GENERALIZED MAGNETO-HYDRODYNAMIC FORMULAE

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ABSTRACT. The formulae for current and force which were employed in some of the initial work on magneto-hydrodynamic theory are not valid over the whole range of phenomena which the subject itself embraces. Generalized formulae, suitable for wider application, are derived by an extension of the magneto-ionic approach, and are applied to the simple case of plane-wave propagation in the direction of a primary magnetic field. The dispersion equation which results, as well as the basic current and force formulae, are reduced by approximations in a variety of limiting circumstances. Some earlier results are thereby regained and, in particular, the region of known validity of Alfvén's propagation relation is extended.

1. *Introduction.* The study of magneto-hydrodynamics involves the effect on the motion of a conducting fluid produced by a current flowing through it, the current being generated by the motion and by the presence of electromagnetic fields. The mathematical treatment of the subject has been advanced with the use, on occasion, of the relations

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}) \tag{1}$$

and

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{2}$$

(See Alfvén (1, 2), Batchelor (4), and Herlofson (8), for example.) In these, $\mathbf{J}$ is the current density, $\mathbf{F}$ the corresponding force density, $\mathbf{E}$ the electric field intensity, $\mathbf{B}$ the magnetic induction, σ the conductivity, and $\mathbf{U}$ the velocity of the fluid, all in m.k.s. units. It is well known, however, that these formulae do not always hold and that the restrictions on them, as well as more general relations, can be determined by a study of charge diffusion in a composite medium.

This approach has been developed in great detail and generality by Chapman and Cowling (5) and Cowling (6) with the use of Boltzmann statistics, but their methods do not appear to have been applied in magneto-hydrodynamic problems as such. Indeed, it is doubtful whether the subject warrants a treatment of such accuracy and complexity, at least in its present stage of development.

Instead, it seems more appropriate to extend magneto-hydrodynamic formalism by the simpler methods of magneto-ionic theory, in which mean velocities and forces are employed, and non-linearities are avoided by a perturbation treatment. Accordingly, we shall adopt this approach in the present paper.

For simplicity of notation and calculation, we shall consider only those forces which result from macroscopic electromagnetic effects on charge and from friction-like action and reaction (of equal magnitude, opposite direction) between the free charge and the uncharged background material. Except in the limiting case when all such frictions are negligible, the omission of a corresponding inter-charge effect will limit our scope to media having only a slight degree of ionization; in particular, our formalism will not apply to any astrophysical problems in which collisions between charged

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particles play an effective part. A direct generalization could be made if the number of charge complexes were, for practical reasons, sufficiently restricted, but we shall not undertake such an extension here.

The perturbations will be confined, except by Fourier integration, to time variations of the form $\exp i\omega t$. In order that no similar restriction (e.g. to plane waves) shall be imposed on the spatial variations, and for further simplicity, we shall assume an unperturbed state in which a constant magnetic induction is the only non-vanishing field vector. It is for these same reasons that we neglect partial pressure gradients, even though they generally occur in the perturbed state; their effects become small at low temperatures, and vanish in the one problem to which we shall apply our results —that of plane-wave propagation along the primary magnetic field. If plane waves are assumed from the start these limitations can be avoided, though the formal complexity is then substantially increased (see Bailey (3)).

The use of mean values, as always in these macroscopic theories, is only legitimate in application if the numerical density of particles is sufficiently large.

2. *Magneto-hydrodynamic charge velocities.* Consider a charge complex having constant charge/mass ratio e/M, moving with velocity $\mathbf{V}$ in a field of electric intensity $\mathbf{E}$ and magnetic induction $\mathbf{B}$ through an uncharged medium having velocity $\mathbf{U}$. Its differential equation of motion is given, in the simple magneto-ionic theory, by

$$\partial\mathbf{V}/\partial t + (\mathbf{V}.\nabla)\,\mathbf{V} = (e/M)\,(\mathbf{E}+\mathbf{V}\times\mathbf{B}) + K(\mathbf{U}-\mathbf{V}), \tag{3}$$

where K is a suitably defined 'collision frequency' which may, for present purposes, be treated as a constant.

