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Generalized Analytical Upper Bounds for American Option Prices

San-Lin Chung and Hsieh-Chung Chang*

Abstract

This paper generalizes and tightens Chen and Yeh's (2002) analytical upper bounds for American options under stochastic interest rates, stochastic volatility, and jumps, where American option prices are difficult to compute with accuracy. We first generalize Theorem 1 of Chen and Yeh (2002) and apply it to derive a tighter upper bound for American calls when the interest rate is greater than the dividend yield. Our upper bounds are not only tight, but also converge to accurate American call option prices when the dividend yield or strike price is small or when volatility is large. We then propose a general theorem that can be applied to derive upper bounds for American options whose payoffs depend on several risky assets. As a demonstration, we utilize our general theorem to derive upper bounds for American exchange options and American maximum options on two risky assets.

I. Introduction

One stream of finance research focuses on deriving option pricing bounds.[1] Chen and Yeh (2002) provide upper bounds that are applicable to general types of American options, e.g., American calls (puts) on dividend-paying stocks and American calls (puts) on futures. Since Chen and Yeh's (2002) upper bounds are closed form, they are particularly useful when there are multiple state variables. Moreover, their upper bounds rely neither on the distribution of the state variable, nor on continuous time trading. Chaudhury (2004) also offers general properties required of analytical upper bounds for American options without direct reliance on the exercise boundary. Chaudhury's (2004) results are of economic importance because they enlarge the set of bounding claims whose values are always greater

*Chung, chungs@management.ntu.edu.tw, National Taiwan University, Department of Finance, 85, Section 4, Roosevelt Road, Taipei 106, Taiwan, and Chang, s1428007@cc.ncu.edu.tw, National Central University, Department of Finance, 300, Chungda Road, Chungli 320, Taiwan. We thank Gurdip Bakshi (associate editor and referee), Chuang-Chang Chang, Mohammed Chaudhury, Hsuan-Chi Chen, Ren-Raw Chen, Bing-Huei Lin, Paul Malatesta (the editor), Mark Shackleton, and Shih-Kuo Yeh for helpful comments. The authors thank the National Science Council of Taiwan for financial support.

[1]Option pricing bounds are useful because i) they provide qualitative properties of options, ii) they can be used to screen market data for empirical research, iii) they shed light on hedging, and iv) they are generally obtained with the least assumptions on the investor's preferences and the distributions of the underlying asset prices.

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than the American option value. Nevertheless, Chaudhury's upper bounds are only valid for American options on one underlying asset.

Although Chen and Yeh's (2002) analytical upper bounds are very general, they can be applied only in the case where the interest rate is greater than the dividend yield. Their upper bounds may be inadequate for options on several underlying assets because it is likely that some underlying asset will have a dividend yield larger than the risk-free rate. In contrast, this article provides two general theorems that can be used to derive upper bounds for American options under general situations, including the case in which the dividend yield is larger than the risk-free rate. As a demonstration, we apply our general theorems to derive upper bounds for American calls when the interest rate is smaller than the dividend yield for American exchange options and for American maximum options on two risky assets.

We contribute to the literature on option pricing bounds in several ways. First, while previous papers on option bounds concentrate on European options with a single underlying asset or a single state variable, we provide upper bounds for American options whose pricing involves several risky assets and/or several risk factors (e.g., stochastic interest rates, stochastic volatility, and jumps) for each asset price process. Second, our upper bounds are not only tight, but also converge to accurate American call option prices when the dividend yield or strike price is small or when volatility is large. Third, we correct typos in Chen and Yeh (2002) and provide numerical results to investigate the tightness of their upper bounds and the tightness of ours. The numerical results indicate that our upper bounds are generally tighter than those of Chen and Yeh (2002).

The rest of this article proceeds as follows. Section II provides a general analysis for obtaining upper bounds of American options. In this section, two general theorems for developing upper bounds of American option prices are introduced. Section III discusses upper bounds under the stochastic interest rates, stochastic volatility, and jumps model using the inversion Fourier method. Heston (1993), Scott (1997), Bakshi, Cao, and Chen (1997), and Bakshi and Madan (2000) also use this model. We derive upper bounds for American exchange options and American maximum options under the Black-Scholes economy in this section. Section IV provides numerical results to analyze the tightness of our upper bounds. Section V concludes the paper.

II. Upper Bounds for American Option Prices

Theorem 1 of Chen and Yeh (2002) shows that an American option is bounded from above by the risk-neutral expectation of its maturity payoff if this expectation is greater than the intrinsic value at all times.[2] This theorem is very general, and the only assumptions required are that i) the risk-neutral measure exists, and ii) the nominal risk-free rate is strictly positive and greater than the dividend yield of the underlying asset. We restate Theorem 1 of Chen and Yeh (2002) as follows:

[2]Following Chen and Yeh (2002), all expectations are taken under the risk-neutral measure throughout the paper.

Theorem 1 of Chen and Yeh (2002). An American option is bounded from above by the risk-neutral expectation of its maturity payoff if this expectation is greater than the intrinsic value at all times.

This article extends Chen and Yeh's idea so that it is not necessary to use the maturity payoff $(X(T))$ of the American option to derive upper bounds. Instead, we replace $X(T)$ with other functions that can be applied to derive tighter upper bounds in our generalized Theorem 1. Note that both theorems are proved by a discrete approximation similar to the lattice approach. The results will hold in continuous time as Δt reaches a limit.

A. The Generalized Theorem 1

The Generalized Theorem 1 of Chen and Yeh (2002). Let T be the maturity of the option contract. Define $Y(t,T) = \omega(t,T)X(T)$, where $X(t)$ is the intrinsic value of the option at time t and $\omega(t,T)$ is any function that satisfies: i) $\omega(t,s) \geq \delta(t,s)$ for any $t < s$, where $\delta(t,s)$ is the discount factor from time t to time s, ii) $\omega(t,T) = \omega(t,s)\omega(s,T)$ for any $s \in (t,T)$, and iii) $\omega(t,t) = 1$. If $E_t[Y(t,T)] > X(t)$ for all t, then $E_t[Y(t,T)]$ is an upper bound of the American option value at time t.

