \begin{pmatrix} e^{\widehat{r_1}T} & 0 \\ 0 & e^{\widehat{r_2}T} \end{pmatrix}, \\
\pi_1 &= \sqrt{(\sigma^2(e_1) - \sigma^2(e_2) + \nu - \lambda)^2 + 4\nu\lambda}, \quad \nu > 0,\ \lambda > 0, \\
\widehat{r_1} &= \frac{\sigma^2(e_1) + \sigma^2(e_2) - \lambda - \nu - \pi_1}{2}, \quad \widehat{r_2} = \frac{\sigma^2(e_1) + \sigma^2(e_2) - \lambda - \nu + \pi_1}{2}, \\
M_\alpha &= \begin{pmatrix} -\dfrac{\lambda}{\nu} & 1 \\ 1 & 1 \end{pmatrix}, \quad D_e(T) = \begin{pmatrix} \dfrac{1}{\lambda+\nu}(1 - e^{-(\lambda+\nu)t}) & 0 \\ 0 & t \end{pmatrix}.
\end{aligned} \tag{5.3}$$

Substituting the above expression into (5.2), we see that

PROPOSITION 5.1: *On the two-state space $\mathcal{M}_2$ of Markov chain with positive transition rate $\lambda > 0$, $\nu > 0$, and given $\alpha_0 = e_i$ for some $i \in \{1, 2\}$, the difference of the suggestive volatility σ and the averaged volatility $\bar{\sigma}$ is given by*

$$\sigma - \bar{\sigma} = \left[\sqrt{G_1(T)} - \sqrt{G_2(T)}\right]_{(i)}, \tag{5.4}$$

where

$$\begin{aligned}
G_1(T) &:= \left(\frac{\log(c_1c_3c_4 + c_2c_5c_6)}{T}, \frac{\log(c_1c_4 + c_2c_6)}{T}\right), \\
c_1 &= e^{\frac{T\left(\sigma^2(e_1) + \sigma^2(e_2) - \lambda - \nu - \pi_1\right)}{2}}, \quad c_2 = e^{\frac{T\left(\sigma^2(e_1) + \sigma^2(e_2) - \lambda - \nu + \pi_1\right)}{2}}, \\
c_3 &= \frac{\sigma^2(e_1) - \sigma^2(e_2) - \lambda + \nu - \pi_1}{2\nu}, \quad c_4 = \frac{\sigma^2(e_1) - \sigma^2(e_2) - \lambda - \nu + \pi_1}{2\pi_1}, \\
c_5 &= \frac{\sigma^2(e_1) - \sigma^2(e_2) - \lambda + \nu + \pi_1}{2\nu}, \quad c_6 = \frac{-\sigma^2(e_1) + \sigma^2(e_2) + \lambda + \nu + \pi_1}{2\pi_1}, \\
G_2(T) &= \left(\frac{(\nu\sigma^2(e_1) + \lambda\sigma^2(e_2))}{\lambda+\nu} + \frac{\lambda(\sigma^2(e_2) - \sigma^2(e_1))}{(\lambda+\nu)^2 T}(e^{-(\lambda+\nu)T} - 1),\right. \\
&\qquad \left.\frac{(\nu\sigma^2(e_1) + \lambda\sigma^2(e_2))}{\lambda+\nu} + \frac{\nu(\sigma^2(e_1) - \sigma^2(e_2))}{(\lambda+\nu)^2 T}(e^{-(\lambda+\nu)T} - 1)\right),
\end{aligned}$$

and π_1 is given by (5.3).