We shall be concerned with the consequences of this relation (combined, subsequently, with others) when a constant, real, primary magnetic induction $\mathbf{B}_0$ exists, and when the vectors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{E}$ and $\mathbf{B}$–$\mathbf{B}_0$ are perturbations, varying in time as $\exp i\omega t$, of sufficiently small magnitude that second-order terms in $\exp i2\omega t$ may be neglected. (If Fourier integration is to be employed, this last condition must be extended somewhat.) In these circumstances, (3) reduces to

$$i\omega\mathbf{V} = (e/M)\,(\mathbf{E}+\mathbf{V}\times\mathbf{B}_0) + K(\mathbf{U}-\mathbf{V}), \tag{4}$$

which may readily be solved for $\mathbf{V}$ explicitly.

If $\mathbf{B}_0$ is directed along the positive z axis, for example, one finds

$$V_x = U_x K/K' + [e/M(K'^2+G^2)]\,[K'(E_x+U_y B_0 K/K') + G(E_y - U_x B_0 K/K')], \tag{5}$$

$$V_y = U_y K/K' + [e/M(K'^2+G^2)]\,[K'(E_y-U_x B_0 K/K') - G(E_x+U_y B_0 K/K')], \tag{6}$$

$$V_z = U_z K/K' + [e/MK']\,E_z, \tag{7}$$

where $K' = K + i\omega$ and $G = B_0 e/M = \pm$ the gyrofrequency, according as the charge is positive or negative.

Alternatively, these results can be combined into the single vector relation

$$\mathbf{V} = \mathbf{U} + [1/MK']\,\mathbf{F}^* + [e/M^2(K'^2+G^2)]\,\mathbf{F}^*\times\mathbf{B}_0 + [e^2/M^3K'(K'^2+G^2)]\,[\mathbf{F}^*\times\mathbf{B}_0]\times\mathbf{B}_0, \tag{8}$$

in which

$$\mathbf{F}^* = e\mathbf{E} + e\mathbf{U}\times\mathbf{B}_0 - i\omega M\mathbf{U} \tag{9}$$

is the total force—electromagnetic plus inertial—which would act on the charge if it moved with the velocity $\mathbf{U}$ of the background material.

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The essentially magneto-hydrodynamic aspect of these formulae is to be found wherever **U** appears, for in the earlier applications of the magneto-ionic approach—e.g. in deriving the Appleton-Hartree relations—the background material was taken to be immobile.

3. *Generalized current and force densities.* We now consider a conducting fluid containing any number of charge complexes governed by equations of the type just obtained; they exert no influence on one another except indirectly, through their effect on the macroscopic electromagnetic field. A subscript r will be appended to the various parameters dependent on the charge, relating them to the general rth complex. Any summations which occur will be indicated by Σ, and are to be extended over all values of r. In the unperturbed state we have $\mathbf{E} = 0$, so there can be no net space-charge; if $N_r e_r$ is the unperturbed charge density of the rth complex, then $\Sigma N_r e_r = 0$.

In these circumstances, the current is found to have a density

$$\begin{aligned}\mathbf{J} &= \Sigma N_r e_r \mathbf{V}_r \\ &= \Sigma\{[N_r e_r/M_r K_r'] \mathbf{F}_r^* + [N_r e_r^2/M_r^2(K_r'^2+G_r^2)] \mathbf{F}_r^* \times \mathbf{B}_0 \\ &\qquad + [N_r e_r^3/M_r^3 K_r'(K_r'^2+G_r^2)] [\mathbf{F}_r^* \times \mathbf{B}_0] \times \mathbf{B}_0\}, \quad (10)\end{aligned}$$

and the frictional reaction on the background material, exerted by the charge in its motion, has a density

$$\begin{aligned}\mathbf{F} &= \Sigma N_r M_r K_r(\mathbf{V}_r - \mathbf{U}) \\ &= \Sigma\{[N_r K_r/K_r'] \mathbf{F}_r^* + [N_r e_r K_r/M_r(K_r'^2+G_r^2)] \mathbf{F}_r^* \times \mathbf{B}_0 \\ &\qquad + [N_r e_r^2 K_r/M_r^2 K_r'(K_r'^2+G_r^2)] [\mathbf{F}_r^* \times \mathbf{B}_0] \times \mathbf{B}_0\}. \quad (11)\end{aligned}$$

We are here neglecting, as usual, second-order perturbation effects.