Proof. Following Chen and Yeh (2002), we prove this theorem using a discrete approximation similar to the lattice approach. At time $T - \Delta t$, consider the function $Y(T-\Delta t, T)$ defined in the generalized Theorem 1. It is true that $E_{T-\Delta t}[Y(T-\Delta t,T)]$ is larger than the discounted terminal value,

$$
\begin{aligned}
(1) \qquad E_{T-\Delta t}[Y(T-\Delta t,T)] &= E_{T-\Delta t}[\omega(T-\Delta t,T)X(T)] \\
&> E_{T-\Delta t}[\delta(T-\Delta t,T)X(T)],
\end{aligned}
$$

where the second inequality comes from condition i) of the generalized Theorem 1. By constraint, $E_{T-\Delta t}[Y(T-\Delta t,T)]$ is also larger than the intrinsic value $(X(T-\Delta t))$ and thus is an upper bound of the American option price at time $T-\Delta t$. Since $E_{T-\Delta t}[Y(T-\Delta t,T)] \geq X(T-\Delta t)$ is a fact, it is true that $E_{T-2\Delta t}[Y(T-2\Delta t,T)]$ is greater than the continuation value of the American option at time $T-2\Delta t$,

$$
\begin{aligned}
(2) \qquad E_{T-2\Delta t}[Y(T-2\Delta t,T)] &= E_{T-2\Delta t}[\omega(T-2\Delta t,T)X(T)] \\
&= E_{T-2\Delta t}[\omega(T-2\Delta t,T-\Delta t)E_{T-\Delta t}[\omega(T-\Delta t,T)X(T)]] \\
&= E_{T-2\Delta t}[\omega(T-2\Delta t,T-\Delta t)E_{T-\Delta t}[Y(T-\Delta t,T)]] \\
&> E_{T-2\Delta t}[\delta(T-2\Delta t,T-\Delta t)E_{T-\Delta t}[Y(T-\Delta t,T)]] \\
&> E_{T-2\Delta t}[\delta(T-2\Delta t,T-\Delta t) \\
&\qquad \times \max\{E_{T-\Delta t}[\delta(T-\Delta t,T)X(T)], X(T-\Delta t)\}].
\end{aligned}
$$

The second line follows from condition ii) and the fourth line follows from condition i) of the generalized Theorem 1. By constraint, $E_{T-2\Delta t}[Y(T-2\Delta t,T)]$ is also greater than the intrinsic value of the American option and thus is an upper bound of the American option value at time $T-2\Delta t$. By mathematical induction, it is straightforward to show that $E_t[Y(t,T)]$ is an upper bound of the American option value. The result holds in continuous time when Δt approaches zero. □

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The main difference between Chen and Yeh's (2002) Theorem 1 and our generalized Theorem 1 is that we multiply the maturity payoff function by a function $\omega(t,T)$. Therefore, their upper bound is a special case of ours, where $\omega(t,T)=1$. As long as we can find an appropriate function $\omega(t,T)$ that is smaller than one and satisfies the criteria in the generalized Theorem 1, then $E_t[Y(t,T)]$ is an upper bound that is tighter than Chen and Yeh's.

It should be noted that when the function $\omega(t,T)$ is always smaller than one, then $E_t[Y(t,T)] \leq E_t[E_s[Y(s,T)]]$ for any $s \in (t,T)$. In this case, our upper bound is a submartingale process. In contrast, the upper bound of Chen and Yeh is a martingale process. Nevertheless, as with the discounted American option prices, the discounted processes of both upper bounds are supermartingale processes, i.e.,

$$\begin{aligned} E_t[Y(t,T)] &\geq E_t[\delta(t,s)E_s[Y(s,T)]], \\ E_t[X(T)] &\geq E_t[\delta(t,s)E_s[X(T)]]. \end{aligned} \tag{3}$$

B. A Further Extension: Theorem 2

The generalized Theorem 1 is actually still restrictive in the sense that the upper bound is related to the maturity payoff of the American option. If we extend our concept to allow $Y(t,T)$ to be any random variable that satisfies similar (or the same) criteria in the generalized Theorem 1, then it is possible to derive upper bounds for general types of American options. We next establish our Theorem 2 and give three applications later in the paper.

Theorem 2. Let T be the maturity of the option contract. Define $Y(t,T)$ as a random variable at time t that satisfies: i) $Y(T,T) \geq X(T)$, ii) $E_t[Y(t,T)] \geq E_t[\delta(t,t+\Delta t)Y(t+\Delta t,T)]$ for any $t \in [0, T-\Delta t]$, and iii) $E_t[Y(t,T)] \geq X(t)$ for all $t \in [0,T]$, where $X(t)$ is the intrinsic value of the option at time t. Thus, $E_t[Y(t,T)]$ is an upper bound of the American value.[3]

Proof. At time $T-\Delta t$, it is true that $E_{T-\Delta t}[Y(T-\Delta t,T)]$ is larger than the discounted terminal value,

$$\begin{aligned} E_{T-\Delta t}[Y(T-\Delta t,T)] &\geq E_{T-\Delta t}[\delta(T-\Delta t,T)Y(T,T)] \\ &= E_{T-\Delta t}[\delta(T-\Delta t,T)X(T)], \end{aligned} \tag{4}$$

where the first inequality comes from condition ii) of Theorem 2. From condition iii), $E_{T-\Delta t}[Y(T-\Delta t,T)]$ is also larger than the intrinsic value and hence is an upper bound of the American option price at time $T-\Delta t$.

[3]Following Chaudhury (2004), we can provide an economic meaning for our upper bound. Define a contingent claim whose value at time t, V_t, is $V_t = \max\{E_t[Y(t,T)], \delta(t,t+\Delta t)V_{t+\Delta t}\}$, $V_T = Y(T,T)$. It is easy to see that the defined contingent claim is always more valuable than the corresponding American option. In this case, the defined contingent claim satisfies the same property of the generalized claim as defined in Lemma 2 of Chaudhury (2004).

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Similarly, it is true that $E_{T-2\Delta t}[Y(T-2\Delta t,T)]$ is greater than the continuation value of the American option at time $T-2\Delta t$,

$$
\begin{aligned}
(5)\quad E_{T-2\Delta t}[Y(T-2\Delta t,T)] &\geq E_{T-2\Delta t}[\delta(T-2\Delta t,T-\Delta t)Y(T-\Delta t,T)] \\
&= E_{T-2\Delta t}[\delta(T-2\Delta t,T-\Delta t)E_{T-\Delta t}[Y(T-\Delta t,T)]] \\
&\geq E_{T-2\Delta t}[\delta(T-2\Delta t,T-\Delta t) \\
&\qquad \times \max\{E_{T-\Delta t}[\delta(T-\Delta t,T)X(T)],X(T-\Delta t)\}].
\end{aligned}
$$

By constraint, $E_{T-2\Delta t}[Y(T-2\Delta t,T)]$ is also greater than the intrinsic value of the American option and thus is an upper bound of the American option value at time $T-2\Delta t$. By mathematical induction, it is straightforward to show that $E_t[Y(t,T)]$ is an upper bound of the American option value at time t. The result holds in continuous time when Δt approaches zero. □

The idea of our Theorem 2 is quite intuitive and can be reasoned as follows. Condition i) is the terminal condition that has to be fulfilled by any upper bound. Condition ii) implies that the upper bound for this period is larger than the discounted value of the upper bound in the next period. Condition iii) is a necessary condition for an upper bound, i.e., an upper bound must be greater than the intrinsic value of the option at all times. Combining all three conditions will guarantee that $E_t[Y(t,T)]$ is an upper bound of the American option price.