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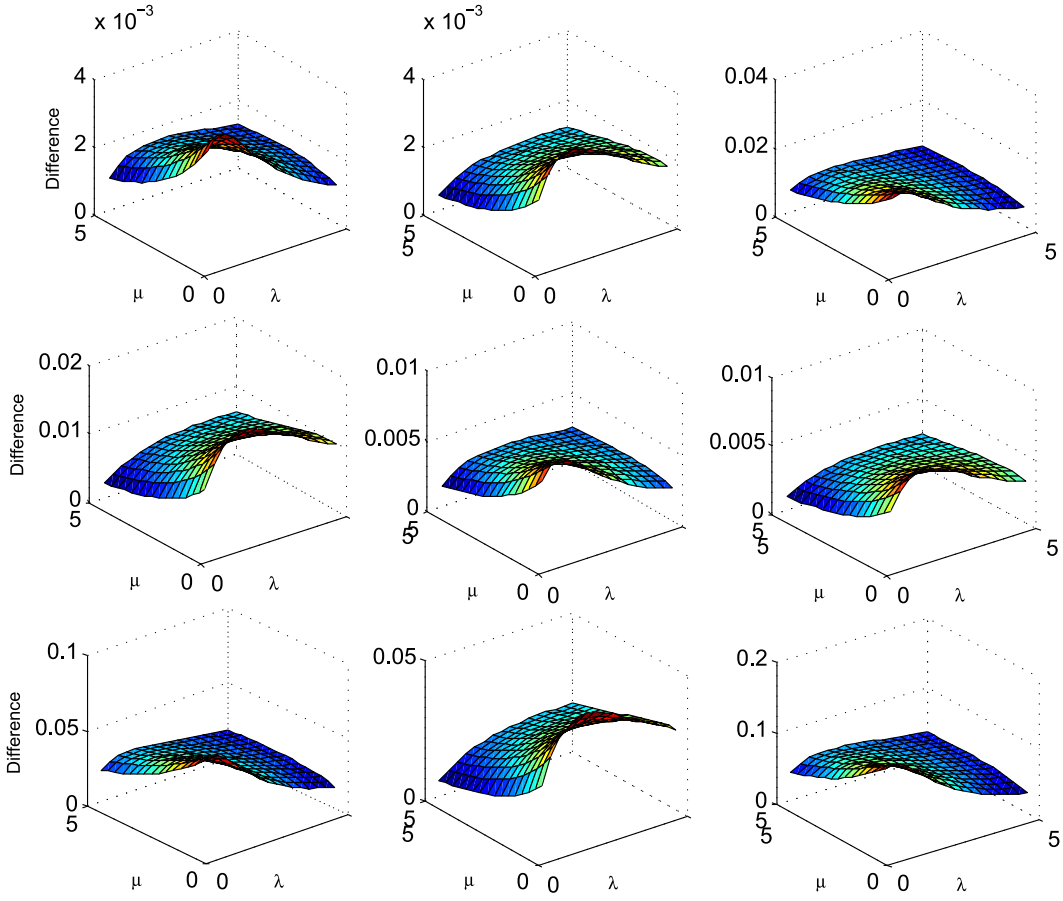


FIGURE 1. $\sigma - \bar{\sigma}$.

Next, applying Proposition 5.1, we implement the calculation of $\sigma - \bar{\sigma}$ numerically. Firstly, let $T = 1$ for all cases below, then let two horizontal axes denote λ and ν, respectively, and the vertical axes record the difference $\sigma - \bar{\sigma}$. Hence, we plot nine graphs in Figure 1, where the parameters of each are taken from the entry at the corresponding place of Table 1. Besides, a clear observation (let $\sigma(e_1) = 0.2, \sigma(e_2) = 0.8$) is displayed in Figure 2, the left one of which shows the $\sigma - \bar{\sigma}$ values for different λ and ν, and the right one records the $\sigma - \bar{\sigma}$ values for the cases that $\lambda = \nu$. Some properties are observed from Figures 1 and 2:

1. Both Figures 1 and 2 lead to conclude that $\sigma - \bar{\sigma}$ is significantly subject to the transition rate of Markov chains. That is, the model volatility σ is partially caused by the transition of Markov states, namely the regime-switchings.
2. When both $\sigma(e_1)$ and $\sigma(e_2)$ are small, variations caused by Brownian randomness and regime-switching are small. This is illustrated by the first two graphs in Figure 1, where we see both the value of $\sigma - \bar{\sigma}$ are below 0.004. Beside, σ equals $\bar{\sigma}$ when $\sigma(e_1) = \sigma(e_2)$, so there is no need to plot the cases of $\sigma(e_1) = \sigma(e_2)$, and it also explains the phenomenon in the fifth and sixth graphs in Figure 1, where both the value of $\sigma - \bar{\sigma}$ are relatively small (below 0.005 for most part) when the $\sigma(e_1)$ and $\sigma(e_2)$ are close.