Although formally compact, (10) and (11) are not normally convenient in application; it is frequently better to employ relations in which the longitudinal and transverse parts of **J**, **F**, **E** and **U** (relative to $\mathbf{B}_0$) appear explicitly. Denoting them by the subscripts L and T respectively, one obtains

$$\mathbf{J}_L = \Sigma[N_r e_r^2/M_r K_r'] \mathbf{E}_L + \Sigma[N_r e_r K_r/K_r'] \mathbf{U}_L, \quad (12)$$

$$\begin{aligned}\mathbf{J}_T &= \Sigma\left[\frac{N_r e_r^2 K_r'}{M_r(K_r'^2+G_r^2)}\right] \mathbf{E}_T + \Sigma\left[\frac{N_r e_r^3}{M_r^2(K_r'^2+G_r^2)}\right] \mathbf{E}_T \times \mathbf{B}_0 \\ &\qquad + \Sigma\left[\frac{N_r e_r K_r K_r'}{K_r'^2+G_r^2}\right] \mathbf{U}_T + \Sigma\left[\frac{N_r e_r^2 K_r}{M_r(K_r'^2+G_r^2)}\right] \mathbf{U}_T \times \mathbf{B}_0, \quad (13)\end{aligned}$$

$$\mathbf{F}_L = \Sigma[N_r e_r K_r/K_r'] \mathbf{E}_L - i\omega\Sigma[N_r M_r K_r/K_r'] \mathbf{U}_L, \quad (14)$$

and

$$\begin{aligned}\mathbf{F}_T &= \Sigma\left[\frac{N_r e_r K_r K_r'}{K_r'^2+G_r^2}\right] \mathbf{E}_T + \Sigma\left[\frac{N_r e_r^2 K_r}{M_r(K_r'^2+G_r^2)}\right] \mathbf{E}_T \times \mathbf{B}_0 \\ &\qquad - \Sigma\left[N_r M_r K_r \frac{i\omega K_r' + G_r^2}{K_r'^2+G_r^2}\right] \mathbf{U}_T + \Sigma\left[\frac{N_r e_r K_r^2}{K_r'^2+G_r^2}\right] \mathbf{U}_T \times \mathbf{B}_0, \quad (15)\end{aligned}$$

in which some reduction has been effected by use of the $\Sigma N_r e_r = 0$ condition.

As they stand, these relations hold only for $\exp i\omega t$ variations, but this limitation can be removed by Fourier integration. The other restrictions which have been imposed, to remove non-linearities and some complicating physical processes, are not necessarily confined to our present development; in effect, most apply implicitly even

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when the simpler relations, (1) and (2), are employed, and those that do not are often added for simplicity in application. Formulae (10)–(15) provide, then, the desired generalizations for current and force densities in conducting fluids.

4. *Simple magneto-hydrodynamic waves.* We shall now apply these results to a simple case of magneto-hydrodynamic propagation, in which plane waves travel in the direction of $\mathbf{B}_0$ through an otherwise homogeneous conducting fluid.

The only further relations we need for the determination of the dispersion equation, which will be our main concern, are the differential equation of motion for the background material,

$$\rho\,\partial\mathbf{U}/\partial t = \mathbf{F}, \tag{16}$$

and Maxwell's curl equations,

$$\operatorname{curl}\mathbf{E} = -\partial\mathbf{B}/\partial t \tag{17}$$

and

$$\operatorname{curl}\mathbf{B} = \mu\mathbf{J} + \epsilon\mu\,\partial\mathbf{E}/\partial t. \tag{18}$$

In (16), ρ is the unperturbed mass density of the background matter; the frictional reaction of the charge is assumed to provide the only force, and second-order terms are again neglected. Conservation equations could be added, and would generally indicate the extent of the perturbations in mass and charge density, but for present purposes they are redundant. (The densities stay constant, in our simple case.)