It is worth noting that Theorem 1 of Chen and Yeh (2002), the generalized Theorem 1, and Theorem 2 are sustained even if the payoff function $X(T)$ or our general function $Y(t,T)$ depends on prices of multiple underlying assets. However, Theorem 1 of Chen and Yeh is applicable only when dividend yields of all assets are smaller than the risk-free rate.[4] We now show some applications of the generalized Theorem 1 and Theorem 2 in the following subsections.

C. Applications of the Generalized Theorems 1 and 2

1. American Calls on Dividend-Paying Stocks (when $r > q$)

According to the generalized Theorem 1, if one can find a suitable $Y(t,T)$ that satisfies our criteria, then $E_t[Y(t,T)]$ is an upper bound of the American option price. Let $Y(t,T)=e^{\int_t^T(q_u-r_u)du}\max\{S_T-K,0\}$, where S is the stock price, K is the strike price, r is the interest rate, and q is the dividend yield of the stock. It is easy to show that $\omega(t,T)=e^{\int_t^T(q_u-r_u)du}$ satisfies three criteria in the generalized Theorem 1. Moreover, we can verify that $E_t[Y(t,T)] > X(t)$ for all t,

$$
\begin{aligned}
(6)\quad & E_t\left[e^{\int_t^T(q_u-r_u)du}\max\{S_T-K,0\}\right] \\
&\geq \max\left\{E_t\left[S_T e^{\int_t^T(q_u-r_u)du}-Ke^{\int_t^T(q_u-r_u)du}\right],0\right\} \\
&> \max\{S_t-K,0\}.
\end{aligned}
$$

[4]Chen and Yeh's upper bound may be very restrictive for certain types of options on multiple underlying assets. For example, $E_t[X(T)]$ is an upper bound of the American exchange option price only when $r \geq q_2 \geq q_1$, where q_1 and q_2 are the dividend yields of the first and second underlying assets, respectively.

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The first line follows from Jensen's inequality and the second line holds since $E_t[S_T e^{\int_t^T (q_u - r_u)du}] = S_t$ and $r > q$. Therefore, $E_t[Y(t,T)]$ is an upper bound of the American call option.[5]

Since $r > q$ is true, our upper bound $(E_t[e^{\int_t^T (q_u - r_u)du} \max\{S_T - K, 0\}])$ is tighter than Chen and Yeh's $(E_t[\max\{S_T - K, 0\}])$. Furthermore, our upper bound converges to the accurate American call option price under some circumstances. For instance, it is well known that when the dividend yield is zero, the American call option price equals the price of its European counterpart (see Merton (1973)). Our upper bound also converges to the European call option price when the dividend yield approaches zero. Moreover, when the strike price is very small or when volatility is very large, both the accurate American call price and our upper bound converge to the stock price.

2. American Call Options on Dividend-Paying Stocks (when $r < q$)

Our Theorem 2 is valid no matter whether the interest rate is larger or smaller than the dividend yield. When the interest rate is smaller than the dividend yield, we define a function $Y(t,T)$ for American call options as

$$Y(t,T) = \max\left\{e^{\int_t^T [q_u - r_u]du} S_T - K, 0\right\}. \tag{7}$$

First of all, $Y(T,T) = \max\{S_T - K, 0\} = X(T)$, which satisfies condition i) of Theorem 2. Second, $Y(t,T)$ also fulfills condition ii) of Theorem 2 as

$$\begin{aligned} & E_t\left[\max\left\{S_T e^{\int_t^T (q_u - r_u)du} - K, 0\right\}\right] \\ > \; & E_t\left[\max\left\{S_T e^{\int_{t+\Delta t}^T (q_u - r_u)du} - K, 0\right\}\right] \\ > \; & E_t\left[\delta(t, t+\Delta t) \max\left\{S_T e^{\int_{t+\Delta t}^T (q_u - r_u)du} - K, 0\right\}\right] \\ = \; & E_t\left[\delta(t, t+\Delta t)\, Y(t+\Delta t, T)\right]. \end{aligned} \tag{8}$$

Finally, $E_t[Y(t,T)]$ is always greater than the intrinsic value at any time t because

$$\begin{aligned} E_t\left[\max\left\{e^{\int_t^T (q_u - r_u)du} S_T - K, 0\right\}\right] &\geq \max\left\{E_t\left[S_T e^{\int_t^T (q_u - r_u)du} - K\right], 0\right\} \\ &= \max\{S_t - K, 0\}. \end{aligned} \tag{9}$$

Thus, from Theorem 2 we know that $E_t[\max\{e^{\int_t^T (q_u - r_u)du} S_T - K, 0\}]$ is indeed an upper bound for American call options when $r < q$.

3. American Exchange Options

An exchange option is an option to exchange one asset for another. The payoff from this option is $X(T) = \max\{S_{1T} - S_{2T}, 0\}$, where S_{1T} and S_{2T} are values of asset one and asset two at time T, respectively. From Chen and Yeh's Theorem 1, it is easy to verify that $E_t[X(T)]$ is an upper bound of the American

[5] From our Theorem 2, we can propose another upper bound for American call options where $Y(t,T) = \max\{e^{\int_t^T [q_u - r_u]du} S_T - K e^{-\int_t^T r_u du}, 0\}$. However, this upper bound is not tighter than the above one and thus is not used in our numerical analysis.

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exchange option when $r \geq q_2 \geq q_1$, where q_1 and q_2 are the dividend yields of asset one and asset two, respectively. Now, we will derive another upper bound of the American exchange option.

Consider a function $Y(t,T)$ as

$$Y(t,T) = \max\left\{e^{\int_t^T [q_{1u}-r_u]du}S_{1T} - e^{\int_t^T [\min(q_{1u},q_{2u})-r_u]du}S_{2T}, 0\right\}. \tag{10}$$

Using a similar procedure, one can easily show that $Y(t,T)$ satisfies conditions i) to iii) of Theorem 2. Thus, $E_t[Y(t,T)]$ is an upper bound of the American exchange option price. It is not difficult to show that our upper bound is tighter than Chen and Yeh's, especially when dividend yields are small. Furthermore, it is true that our upper bound is applicable for any r, q_1, and q_2.