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TABLE 1. Parameters of Figure 1

$(\sigma(e_1) = 0.2, \sigma(e_2) = 0.5)$	$(\sigma(e_1) = 0.5, \sigma(e_2) = 0.2)$	$(\sigma(e_1) = 0.2, \sigma(e_2) = 0.8)$
$(\sigma(e_1) = 0.8, \sigma(e_2) = 0.2)$	$(\sigma(e_1) = 0.5, \sigma(e_2) = 0.8)$	$(\sigma(e_1) = 0.8, \sigma(e_2) = 0.5)$
$(\sigma(e_1) = 0.2, \sigma(e_2) = 1.1)$	$(\sigma(e_1) = 1.1, \sigma(e_2) = 0.2)$	$(\sigma(e_1) = 0.5, \sigma(e_2) = 1.5)$

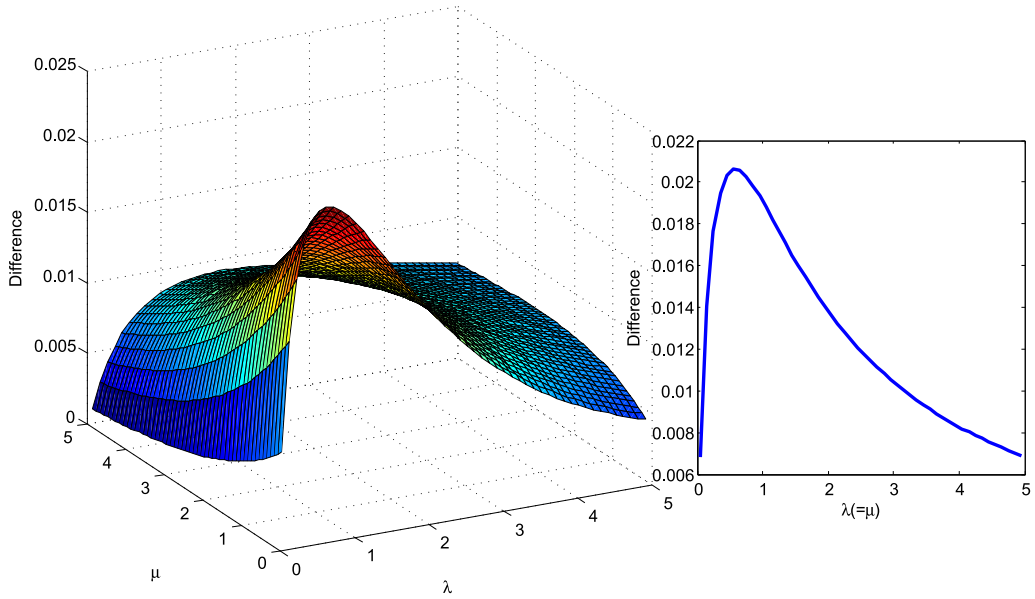


FIGURE 2. $\sigma - \bar{\sigma}$.

3. Checking Figure 2, especially the curve in the left side, we find the previous intuition that the volatility increases by enlarging the transition is incorrect. Indeed, this curve has a turning point (λ is about 0.9) and glides downward after it. Since $\alpha_0 = e_1$, the expected time of first jump is 1 when $\lambda = 1$, volatility rises because the regime will remain or change once before $T = 1$. After this turning point, switching becomes more frequent, the accumulated discrepancy of states and the global variation are eliminated gradually.