Assuming variations in $\mathbf{U}$, $\mathbf{E}$, $\mathbf{B}$–$\mathbf{B}_0$, $\mathbf{J}$ and $\mathbf{F}$ of the form $\exp i(\omega t+\beta z)$, where z is again a coordinate in the direction of $\mathbf{B}_0$, one finds that ω and β must be related by the generalized dispersion equation

$$\beta^2-\omega^2\epsilon\mu = -\omega\mu\,\Sigma\frac{N_r e_r^2}{M_r(\omega\pm G_r-iK_r)}+\frac{\omega\mu\left(\Sigma\dfrac{N_r e_r K_r}{\omega\pm G_r-iK_r}\right)^2}{\omega\rho-i\,\Sigma\dfrac{N_r M_r K_r(\omega\pm G_r)}{\omega\pm G_r-iK_r}}. \tag{19}$$

As is to be expected, the medium is doubly refracting (as the $\pm$ option indicates); two modes are propagated independently, and they are circularly polarized in opposite senses.

5. *Quasi-equilibrium current and force.* The generalized current, force and dispersion relations we have here obtained are too complex in structure to allow easy manipulation or interpretation. In limiting circumstances, however, they naturally tend to simpler formulae which will be useful in certain physical applications, and some of these we shall now investigate. First, let us consider the current and force obtaining when the variations take place extremely slowly in time.

If we set $\omega = 0$ in (12) and (13), and define $\mathbf{b}_0$ as a unit vector in the direction of $\mathbf{B}_0$, we find

$$\mathbf{J}_L = \sigma_L\mathbf{E}_L \tag{20}$$

and

$$\mathbf{J}_T = \sigma_{TD}(\mathbf{E}+\mathbf{U}\times\mathbf{B}_0)_T+\sigma_{TH}(\mathbf{E}+\mathbf{U}\times\mathbf{B}_0)_T\times\mathbf{b}_0, \tag{21}$$

where

$$\sigma_L = \Sigma N_r e_r^2/M_r K_r, \tag{22}$$

$$\sigma_{TD} = \Sigma N_r e_r^2 K_r/M_r(K_r^2+G_r^2), \tag{23}$$

and

$$\sigma_{TH} = \Sigma N_r e_r^2 G_r/M_r(K_r^2+G_r^2) \tag{24}$$

are, respectively, the longitudinal, the transverse direct, and the transverse Hall conductivities. These relations can, of course, be obtained more directly by omitting the acceleration term, $i\omega\mathbf{V}$, from equation (4) at the start. Such a procedure may, to some extent, be justified even for non-vanishing ω's by a quasi-equilibrium treatment;

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the charge is then assumed to reach a state of equilibrium in each region it traverses long before moving on, in time and space, to regions with appreciably different conditions.

The necessary and sufficient conditions under which (20) and (21) provide a good approximation to (12) and (13), even when ω does not vanish exactly, can be found by comparing the two sets of equations. These conditions will normally be satisfied if ω is very small relative to the collision frequencies of all the charge complexes—or at least, of the more prominent ones—although exceptional cases would arise if significant cancellations of $\mathbf{E}$ and $\mathbf{U}$ terms occurred, or if $\mathbf{U}_L$ were too great. For simplicity, we write $|\omega| \ll$ all K_r's as the condition of validity for (20) and (21).

Under these same conditions, and with similar reservations, (14) and (15) lead to

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}_0, \tag{25}$$

with $\mathbf{J}$ given by (21) [and (20)]. This corresponds to the original formula (2), though we are here confined to a constant $\mathbf{B}_0$ and to small perturbations. The first of these restrictions can be removed by the direct quasi-equilibrium approach, but the second remains as a condition that this approach shall itself be legitimate. One might anticipate (25), for in a quasi-equilibrium state the electromagnetic forces on the charge must be balanced by the frictional action, and so must be transferred to the background material as a frictional reaction.