4. American Maximum Options on Two Risky Assets

Options on the maximum or minimum of two risky assets were first introduced by Stulz (1982). For example, the payoff of a European call option on the maximum of two risky assets is

$$X(T) = \max\{\max\{S_{1T}, S_{2T}\} - K, 0\}. \tag{11}$$

Following Theorem 1 of Chen and Yeh (2002), it is straightforward to show that the expected value of the maturity payoff is an upper bound of the American maximum option, because its value is always greater than the intrinsic value, i.e.,

$$\begin{aligned} E_t[\max\{\max\{S_{1T}, S_{2T}\} - K, 0\}] &\geq \max\{E_t[\max\{S_{1T}, S_{2T}\}] - K, 0\} \\ &\geq \max\{\max\{E_t[S_{1T}], E_t[S_{2T}]\} - K, 0\} \\ &> \max\{\max\{S_{1t}, S_{2t}\} - K, 0\}. \end{aligned} \tag{12}$$

However, this upper bound is sustained only when the dividend yields of all risky assets are smaller than the risk-free rate.

If one follows our Theorem 2, it is easy to prove that the expected value of the following function is also an upper bound of the American maximum option,

$$\begin{aligned} Y(t,T) = \max\Big\{&\max\left\{e^{\int_t^T [q_{1u}-r_u]du}S_{1T}, e^{\int_t^T [q_{2u}-r_u]du}S_{2T}\right\} \\ &- Ke^{\int_t^T [\min(q_{1u},q_{2u})-r_u]du}, 0\Big\}, \end{aligned} \tag{13}$$

because

$$\begin{aligned} & E_t\Big[\max\Big\{\max\left\{e^{\int_t^T [q_{1u}-r_u]du}S_{1T}, e^{\int_t^T [q_{2u}-r_u]du}S_{2T}\right\} \\ &\qquad - Ke^{\int_t^T [\min(q_{1u},q_{2u})-r_u]du}, 0\Big\}\Big] \\ &\geq \max\Big\{E_t\left[\max\left\{e^{\int_t^T [q_{1u}-r_u]du}S_{1T}, e^{\int_t^T [q_{2u}-r_u]du}S_{2T}\right\}\right] \\ &\qquad - E_t\left[Ke^{\int_t^T [\min(q_{1u},q_{2u})-r_u]du}\right], 0\Big\} \\ &\geq \max\Big\{\max\left\{E_t\left[e^{\int_t^T [q_{1u}-r_u]du}S_{1T}\right], E_t\left[e^{\int_t^T [q_{2u}-r_u]du}S_{2T}\right]\right\} \\ &\qquad - E_t\left[Ke^{\int_t^T [\min(q_{1u},q_{2u})-r_u]du}\right], 0\Big\} \\ &> \max\{\max\{S_{1t}, S_{2t}\} - K, 0\}. \end{aligned} \tag{14}$$

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There are many points worth discussing. First, under the Black-Scholes economy (where r and q_s are constant), our upper bound is actually the price of a European call option on the maximum of two risky assets with adjusted initial prices $S_{1t}e^{q_1(T-t)}$ and $S_{2t}e^{q_2(T-t)}$, and adjusted strike price $Ke^{\min(q_1,q_2)\times(T-t)}$. Second, although $Y'(t,T) = \max\{\max\{e^{\int_t^T[q_{1u}-r_u]du}S_{1T}, e^{\int_t^T[q_{2u}-r_u]du}S_{2T}\} - K, 0\}$ also satisfies conditions i) and iii) of Theorem 2, it does not satisfy condition ii) of Theorem 2, and thus it is not an upper bound of the American maximum call option. Third, our upper bound is valid for any r, q_1, and q_2 as long as $r > \min(q_1, q_2)$, e.g., $q_1 \geq r \geq q_2$. Finally, our upper bound is not necessarily smaller than Chen and Yeh's (2002) due to the fact that the strike price in our upper bound $(Ke^{\int_t^T[\min(q_{1u},q_{2u})-r_u]du})$ is smaller than the strike price in Chen and Yeh's upper bound (K).[6]

III. Bounds for a Class of Stock Price Processes

Both Chen and Yeh's (2002) upper bounds and our upper bounds have analytical solutions under many asset price models. In fact, as long as a European option has an analytical solution under a model, both upper bounds will also have analytical solutions under the same model because both upper bounds can be regarded as European options with an adjusted maturity payoff.[7] For example, a European option has an analytical solution under the stochastic volatility model of Heston (1993) and so does our upper bound.

To achieve a comparison with Chen and Yeh (2002), we derive analytical solutions for our upper bounds under stochastic interest rates, stochastic volatility, and the jumps (SVSIJ) model in the single asset cases.[8] However, we apply the Black-Scholes model in the multiple asset cases for simplicity.

A. American Calls on Dividend-Paying Stocks

Following Chen and Yeh (2002), we assume that under the risk-neutral measure the stock price S_t and its components are given by the following SVSIJ model,

$$
\begin{aligned}
dS_t &= (y_t S_t - \lambda_J \mu_J S_t)dt + S_t\sqrt{v_t}dW_t^S + J_t S_t d\xi_t, \\
dv_t &= \alpha(\beta - v_t)dt + \gamma\sqrt{v_t}dW_t^v, \\
dy_t &= a(b - y_t)dt + g\sqrt{y_t}dW_t^r, \\
\ln[1+J_t] &\sim N\left(\ln[1+\mu_J] - \frac{1}{2}\sigma_J^2, \sigma_J^2\right),
\end{aligned} \tag{15}
$$

where $y_t = r_t - q_t$; λ_J is the frequency of jumps per year; v_t is the diffusion component of return variance (conditional on no jump occurring); W_t^S, W_t^v, and W_t^r are

[6] When both upper bounds are applicable (i.e., r is larger than q_1 and q_2), one can take the minimum of both upper bounds as the upper bound of the American maximum option price.

[7] For example, Chen and Yeh's upper bound $E_t[X(T)]$ is actually the price of a European option with an adjusted payoff $e^{\int_t^T r_u du}X(T)$.