Based on the analysis above, it has been demonstrated that behavior of regime-switching contributes to volatility as claimed by Hypothesis 3.1.

5.2. For Two-State Markov Chain with an Absorbing State

In this section, we consider the Markov chain $(\alpha_t)_{t\in\mathbb{R}}$ on two-state space $\mathcal{M}_2 := \{e_1, e_2\}$ with its transition rate matrix that

$$Q_3 := \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix},$$

where $\lambda > 0$, namely letting $\nu = 0$ in the transition rate matrix Q_2 in Section 5.2. Hence, state e_2 becomes an absorbing state. Applying the similar approach as for Proposition 5.1, we have Proposition 5.2 as follows:

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PROPOSITION 5.2: *On the two-state space $\mathcal{M}_2$, consider the Markov chain $(\alpha_t)_{t\in\mathbb{R}}$ with the transition rate matrix Q_3. Given $\sigma^2(e_1)-\sigma^2(e_2)\neq\lambda$ and $\alpha_0=e_i$ for some $i\in\{1,2\}$, the difference of the suggestive volatility σ and the averaged volatility $\bar{\sigma}$ is given by*

$$\sigma-\bar{\sigma}=\left[\sqrt{V_1(T)}-\sqrt{V_2(T)}\right]_{(i)}, \tag{5.5}$$

where

$$V_1(T)=\left(\frac{1}{T}\log\left(\frac{(\sigma^2(e_1)-\sigma^2(e_2))e^{(\sigma^2(e_1)-\lambda)T}-\lambda e^{\sigma^2(e_2)T}}{\sigma^2(e_1)-\sigma^2(e_2)-\lambda}\right),\sigma^2(e_2)\right),$$

$$V_2(T)=\left(\sigma^2(e_2)+\frac{1}{\lambda T}(\sigma^2(e_2)-\sigma^2(e_1))(e^{-\lambda T}-1),\sigma^2(e_2)\right).$$

Notice that when $\sigma^2(e_1)-\sigma^2(e_2)=\lambda$, we also have (4.9), with $A_\sigma:=\begin{pmatrix}\sigma^2(e_1)-\lambda & \lambda\\ 0 & \sigma^2(e_2)\end{pmatrix}$, which fails to be similarity diagonalized, hence a similar expression as in (5.2) and (5.3) is infeasible, see also that the denominator π_1 in (5.3) and (5.4) becomes 0.

However, what makes things easier is the fact that $\exp\left(t\begin{pmatrix}a & b\\ 0 & a\end{pmatrix}\right)=e^{at}\begin{pmatrix}1 & bt\\ 0 & 1\end{pmatrix}$ for any real number a, b, t. Hence, with the condition $\sigma^2(e_1)-\sigma^2(e_2)=\lambda$, we have $e^{A_\sigma T}=e^{\sigma^2(e_2)T}\begin{pmatrix}1 & \lambda T\\ 0 & 1\end{pmatrix}$. Substituting this formula into (4.9) and letting $\nu=0$ in $G_2(T)$ of (5.4), we see that

$$\sigma-\bar{\sigma}=\left[\sqrt{H_1(T)}-\sqrt{H_2(T)}\right]_{(i)}, \tag{5.6}$$

where $\alpha_0=e_i$ for some $i\in\{1,2\}$,

$$H_1(T)=\left(\sigma^2(e_2)+\frac{1}{T}\log(1+\lambda T),\sigma^2(e_2)\right)\quad\text{and}\quad H_2(T)=\left(\sigma^2(e_2)-\frac{e^{-\lambda T}-1}{T},\sigma^2(e_2)\right). \tag{5.7}$$

Connections between the cases with and without an absorbing state: It is easy to check that when $\nu>0$ approaches 0, $(\sigma,\bar{\sigma})$ with their expressions $(\sqrt{G_1(T)},\sqrt{G_2(T)})$ in Proposition 5.1 converges to $(\sqrt{V_1(T)},\sqrt{V_2(T)})$ (for $\sigma^2(e_1)-\sigma^2(e_2)\neq\lambda$) and $(\sqrt{H_1(T)},\sqrt{H_2(T)})$ (for $\sigma^2(e_1)-\sigma^2(e_2)=\lambda$) for the case with an absorbing state (namely $\nu=0$). Hence ,we conclude that both mappings $\nu\to\sigma$ and $\nu\to\bar{\sigma}$ are right-continuous at $\nu=0$.