If $|G_r| \ll K_r$ for all r's, (20) and (21) reduce further to

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}_0), \tag{26}$$

with $\sigma = \sigma_L$ as given by (22). This, of course, corresponds to the original relation (1). Again, if $|G_r| \gg K_r$ for all r's, we find

$$\mathbf{J}_L = \sigma \mathbf{E}_L \tag{27}$$

and

$$\mathbf{J}_T = \sigma^*(\mathbf{E} + \mathbf{U} \times \mathbf{B}_0)_T, \tag{28}$$

where

$$\sigma^* = \Sigma N_r M_r K_r / B_0^2 \ll \sigma.$$

(In deriving these formulae, it is often necessary to recall that $\Sigma N_r e_r = 0$, and that $G_r = B_0 e_r / M_r$. Thus, while individual terms in σ_{TH} are large relative to corresponding terms in σ_{TD}, if $|G_r| \gg K_r$, on summing there occurs a significant cancellation in σ_{TH} which leaves it small relative to σ_{TD}. Cancellations of this type will be found frequently on making various approximations to the general formulae. In this connexion it should be noted that electrons cannot, under the general conditions contemplated, be considered as the sole contributors to the current even in media for which such an approximation is normally valid, as the behaviour of σ_{TH} here illustrates.)

Although the general relations (10)–(15) were valid only for a single (complex) frequency, no such restriction applies to the results of this section. The various conductivities, being independent of ω, can be taken outside of any Fourier integrals which might be introduced. The $\mathbf{E}$ and $\mathbf{U}$ which now occur are, then, restricted only by the condition that $|\omega| \ll K_r$ for all their significant components, and by the perturbation condition.

The formulae which we have obtained here are equivalent to those derived by the more fundamental methods of a free-path kinetic theory under similar conditions (cf. Alfvén (2), or Chapman and Cowling (5)).

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6. *Approximations for large collision frequencies.* As we have just seen, the simple relations (1) and (2) can emerge if the collision frequencies are sufficiently large. It will be well to consider this case in some greater detail, to allow for comparison with some earlier results and to provide simplified formulae for application to mercury-like fluids in normal laboratory conditions.

Under the conditions $|\omega| \ll K_r \gg |G_r|$ for all r's—and again, with some reservations —the relations (12)–(15) reduce to

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}_0) - i\omega\tau\mathbf{U} \tag{29}$$

and

$$\mathbf{F} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}_0) \times \mathbf{B}_0 - i\omega\tau\mathbf{E} - i\omega\delta\mathbf{U} \tag{30}$$

where $\sigma = \Sigma N_r e_r^2/M_r K_r$ (as before), $\tau = \Sigma N_r e_r/K_r$ and $\delta = \Sigma N_r M_r$ = the total mass density of the charge. These relations are obtained, in effect, by expanding the exact formulae as series in ω/K_r and G_r/K_r, and retaining only the lowest-order non-cancelling coefficients of $\mathbf{E}$ and $\mathbf{U}$ separately; we assume that no unsuspected cancellations of significance occur (though they might in τ, between $\mathbf{E}$ and $\mathbf{U}$ terms, or between the $\mathbf{U}$ terms of (30)).

We now see that it is only if

$$|\omega\tau| \ll \sigma B_0 \quad \text{and} \quad |\omega|\,\delta \ll \sigma B_0^2$$

—or, loosely, if $|\omega/G_r| \ll |G_r/K_r|$—that the simple relations

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}_0) \tag{31}$$

and

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}_0 \tag{32}$$

[(26) and (25)] result from (29) and (30); the conditions $|\omega| \ll K_r$ imposed previously to validate the quasi-equilibrium formulae are not sufficiently restrictive if $|G_r| \ll K_r$.