[8] The Black-Scholes model has been extended to the stochastic volatility, stochastic interest rate, and jumps model; see, for example, Heston (1993), Scott (1997), and Bakshi, Cao, and Chen (1997).

each a standard Brownian motion with $dW_t^S dW_t^v = \rho dt$; W_t^r is uncorrelated with W_t^S and W_t^v; α, β, γ, a, b, and g are parameters associated with the processes; and J_t is the percentage jump size (conditional on a jump occurring) that is log-normally, identically, and independently distributed over time with unconditional mean μ_J. The standard deviation of $\ln[1 + J_t]$ is σ_J; ξ_t is a Poisson jump counter with intensity λ_J, that is, $\Pr\{d\xi_t = 1\} = \lambda_J dt$ and $\Pr\{d\xi_t = 0\} = 1 - \lambda_J dt$; and ξ_t and J_t are uncorrelated with each other or with W_t^S, W_t^v, and W_t^r.

The traditional approach for solving our upper bound under the above SVSIJ model requires a correct conjecture of the solution. Substituting the conjecture into the valuation partial differential equations generally produces at most four valuation equations or corresponding conditional expectations. Moreover, one must again, by trial and error, guess a solution to each of the valuation equations.[9] Based on the characteristic function methodology of Bakshi and Madan (2000), our upper bound of the American call option is

$$\begin{aligned} (16)\quad U^C &= E\left[e^{\int_t^T(-y_u)du}\max\{(S_T-K),0\}\right] = G(t,T)\Pi_1 - KM(t,T)\Pi_2, \\ \Pi_j &= \frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-iu\ln K}f_j(u)}{iu}\right]du, \qquad j = 1,2, \end{aligned}$$

where $G(t,T)$, $M(t,T)$, and the characteristic functions (f_1 and f_2) are shown in Appendix B.

To derive the upper bound of American calls when $r < q$, we assume the same SVSIJ model as in the previous subsection except that

$$(17)\qquad dy_t' = a(b - y_t')dt + g\sqrt{y_t'}dW_t^r,$$

where $y_t' = q_t - r_t > 0$. Following a similar procedure, one can derive the upper bound of the American call options as

$$(18)\qquad U'^C = E_t\left[\max\left\{e^{\int_t^T(y_u')du}S_T - K, 0\right\}\right] = S_t\Pi_1' - K\Pi_2'.$$

The two probabilities Π_1' and Π_2' in equation (18) follow the same formulas as equation (16), but with two different characteristic functions ($f_1'(u)$ and $f_2'(u)$) that are available from the authors.

B. American Exchange Options on Dividend-Paying Stocks

Margrabe (1978) valued an option to exchange one asset for another, which is called an exchange option. We state his model in this subsection and use his setup to derive a closed-form solution for the upper bound of the American exchange option price. Following Margrabe (1978), we assume that the two asset prices follow

$$(19)\qquad dS_{it} = y_i S_{it}dt + \sigma_i S_{it}dW_{it}, \quad i = 1,2,$$

[9] See Heston (1993), Scott (1997), and Bakshi, Cao, and Chen (1997) for the derivations of option pricing formulas using the traditional approach.

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where $y_i = r - q_i$, q_i is the dividend yield of asset i, σ_i is the standard deviation of return of asset i, and dW_{it} is the Brownian motion of asset i.

For simplicity, we follow Margrabe (1978) to assume that both the risk-free rate and dividend yields are constant. Under this simplified assumption, Margrabe (1978) shows that the price of the European exchange option is

$$w_t = S_{1t}e^{-q_1\tau}N(z_1) - S_{2t}e^{-q_2\tau}N(z_2), \tag{20}$$

where

$$z_1 = \frac{\ln\left(\frac{S_{1t}}{S_{2t}}\right) + \left(q_2 - q_1 + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad z_2 = z_1 - \sigma\sqrt{\tau}, \quad \tau = T - t,$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2, \quad \text{and} \quad \rho_{12} = \text{Corr}\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right).$$

Our upper bound for the American exchange option can be regarded as the value of an European exchange option, where the initial value of asset one is $\zeta_{1t} = S_{1t}e^{q_1\tau}$ and the initial value of asset two is $\zeta_{2t} = S_{2t}e^{\min(q_1,q_2)\times\tau}$. Therefore, our upper bound has the closed-form solution,

$$E_t\left[\max\left\{e^{\int_t^T[q_1-r]ds}S_{1T} - e^{\int_t^T[\min(q_1,q_2)-r]ds}S_{2T}, 0\right\}\right] = S_{1t}N\left(z_1^*\right) - S_{2t}e^{(\min(q_1,q_2)-q_2)\tau}N\left(z_2^*\right), \tag{21}$$

where

$$z_1^* = \frac{\ln\left(\frac{S_{1t}}{S_{2t}}\right) + \left(q_2 - \min(q_1,q_2) + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad z_2^* = z_1^* - \sigma\sqrt{\tau}.$$

C. American Options on the Maximum of Two Risky Assets

Under the Black-Scholes economy, Stulz (1982) derives the closed-form solution for a European call option on the maximum of two risky assets as

$$\begin{aligned} &C_{\max}(S_{1t}, S_{2t}, K, r, q_1, q_2, \sigma_1, \sigma_2, \rho_{12}\sigma_1\sigma_2, \tau) \\ &\quad = C_{BS}(S_{1t}, K, r, q_1, \sigma_1, \tau) + C_{BS}(S_{2t}, K, r, q_2, \sigma_2, \tau) \\ &\qquad - C_{\min}(S_{1t}, S_{2t}, K, r, q_1, q_2, \sigma_1, \sigma_2, \rho_{12}\sigma_1\sigma_2, \tau), \end{aligned} \tag{22}$$

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where $C_{BS}(.)$ are Black-Scholes formulas for European call options and $C_{\min}(.)$ is the price of a European call option on the minimum of two risky assets,

$$
\begin{aligned}
(23)\quad & C_{\min}(S_{1t}, S_{2t}, K, r, q_1, q_2, \sigma_1, \sigma_2, \rho_{12}\sigma_1\sigma_2, \tau) \quad = \\
& S_{1t}e^{-q_1\tau}N_2\left(d_1+\sigma_1\sqrt{\tau}, \frac{\left(\ln\frac{S_{1t}}{S_{2t}}+\left(q_1-q_2-\frac{\sigma^2}{2}\right)\tau\right)}{\sigma\sqrt{\tau}}, \frac{\rho_{12}\sigma_2-\sigma_1}{\sigma}\right) \\
& +S_{2t}e^{-q_2\tau}N_2\left(d_2+\sigma_2\sqrt{\tau}, \frac{\left(\ln\frac{S_{2t}}{S_{1t}}+\left(q_2-q_1-\frac{\sigma^2}{2}\right)\tau\right)}{\sigma\sqrt{\tau}}, \frac{\rho_{12}\sigma_1-\sigma_2}{\sigma}\right) \\
& -Ke^{-r\tau}N_2\left(d_1, d_2, \frac{\rho_{12}\sigma_1-\sigma_2}{\sigma}\right),
\end{aligned}
$$

where

$$
d_1 = \frac{\ln\left(\frac{S_{1t}}{K}\right)+\left(r-q_1-\frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, \quad d_2 = \frac{\ln\left(\frac{S_{2t}}{K}\right)+\left(r-q_2-\frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}},
$$

and $\sigma^2=\sigma_1^2+\sigma_2^2-2\rho_{12}\sigma_1\sigma_2$.