5.3. Asymptotic Behavior of Volatility

For the case with two-state Markov chain denotes the limit of suggestive volatility σ and averaged volatility $\bar{\sigma}$ by σ_∞ and $\bar{\sigma}_\infty$, respectively, namely (1) $\sigma_\infty=\lim_{T\to\infty}\sqrt{G_1(T)}$ and $\bar{\sigma}_\infty=\lim_{T\to\infty}\sqrt{G_2(T)}$ when $\lambda>0$, $\nu>0$ or $\nu=0$ but $\sigma^2(e_1)-\sigma^2(e_2)\neq\lambda$ and (2) $\sigma_\infty=\lim_{T\to\infty}\sqrt{H_1(T)}$ and $\bar{\sigma}_\infty=\lim_{T\to\infty}\sqrt{H_2(T)}$ when $\nu=0$ and $\sigma^2(e_1)-\sigma^2(e_2)=\lambda$. In this section, we assume that $\alpha_0=e_i$ for some $i\in\{1,2\}$.

First, for all cases (with or without an absorbing state) on two-state space $\mathcal{M}_2$, it is easily checked that for both $\alpha_0=e_1$ and $\alpha_0=e_2$, $\bar{\sigma}_\infty=\sqrt{(\nu\sigma^2(e_1)+\lambda\sigma^2(e_2))/(\lambda+\nu)}$. Next, to analyze σ_∞, we discuss based on classification as follows:

1. For the case $\lambda>0$, $\nu>0$:

$$\sigma_\infty=\left[\sqrt{\frac{\sigma^2(e_1)+\sigma^2(e_2)-\lambda-\nu+\pi_1}{2}},\sqrt{\frac{\sigma^2(e_1)+\sigma^2(e_2)-\lambda-\nu+\pi_1}{2}}\right]_{(i)}.$$

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2. For the case $\lambda > 0$, $\nu = 0$, $\sigma^2(e_1) - \sigma^2(e_2) < \lambda$: $\sigma_\infty = [\sigma(e_2), \sigma(e_2)]_{(i)}$.
3. For the case $\lambda > 0$, $\nu = 0$, $\sigma^2(e_1) - \sigma^2(e_2) > \lambda$:

$$\sigma_\infty = \left[\sqrt{\sigma^2(e_1) - \lambda}, \sqrt{\sigma^2(e_1) - \lambda}\right]_{(i)}.$$

4. For the case $\lambda > 0$, $\nu = 0$, $\sigma^2(e_1) - \sigma^2(e_2) = \lambda$: $\sigma_\infty = [\sigma(e_2), \sigma(e_2)]_{(i)}$.

The last item indicates that, for the case with an absorbing state and under the condition $\sigma^2(e_1) - \sigma^2(e_2) = \lambda$, the difference $\bar{\sigma} - \sigma$ converges to 0. This fact is also achieved by passing T to infinity in formula (5.6).

As we find in the third item above, when $\sigma^2(e_1) - \sigma^2(e_2) > \lambda$, σ will not converges to $\sigma(e_2)$ when T approaches infinity. We notice that $\bar{\sigma}_\infty = \sigma(e_2)$ when $\nu = 0$ because the equilibrium state of the chain $(\alpha_t)_{t\in\mathbb{R}_+}$ is $\sigma(e_2)$ when the transition rate of state e_2 is 0, but the transition rate of Markov process (X_t, α_t) is not 0 when we analysis σ in Section 4.1. Instead, the diagonal elements feature matrix A (which servers equivalently as the transition rate matrix) in (4.4) of infinitesimal generator $\mathcal{A}$ is $\sigma^2(e_1) - \lambda$ and $\sigma^2(e_2)$ (when $\nu = 0$), when $\sigma^2(e_1) - \sigma^2(e_2) > \lambda$, the equilibrium state deviates from state e_2.