If we slightly relax the conditions just imposed, so that now $|\omega| \ll |G_r|$ and hence $|\omega\tau| \leqslant \sigma B_0$—but not necessarily $|\omega|\delta \ll \sigma B_0^2$—we can again obtain (31) and (32), but with a new interpretation of $\mathbf{F}$: it must now be taken as acting on a fluid of mass density $\rho + \delta$ moving with the velocity $\mathbf{U}$. This is, at best, a rule of thumb; it breaks down, or leads to confusion, if other forces are acting on the background material. In any case, we see that formulae of the simple type (1) and (2) are valid, from the present point of view, only when severe restrictions are imposed on the systems.

Under the alternative limiting conditions $|\omega| \gg |G_r|$, (29) and (30) become

$$\mathbf{J} = \sigma\mathbf{E} - i\omega\tau\mathbf{U} \tag{33}$$

and

$$\mathbf{F} = -i\omega\tau\mathbf{E} - i\omega\delta\mathbf{U}, \tag{34}$$

respectively.

Reverting to our more general conditions, $|\omega| \ll K_r \gg |G_r|$, the dispersion equation (19) reduces to

$$\beta^2 - \omega^2\epsilon\mu = \omega^2[B_0^2/\mu(\rho + \delta) + i\omega/\mu\sigma]^{-1}, \tag{35}$$

which differs in no essential respect from the relation given by Alfvén (2) for magneto-hydrodynamic wave propagation along $\mathbf{B}_0$. (But we have included here the effect of the displacement current, and the units and notation are different.) However, the present development actually *extends* the known range of validity of this formula for, as we have just seen, Alfvén was essentially committed to the conditions

$$|\omega| \ll |G_r| \ll K_r$$

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(or to some others, which will appear later) in his use of the restricted current and force relations, (1) and (2), whereas the conditions now imposed are less stringent.

If we do add the condition $|\omega/G_r| \ll |G_r/K_r|$, or simply $|\omega|\delta \ll \sigma B_0^2$, then (35) reduces in turn to

$$\beta^2 - \omega^2\epsilon\mu = \omega^2[1+\delta/\rho][B_0^2/\mu\rho + i\omega/\mu\sigma]^{-1}, \tag{36}$$

while if $|\omega|(\rho+\delta) \ll \sigma B^2$ and displacement currents are neglected, it yields the well-known 'infinite conductivity' result

$$|\omega/\beta| = S \equiv B_0(\mu\rho)^{-\frac{1}{2}}, \tag{37}$$

giving a phase speed S if ω is real.

Alternatively, if $|\omega| \gg |G_r|$, we obtain another limiting form of (35),

$$\beta^2 - \omega^2\epsilon\mu = -i\omega\mu\sigma \tag{38}$$

approximately.

The dispersion equations which result from a solution of (16), (17) and (18) with the *approximate* current and force formulae, (29) and (30), or (33) and (34), differ from the corresponding relations, (35) or (38), by terms which are negligible under the relevant conditions. But this is not true, in the same sense, if the approximations (31) and (32) are used for the current and force; the $1+\delta/\rho$ factor of (36) is lacking in the dispersion equation to which they give rise. We are, of course, confined to slightly ionized media, so we may expect that δ/ρ must here be negligible relative to unity, and no real error occurs. But this has never been imposed as a mathematical condition, so the result illustrates the fact—and gives corresponding warning—that our approximation procedure need not commute with algebraic operations. When δ is not negligible relative to ρ (all physics aside), the fault can be traced to a failure of (32); although $-i\omega\delta\mathbf{U}$ may be negligible relative to $-\sigma B_0^2\mathbf{U}$ [in (30)], it is nevertheless comparable to $\sigma\mathbf{E}\times\mathbf{B}_0 - \sigma B_0^2\mathbf{U}$, since this combination is then of the order $i\omega\rho\mathbf{U}$. An unsuspected cancellation of $\mathbf{E}$ and $\mathbf{U}$ terms produces the error.

This is by no means solely an academic question. It is conceivable that similar cancellations may occur in other approximations, in cases where the effect would not be negligible: one should properly always conduct an *a posteriori* test for self-consistency. Certainly, in a more general treatment which accounts for inter-charge collisions, and which is therefore independent of the value of δ/ρ, this same situation will arise; the force formula would be more complicated than (30), but the term $-i\omega\delta\mathbf{U}$ would again occur and would again, under suitable conditions, be negligible compared with other terms appearing. But, if δ were not negligible relative to ρ, these other terms would cancel to such an extent that $-i\omega\delta\mathbf{U}$ would become significant, and its omission quite erroneous.