It is apparent that our upper bound for the American maximum call option is actually the price of a European maximum call option with adjusted initial stock prices and an adjusted strike price. Therefore, it follows a similar formula, i.e.,

$$
(24)\quad C_{\max}(S_{1t}e^{q_1\tau}, S_{2t}e^{q_2\tau}, Ke^{\min(q_1,q_2)\times\tau}, r, q_1, q_2, \sigma_1, \sigma_2, \rho_{12}\sigma_1\sigma_2, \tau).
$$

IV. Numerical Results

A. Correct Numerical Results in Chen and Yeh (2002)

The pricing formulas of Chen and Yeh's (2002) upper bounds under the SVSIJ model are not correct, and thus the numerical results in Chen and Yeh (2002) are misleading.[10] The correct formulas are shown in Appendix A of this paper. We report the correct numbers in their Tables 1 and 2 to show the tightness of their upper bounds.

Panel A of Table 1 shows Chen and Yeh's upper bounds of calls and puts under a stochastic volatility environment (SV model). In the SV model, the risk-free rate and the dividend yield remain constant and no jumps occur in the stock price. Our results indicate that the original upper bounds of Chen and Yeh (2002) (see their Table 1) are even smaller than the European option values, thus confirming that their formulas are doubtful.

Panel B of Table 1 shows the upper bounds of Chen and Yeh for American calls on dividend-paying stocks and their counterpart European option values

[10]To save space, we only correct Chen and Yeh's (2002) typos for American calls on dividend-paying stocks in Appendix A. The correct formulas of Chen and Yeh (2002) for other types of American options can be obtained from the authors.

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under the SVSIJ model.[11] Following Chen and Yeh (2002), we compute the European option values using the pricing formulas of Bakshi, Cao, and Chen (1997) since these numbers serve as lower bounds of American option values. From Table 1, we find that Chen and Yeh's upper bounds are generally quite tight, because their values are very close to European option values. The differences between the upper bounds and the counterpart European option values are all within 5.5% of the counterpart European option values.

TABLE 1

Correct Numerical Results in Chen and Yeh (2002)

Panel A of Table 1 shows Chen and Yeh's upper bounds of American calls and puts under a stochastic volatility environment (SV model). The top numbers are Chen and Yeh's upper bound values and the bottom numbers are European values. The parameters are $S_t = 100$, $v_t = 0.04$, $r = 0.05$, $q = 0.03$, $\alpha = 1.5$, $\beta = 0.04$, $\gamma = 0.1$, $\rho = -0.5$, and $\tau = 1$. Panel B shows Chen and Yeh's upper bounds of American calls on dividend-paying stocks under the SVSIJ model. The top numbers are Chen and Yeh's upper bound values and the bottom numbers are European values. The parameters are $S_t = 100$, $v_t = 0.04$, $r_t = 0.05$, $q = 0.03$, $\alpha = 1.5$, $\beta = 0.04$, $\gamma = 0.1$, $\rho = -0.5$, $a = 0.6$, $b = 0.02$, $g = 0.05$, $\mu_J = 0$, $\sigma_J^2 = 0.1$, and $\lambda_J = 0.6$.

Panel A. Correct Numbers in Chen and Yeh's Table 1

K	Call Option	Put Option
110	4.86	14.05
	4.63	12.22
105	6.73	10.72
	6.40	9.23
100	9.06	7.86
	8.62	6.70
95	11.90	5.50
	11.32	4.64
90	15.22	3.65
	14.47	3.04

Panel B. Correct Numbers in Chen and Yeh's Table 2 (calls on dividend-paying stocks)

	ρ			λ_J		
K	-0.4	0.0	0.4	0.2	0.4	0.6
110	8.79	8.86	8.94	6.25	7.55	8.77
	8.36	8.43	8.50	5.94	7.18	8.34
105	10.64	10.68	10.72	8.11	9.41	10.62
	10.12	10.16	10.20	7.72	8.95	10.11
100	12.85	12.85	12.86	10.41	11.66	12.85
	12.22	12.23	12.23	9.90	11.10	12.22
95	15.44	15.41	15.38	13.15	14.34	15.45
	14.69	14.66	14.63	12.51	13.64	14.70
90	18.43	18.37	18.30	16.35	17.42	18.44
	17.53	17.47	17.41	15.55	16.57	17.54

B. Our Upper Bounds for American Calls on Dividend-Paying Stocks

We compare the tightness of our upper bound with that of Chen and Yeh's (2002) for American calls on dividend-paying stocks when $r > q$ under the SVSIJ model. We adopt the same parameter values as Panel B of Table 1. From Panel A of Table 2, it is clear that our upper bound of calls is indeed tighter than Chen and Yeh's upper bound when $r > q$. On average, our upper bounds are 3.03%

[11] The parameter values used in Chen and Yeh (2002) are close to those estimated by Bakshi, Cao, and Chen (1997) using S&P 500 index option data.

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larger than European option values while Chen and Yeh's upper bounds are 5.13% larger.

Chen and Yeh's upper bounds are not available for American calls on dividend-paying stocks when $r < q$. On the other hand, our upper bounds are still applicable in this case. To demonstrate the tightness of our upper bounds in this situation, we also use the SVSIJ model and adopt the same parameter values as Panel B of Table 1 except that we set $r_t = 0.03$ and $q_t = 0.05$. From Panel B of Table 2, we find that when the dividend yield is larger than the risk-free rate our upper bounds are not as tight as the case where the dividend yield is smaller than the risk-free rate. On average, our upper bounds are 13.82% larger than European option values when $r < q$. The results are expected because our upper bounds work best (i.e., no error) for American calls when the dividend yield is zero.