As pointed at Section 5.2, the mapping $\nu \to \sigma$ is right-continuous at $\nu = 0$, it implies the existence of an unified expression of σ_∞ for all above discussed cases. Indeed, the above items can be summarized with extension to the case $\lambda = 0$, as the final expression of σ_∞ in the following proposition:

PROPOSITION 5.3: *For all $\lambda \geq 0$, $\nu \geq 0$ and $\lambda + \nu > 0$, and for both $\alpha_0 = e_1$ and $\alpha_0 = e_2$, we have*

$$\sigma_\infty = \sqrt{\frac{\sigma^2(e_1) + \sigma^2(e_2) - \lambda - \nu + \pi_1}{2}}, \quad \bar{\sigma}_\infty = \sqrt{\frac{\nu\sigma^2(e_1) + \lambda\sigma^2(e_2)}{\lambda + \nu}}, \tag{5.8}$$

and the following inequality holds:

$$\sigma_\infty \geq \bar{\sigma}_\infty, \tag{5.9}$$

and the strict inequality $\sigma_\infty > \bar{\sigma}_\infty$ holds for three situations below:

1. *$\lambda \neq 0$, $\nu \neq 0$ and $\sigma^2(e_1) \neq \sigma^2(e_2)$,*
2. *$\lambda = 0$, $\nu \neq 0$ and $\sigma^2(e_2) - \sigma^2(e_1) > \nu$,*
3. *$\nu = 0$, $\lambda \neq 0$ and $\sigma^2(e_1) - \sigma^2(e_2) > \lambda$,*

where indeed the first item has already covered the majority of possible values of (λ, ν).

A brief proof of Proposition 5.3 is presented in Appendix. This proposition implies that, for general cases with regime-switching governed by two-state Markov chain, the difference between suggestive volatility σ and averaged volatility $\bar{\sigma}$ always exists and reflects the influences caused by Markov chains as claimed by Proposition 3.3. This finding further enhances the acceptance of Hypothesis 3.1.

6. APPLICATION OF THE ABOVE OBTAINED PROPOSITIONS

The above claimed arguments, especially in Hypothesis 3.1 provides insights for model parameter estimation and related statistical work. As for parameter estimation and settings, Zhang [14] designs an integrated procedure and supported by [13]. This approach applies the historical volatility defined as in [5]. The existed methods usually estimate the drift parameters $\mu(\cdot)$ and $\sigma(\cdot)$ of all regimes through the observations, which contains both the price process and regime changes through all time. Regime detection and recognition seem

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not easy and can be biased. Therefore, it is attempted to recognize the regime and determine the parameters simultaneously based on the data of price process only. With our results of suggestive volatility, it is feasible to achieve that.

Assume that an observed price process $(Z_t)_{t\in[0,T]}$ follows the regime-switching model below:

$$dZ_t = \mu(\alpha_t)Z_t dt + \sigma(\alpha_t)Z_t dB_t, \tag{6.1}$$

where α_t follows a Markov chain on the state space $\{e_1, e_2\}$ with its transition rates $\lambda > 0$ and $\nu > 0$. We aim to figure out the parameters including $\mu(e_1)$, $\mu(e_2)$, $\sigma(e_1)$, $\sigma(e_2)$, λ and ν by applying the formula for suggestive volatility σ. However, it is not obvious to find an unbiased estimator for the σ or even for $e^{\sigma^2 T}$. Based on the discretization of the time interval $[0,T]$ that $t_i = iT/n$ for $i \in \{0,1,\ldots,n\}$ and $n \in \mathbb{N}_+$, we consider the ratio $\hat{Y}_{t_i}/\hat{Y}_{t_{i-1}}$ denoted as x_i. Based on the relation between first and second moment of x_i, $E[\exp(\int_{t_{i-1}}^{t_i} \sigma^2(\alpha_s)ds)]$ is estimated by $E[x_i^2]/E[x_i]^2$, hence $E[\exp(\int_0^T \sigma^2(\alpha_s)ds)]$ is estimated by $(n\sum_{i=0}^n x_i^2/(\sum_{i=0}^n x_i)^2)^n$ and we choose the following estimator $\hat{\sigma}(T,n)$ for σ:

$$\hat{\sigma}(t,k) := \sqrt{\frac{k}{t}\log\frac{k\sum_{i=0}^k x_i^2}{(\sum_{i=0}^k x_i)^2}}, \quad t \in \mathbb{R}_+,\ k \in [1,\ldots,n].$$

Recalling the results in Proposition 5.1, we have $\sigma = \sqrt{G_1(T,\alpha_0)}$ with

$$G_1(t,\alpha_0) = \left[\frac{\log(c_1c_3c_4 + c_2c_5c_6)}{t}, \frac{\log(c_1c_4 + c_2c_6)}{t}\right]_{\alpha_0}, \quad t \in (0,T],$$

where

$$c_1 = e^{\frac{t\left(\sigma^2(e_1)+\sigma^2(e_2)-\lambda-\nu-\pi_1\right)}{2}}, \quad c_2 = e^{\frac{t\left(\sigma^2(e_1)+\sigma^2(e_2)-\lambda-\nu+\pi_1\right)}{2}},$$
$$c_3 = \frac{\sigma^2(e_1)-\sigma^2(e_2)-\lambda+\nu-\pi_1}{2\nu}, \quad c_4 = \frac{\sigma^2(e_1)-\sigma^2(e_2)-\lambda-\nu+\pi_1}{2\pi_1},$$
$$c_5 = \frac{\sigma^2(e_1)-\sigma^2(e_2)-\lambda+\nu+\pi_1}{2\nu}, \quad c_6 = \frac{-\sigma^2(e_1)+\sigma^2(e_2)+\lambda+\nu+\pi_1}{2\pi_1}.$$

Immediately, we obtain one equation $\sqrt{G_1(T,\alpha_0)} = \hat{\sigma}(T,n)$ with all six unknowns, obviously it is not enough to solve. Therefore, we establish a system of six equations $\sqrt{G_1(iT/6,\alpha)} = \hat{\sigma}(iT/6,[i\cdot n/6])$ for $i = 1,2,\ldots,6$, where $[\cdot]$ denotes the integer after rounding the value in the bracket. Finally, they will be solved for unknown parameters $\mu(e_1)$, $\mu(e_2)$, $\sigma(e_1)$, $\sigma(e_2)$, λ and ν.

To conclude, this methods have some merits compared to other ways:

1. It is extendable for the model with multi-state. Slipping the time horizon $[0,T]$ into more unequal periods generates more equations to be solved for more unknown parameters.
2. In the situations that regimes are not obvious to define or not easy to observe, this method makes the estimation of transition rates available.
3. This method defines a new concept called implied regime-switching, to detect the existence of underlying regime-switching as well as how does it perform.

As for the implied regime-switching newly defined in above item (3), further research is required after we conclude the analysis on volatility in this paper.