This point has been laboured simply to emphasize the caution which must sometimes be exercised in treating approximations of this type.

It has been noted that (35) is valid over a wider range than that covered in Alfvén's earlier work. This does not mean, however, that the solution as a whole remains unaltered over this wider range, as is immediately evident from the basic relations themselves. A more graphic illustration of this fact is obtained by considering the plane-polarized waves which can be propagated in limiting cases; when (31) and (32) apply, these are such that the vector groups $(\mathbf{E}, \mathbf{J})$, $(\mathbf{B}-\mathbf{B}_0, \mathbf{F}, \mathbf{U})$, and $(\mathbf{B}_0)$ are mutually orthogonal (cf. Alfvén (2)), but for (33) and (34) the grouping is $(\mathbf{E}, \mathbf{J}, \mathbf{F}, \mathbf{U})$, $(\mathbf{B}-\mathbf{B}_0)$, $(\mathbf{B}_0)$.

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7. *Approximations for low collision frequencies.* For application to tenuous gases, such as those in the ionosphere and in interstellar clouds, approximations for low collision frequencies will often be more appropriate. When $K_r \ll |\omega \pm G_r|$ for all r's—choosing the sign so as to make the right-hand side a minimum—(13) and (15) reduce to

$$\mathbf{J}_T = \Sigma\left[\frac{N_r e_r^2 i\omega}{M_r(G_r^2-\omega^2)}\right]\mathbf{E}_T + \Sigma\left[\frac{N_r e_r^3}{M_r^2(G_r^2-\omega^2)}\right]\mathbf{E}_T \times \mathbf{B}_0 + \Sigma\left[\frac{N_r e_r K_r i\omega}{G_r^2-\omega^2}\right]\mathbf{U}_T + \Sigma\left[\frac{N_r e_r^2 K_r}{M_r(G_r^2-\omega^2)}\right]\mathbf{U}_T \times \mathbf{B}_0 \quad (39)$$

and

$$\mathbf{F}_T = \Sigma\left[\frac{N_r e_r K_r i\omega}{G_r^2-\omega^2}\right]\mathbf{E}_T + \Sigma\left[\frac{N_r e_r^2 K_r}{M_r(G_r^2-\omega^2)}\right]\mathbf{E}_T \times \mathbf{B}_0 - \Sigma[N_r M_r K_r]\,\mathbf{U}_T, \quad (40)$$

respectively, assuming that there are no unsuspected cancellations of significance. For the longitudinal components, we must generally retain (12) and (14) as they stand, since the K_r's may be comparable to ω.

If $|\omega| \ll$ all $|G_r|$'s, we can obtain the further reduction

$$\mathbf{J}_T = \sigma^*(\mathbf{E} + \mathbf{U} \times \mathbf{B}_0)_T + (i\omega\delta/B_0^2)\,\mathbf{E}_T \quad (41)$$

and

$$\mathbf{F}_T = \sigma^*(\mathbf{E} + \mathbf{U} \times \mathbf{B}_0)_T \times \mathbf{B}_0 \quad (42)$$

with $\sigma^* = \Sigma N_r M_r K_r/B_0^2$ and $\delta = \Sigma N_r M_r$ as before. If, further, $|\omega| \ll$ all K_r's, these lead in turn to

$$\mathbf{J}_T = \sigma^*(\mathbf{E} + \mathbf{U} \times \mathbf{B}_0)_T \quad (43)$$

and

$$\mathbf{F}_T = \mathbf{J} \times \mathbf{B}_0. \quad (44)$$

These agree with the values given by (28) and (25) for the transverse components in the quasi-equilibrium approximation; complete agreement will, of course, depend on $\mathbf{E}_L$ and $\mathbf{U}_L$ being sufficiently small.