TABLE 2

Our Upper Bound for American Calls on Dividend-Paying Stocks

The top numbers in Table 2, Panel A are our upper bound values and the bottom numbers are European values. The parameter values are the same as Panel B of Table 1. The top numbers in Panel B are our upper bound values and the bottom numbers are European values. The parameter values are the same as Panel B of Table 1 except that $r = 0.03$ and $q = 0.05$.

	ρ			λ_J		
K	−0.4	0.0	0.4	0.2	0.4	0.6
Panel A. $r > q$						
110	8.61	8.69	8.76	6.12	7.40	8.59
	8.36	8.43	8.50	5.94	7.18	8.34
105	10.43	10.47	10.51	7.95	9.22	10.41
	10.12	10.16	10.20	7.72	8.95	10.11
100	12.59	12.60	12.60	10.20	11.43	12.59
	12.22	12.23	12.23	9.90	11.10	12.22
95	15.14	15.11	15.07	12.89	14.05	15.14
	14.69	14.66	14.63	12.51	13.64	14.70
90	18.06	18.00	17.94	16.03	17.08	18.07
	17.53	17.47	17.41	15.55	16.57	17.54
Panel B. $r < q$						
110	7.88	7.97	8.06	5.40	6.67	7.86
	6.90	7.00	7.09	4.58	5.76	6.88
105	9.58	9.64	9.70	7.09	8.37	9.57
	8.42	8.49	8.55	6.05	7.26	8.40
100	11.64	11.66	11.68	9.20	10.46	11.63
	10.27	10.30	10.34	7.93	9.13	10.26
95	14.08	14.07	14.05	11.77	12.96	14.09
	12.49	12.49	12.48	10.24	11.40	12.49
90	16.93	16.88	16.82	14.81	15.90	16.94
	15.10	15.06	15.02	13.02	14.09	15.11

C. American Exchange Options

To analyze the tightness of our upper bounds for American exchange options, we need benchmark values of them. We apply the adaptive mesh model of Figlewski and Gao (1999) to calculate the American exchange option price using the one-dimensional solution of Bjerksund and Stensland (1993).[12] The

[12]Pricing American exchange options is originally a two-dimensional stochastic problem under the Black-Scholes economy. Bjerksund and Stensland (1993) prove that the above two-dimensional

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number of time steps in the adaptive mesh model is 10,000. The parameter values are adopted from Chen, Chung, and Yang (2002) as follows: $S_1 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, and S_2, q_1, and q_2 are varied.

We let the risk-free rate be greater than the dividend yields of both assets in Panel A of Table 3 and thus the upper bound of Chen and Yeh is applicable. From Panel A of Table 3, it is apparent that our upper bounds are very tight, i.e., very close to the accurate prices of the American exchange options. Our upper bounds are about 1% larger than the accurate prices in general. It is also true that our upper bounds are tighter than Chen and Yeh's (2002).

In Panel B of Table 3, where the dividend yield of asset one is larger than the risk-free rate, Chen and Yeh's upper bound is not available while ours is still workable. However, our upper bounds are not as tight as the case in Panel A. The results are consistent with our argument that our upper bounds are tight when dividend yields of the underlying assets are small.

TABLE 3

Upper Bounds of American Exchange Options on Dividend-Paying Stocks

In Panel A of Table 3, the parameter values are $S_1 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, $q_1 = 0.02$, and $q_2 = 0.03$. In Panel B, the parameter values are $S_1 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, $q_1 = 0.08$, and $q_2 = 0.03$.

S_2	Chen and Yeh	Our Upper Bound	American Values	European Values
Panel A. $r > q_2 > q_1$				
30	10.5850	10.4014	10.2882	10.2807
35	6.4077	6.2965	6.2254	6.2235
40	3.3862	3.3274	3.2904	3.2888
45	1.5759	1.5486	1.5330	1.5306
50	0.6582	0.6468	0.6411	0.6393
Panel B. $q_1 > r > q_2$				
30	N.A.	10.2427	10.0000	8.8666
35	N.A.	6.1506	5.4948	5.0592
40	N.A.	3.2191	2.6432	2.4938
45	N.A.	1.4827	1.1259	1.0776
50	N.A.	0.6128	0.4328	0.4177

D. American Maximum Call Options on Two Risky Assets

We finally investigate the tightness of our upper bounds for American maximum call options on two dividend-paying stocks. We use the lattice model of Chen, Chung, and Yang (2002) to calculate the accurate price of American maximum call options. The number of time steps in the lattice is limited to 1,000 because it is a two-dimensional lattice.[13] The parameter values used here are also

stochastic problem can be simplified to a one-dimensional stochastic problem. They show that the price of an American exchange option equals the price of a standard American call with initial stock price S_{1t}, dividend yield q_1, volatility σ, exercise price S_{2t}, time to maturity τ, and risk-free rate q_2.

[13]To correct the numerical errors due to the chosen medium number of time steps, we employ the control variate technique of Hull and White (1988) and set the accurate price of American maximum call options as $P^A_{\max} = \tilde{P}^A_{\max} + (P^E_{\max} - \tilde{P}^E_{\max})$, where $\tilde{P}^A_{\max}$ and $\tilde{P}^E_{\max}$ are prices of American and European maximum call options calculated from the lattice model of Chen, Chung, and Yang (2002) with 1,000 time steps, respectively, and $P^E_{\max}$ is the closed-form solution of the European maximum call option.

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from Chen, Chung, and Yang (2002) as follows: $S_1 = S_2 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, and K, q_1, and q_2 are varied.

From Panel A of Table 4, we find that our upper bounds are also quite tight for in-the-money American maximum call options. Our upper bounds are about 2% larger than the accurate prices in this case. As the options become out-of-the-money, our upper bounds are looser than the previous case, but the error is still smaller than 6%. It is also interesting to know that our upper bounds work well for in-the-money cases while Chen and Yeh's (2002) work well for out-of-the-money cases.

Chen and Yeh's upper bound is not available, however, while ours is still workable in Panel B where the dividend yield of asset one is larger than the risk-free rate. Generally speaking, the results in Table 4 confirm our previous claim that our upper bounds are tight when dividend yields of the underlying assets are small.

TABLE 4

Upper Bounds of American Maximum Call Options on Two Dividend-Paying Stocks

In Panel A of Table 4, the parameter values are $S_1 = S_2 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, $q_1 = 0.02$, and $q_2 = 0.03$. In Panel B, the parameter values are $S_1 = S_2 = 40$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho_{12} = 0.5$, $r = 0.05$, $\tau = 0.5833$, $q_1 = 0.08$, and $q_2 = 0.03$.