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References

1. Darroch, J.N., & Morris, K.W. (1968). Passage-time generating functions for continuous-time finite Markov chains. *Journal of Applied Probability* 5(2): 414–426.
2. Fink, H., Klimova, Y., Czado, C., & Stober, J. (2017). Regime switching vine copula models for global equity and volatility indices. *Econometrics* 5(3): 1–38.
3. Hamilton, J.D. (1989). A new approach to the economic analysis of non-stationary time series. *Econometrica* 57: 357–384.
4. Heidergott, B., Leahu, H., Löpker, A., & Pflug, G. (2016). Perturbation analysis of inhomogeneous finite Markov chains. *Advances in Applied Probability* 48(1): 255–273.
5. Hull, J.C. (2000). *Options, futures, and other derivatives*, 4th ed. Upper Saddle River, NJ: Prentice Hall.
6. Liu, Y., & Nicolas, P. (2018). A recursive algorithm for selling at the ultimate maximum in regime-switching models. *Methodology and Computing in Applied Probability* 20(1): 369–384.
7. Liu, Y., Yang, A.J., & Lin, J.G. (2019). Option pricing of carbon emission allowance based on a regime switching model. *Journal of Applied Statistics and Management* 38(2): 225–234.
8. Ma, J., & Zhou, Z. (2018). Convergence analysis of iterative Laplace transform methods for the coupled PDEs from regime-switching option pricing. *Journal of Scientific Computing* 75(3): 1656–1674.
9. Ma, F., Wahab, M.I.M., Huang, D.S., & Xu, W.J. (2017). Forecasting the realized volatility of the oil futures market: a regime switching approach. *Energy Economics* 67: 136–145.
10. Øksendal, B. (2000). *Stochastic differential equations*, 5th ed. Berlin: Springer-Verlag.
11. Pedler, P.J. (1971). Occupation times for two state Markov chains. *Journal of Applied Probability* 8: 381–390.
12. Privault, N. (2013). *Understanding Markov chains – examples and applications,* 1st ed. Springer Undergraduate Mathematics Series. Singapore: Springer Science+Business Media.
13. Yao, D.D., Zhang, Q., & Zhou, X.Y. (2006). A regime-switching model for European options. *International Series in Operation Research and Management Science* 94: 281–300.
14. Zhang, Q. (2001). Stock trading: an optimal selling rule. *SIAM Journal on Control and Optimization* 40: 64–87.
15. Zhang, Y.J., & Zhang, L. (2015). Interpreting the crude oil price movements: evidence from the Markov regime switching model. *Applied Energy* 143: 96–109.

APPENDIX. A BRIEF PROOF FOR PROPOSITION 5.3

First, we obtain an unified expression of σ_∞ by checking the discussed four items of σ_∞ in Section 5.3 and applying the fact that $\pi_1 = |\sigma^2(e_1) - \sigma^2(e_2) - \lambda|$ when $\nu = 0$. The remaining work for this expression of σ_∞ is to justify the positivity of $\sigma^2(e_1) + \sigma^2(e_2) - \lambda - \nu + \pi_1$ denoted as $f_1(\lambda, \nu)$. Denote $\sigma^2(e_1) - \sigma^2(e_2)$ as η, we have $f_1(\lambda, \nu) > |\eta| - (\lambda + \nu) + \sqrt{x^2 + 2(\nu - \lambda)\eta + (\lambda + \nu)^2}$. Comparing $(\eta - (\lambda + \nu))^2$ with those under square root, we conclude that $f_1(\lambda, \nu) > 0$.

Next, we proof $\sigma_\infty \geq \bar{\sigma}_\infty$. Recalling the formulas of σ_∞ and $\bar{\sigma}_\infty$ in (5.8), we define $f_2(\lambda, \nu) := ((\lambda - \nu)/(\lambda + \nu))\eta - (\lambda + \nu) + \sqrt{\eta^2 + 2(\nu - \lambda)\eta + (\lambda + \nu)^2}$. Comparing $(((\lambda - \nu)/(\lambda + \nu))\eta - (\lambda + \nu))^2$ with those under square root, we conclude that $f_2(\lambda, \nu) \leq 0$, equivalently $\sigma_\infty \geq \bar{\sigma}_\infty$. Meanwhile, we notice that this inequality becomes equality only under at least one of the situations as follows:

1. $\sigma^2(e_1) = \sigma^2(e_2)$,
2. $\lambda = 0$ and $\sigma^2(e_2) - \sigma^2(e_1) < \nu$,
3. $\nu = 0$ and $\sigma^2(e_1) - \sigma^2(e_2) < \lambda$.

Hence, we complete the proof for Proposition 5.3.

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