Alternatively, if $|\omega| \gg$ all $|G_r|$'s, (12)–(15) combine to give

$$\mathbf{J} = -i\Sigma[N_r e_r^2/M_r\omega]\,\mathbf{E} - i\Sigma[N_r e_r K_r/\omega]\,\mathbf{U} \quad (45)$$

and

$$\mathbf{F} = -i\Sigma[N_r e_r K_r/\omega]\,\mathbf{E} - \Sigma[N_r M_r K_r]\,\mathbf{U} \quad (46)$$

approximately.

In approximating to the dispersion equation (19), care must be taken to allow for the cancellations which can occur if the G_r's become large relative to ω, but which are otherwise of no significance. It is simpler to treat the two cases separately, rather than attempt to combine them in a single formula.

Thus, if $|\omega| \ll$ all $|G_r|$'s, we obtain

$$\beta^2 - \omega^2\epsilon\mu = \omega^2\mu\delta/B_0^2 - i\omega^2\mu\rho\sigma^*/(\omega\rho - i\Sigma N_r M_r K_r), \quad (47)$$

while if ω is not negligible relative to all the G_r's, or at least, relative to those of the predominant complexes, (19) then yields

$$\beta^2 - \omega^2\epsilon\mu = -\omega\mu\Sigma N_r e_r^2/M_r(\omega \pm G_r). \quad (48)$$

It will be seen that, in the intermediate case when ω is small relative to the G_r's but large relative to the K_r's, (47) and (48) agree on the formula

$$\beta^2 - \omega^2\epsilon\mu = \omega^2\mu\delta/B_0^2. \quad (49)$$

The less convenient, but more general, formula

$$\beta^2 - \omega^2\epsilon\mu = -\omega\mu\Sigma\frac{N_r e_r^2(\omega \mp G_r - iK_r)}{M_r(\omega^2 - G_r^2)} + \frac{\omega\mu\left(\Sigma\dfrac{N_r e_r K_r}{\omega \pm G_r}\right)^2}{\omega\rho - i\Sigma N_r M_r K_r} \quad (50)$$

does cover all these cases.

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If $|\omega| \ll$ all K_r's, (47) can be replaced by

$$\beta^2 - \omega^2\epsilon\mu = \omega^2[1+\delta/\rho]\,[B_0^2/\mu\rho + i\omega/\mu\sigma^*]^{-1}, \tag{51}$$

which is analogous to the Alfvén-type of relation, (36), as we should expect from the formal similarity of the corresponding current and force (or at least, of their transverse components—the only ones which affect propagation along $\mathbf{B}_0$).

Finally, if $|\omega| \gg$ all $|G_r|$'s, (48) reduces to

$$\beta^2 - \omega^2\epsilon\mu = -\mu\Sigma N_r e_r^2/M_r \tag{52}$$

approximately.

If these dispersion equations are now compared with those resulting from the corresponding current and force approximations, we find, as before, that any differences are of negligible magnitude. (But, it should be noted, the agreement results in some cases from a fortunate choice of the approximations quoted; other relations for **J** and **F**, of equal validity under the relevant conditions, can lead to erroneous dispersion equations.) When $|\omega| \ll$ all K_r's, the agreement again depends on the neglect of δ/ρ relative to unity; the reason for the breakdown is not so obvious here.

8. *Magneto-ionic approximations.* Dungey (7) has applied Bailey's generalized magneto-ionic formulae to obtain 'the dispersion equation for Alfvén's magneto-hydrodynamic waves', but he entirely neglected collisions between the charge and the background material for simplicity. (Bailey had included collisions, but with an unaccelerated background medium.) His formulae for propagation along $\mathbf{B}_0$ can be regained by setting $\epsilon\mu = 1/c^2$ in (48), and restricting the charge to two components.

By taking the limiting form of (19) as $\rho \to \infty$, corresponding to an immobile background material, we find the more general magneto-ionic dispersion equation

$$\beta^2 - \omega^2\epsilon\mu = -\omega\mu\Sigma N_r e_r^2/M_r(\omega \pm G_r - iK_r), \tag{53}$$

which includes collisional effects, for this simple type of propagation.

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