K	Chen and Yeh	Our Upper Bound	American Values	European Values
Panel A. $r > q_2 > q_1$				
30	13.8760	13.7591	13.4800	13.4771
35	9.1213	9.0819	8.8598	8.8591
40	5.1847	5.1969	5.0358	5.0357
45	2.5682	2.5982	2.4944	2.4944
50	1.1553	1.1817	1.1221	1.1221
Panel B. $q_1 > r > q_2$				
30	N.A.	13.5890	12.8706	12.7852
35	N.A.	8.9025	8.3002	8.2494
40	N.A.	5.0450	4.6415	4.6207
45	N.A.	2.4975	2.2960	2.2899
50	N.A.	1.1263	1.0488	1.0474

V. Conclusion

Following the framework of Chen and Yeh (2002), we derive analytical upper bounds of American option prices. These upper bounds are especially useful when there are several state variables involved in the pricing model. Our upper bounds are closed form when the counterpart European option has a closed-form solution. Our upper bounds are very general in the sense that they do not rely on distribution assumptions or continuous trading. Moreover, our upper bounds are not only tight, but also converge to the accurate American call option prices when the dividend yield or strike price is small or when volatility is large.

The only required inputs to implement our upper bounds are the risk-neutral processes of the state variables. This is not a problem because of the recent advances in empirical derivatives research. For example, one can apply the implied binomial tree approach of Rubinstein (1994) and its many extensions, such as

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Derman, Kani, and Chriss (1996) and Britten-Jones and Neuberger (2000), in order to obtain the risk-neutral process of the stock price.

Our upper bound is still feasible even in the extremely complicated case where the pricing of American options depends on multiple risky assets and multiple risk factors (e.g., stochastic interest rates, stochastic volatility, and jumps) for each asset price process. In this case, one can apply standard Monte Carlo simulations to calculate the expected value of our upper bound, which is computationally more efficient (and maybe tighter) than other upper bounds generated by other complicated Monte Carlo methods.[14] This issue is left for future research.

In future research, we would like to empirically compare the tightness of our upper bound with those generated by other approaches, e.g., Carr, Jarrow, and Myneni (1992) and Broadie and Detemple (1996). Although the upper bounds developed by Carr, Jarrow, and Myneni and Broadie and Detemple are based on the Black-Scholes economy, they can be extended to general distributions with slight modifications. Finally, we would like to know if the risk-neutral processes of state variables are implied by European option prices, and how often American option prices may violate our upper bounds. The results therein should shed light on the efficiency of option markets.

Appendix

A. Correcting the Typos of Chen and Yeh (2002)

Recall that Chen and Yeh's (2002) upper bound for the American call option is

$$U^C = E_t[\max\{S_T - K, 0\}], \tag{A.1}$$

where $E_t[.]$ denotes the time t conditional expectation under the risk-neutral measure. To derive solutions for the above equation, define the characteristic function of the state-price density as

$$f(u) = E_t[e^{iuX_T}] = \int_{\Omega} e^{iuX_T} q(v)dv, \tag{A.2}$$

where $X_T = \ln S_T$, $\Omega \equiv \{X_T > -\infty\}$, and $q(v)$ is the risk-neutral (joint) density of the remaining uncertainty: $v \equiv (\int_t^T y_u du, X_T)$. As in Bakshi and Madan (2000), we have

$$U^C = F_{t,T}\Pi_1 - K\Pi_2, \tag{A.3}$$

where $F_{t,T}$ is the time t price of a futures contract maturing at time T, and Π_1 and Π_2 are the time t prices of two Arrow-Debreu securities defined by Bakshi and Madan (2000). The time t price of each Arrow-Debreu security for $j = 1, 2$ is

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left[\frac{e^{-iu\ln K} f_j(u)}{iu}\right] du. \tag{A.4}$$

[14] Monte Carlo methods can provide high biased estimates of the American option prices using the foresight bias (see Broadie and Glasserman (1997)) approach or the duality approach (see Haugh and Kogan (2004)). The computation is generally time consuming because it requires simultaneous determination of the optimal exercise boundary.

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The futures price and the characteristic functions for Arrow-Debreu securities are all determined from $f(u)$ as

$$\begin{aligned} F_{t,T} &= f(-i), \\ f_1(u) &= \frac{f(u-i)}{f(-i)}, \\ f_2(u) &= f(u). \end{aligned} \tag{A.5}$$

Thus, the above valuation equation is simplified to the derivation of $f(u)$ only. Following the approach of Heston (1993) and Bakshi, Cao, and Chen (1997), we derive the formula of $f(u)$ as

$$f(u) = \exp[C(t,T) + D(t,T)v_t + E(t,T)y_t + iuX_t], \tag{A.6}$$

where

$$\begin{aligned} C(t,T) &= -\frac{\alpha\beta}{\gamma^2}\left\{(d_D - b_D)(T-t) + 2\ln\left[1 + \left(\frac{d_D - b_D}{d_D + b_D}\right)e^{-d_D(T-t)}\right]\right. \\ &\quad \left. - 2\ln\left(1 + \frac{d_D - b_D}{d_D + b_D}\right)\right\} \\ &\quad - \frac{ab}{g^2}\left\{(d_E - b_E)(T-t) + 2\ln\left[1 + \left(\frac{d_E - b_E}{d_E + b_E}\right)e^{-d_E(T-t)}\right]\right. \\ &\quad \left. - 2\ln\left(1 + \frac{d_E - b_E}{d_E + b_E}\right)\right\} \\ &\quad + \lambda_J(T-t)\left[(1+\mu_J)^{iu}e^{(iu/2)(iu-1)\sigma_J^2} - 1\right] - \lambda_J iu\,\mu_J(T-t), \\ D(t,T) &= \frac{d_D}{\gamma^2}\left[\frac{\left(\frac{d_D - b_D}{d_D + b_D}\right)e^{-d_D(T-t)} - 1}{\left(\frac{d_D - b_D}{d_D + b_D}\right)e^{-d_D(T-t)} + 1} + \frac{b_D}{d_D}\right], \\ E(t,T) &= \frac{d_E}{g^2}\left[\frac{\left(\frac{d_E - b_E}{d_E + b_E}\right)e^{-d_E(T-t)} - 1}{\left(\frac{d_E - b_E}{d_E + b_E}\right)e^{-d_E(T-t)} + 1} + \frac{b_E}{d_E}\right], \end{aligned}$$

and $b_D = \alpha - \rho\gamma iu$, $d_D = \sqrt{[\alpha - \rho\gamma iu]^2 + \gamma^2(iu + u^2)}$, $b_E = a$, and $d_E = \sqrt{a^2 - 2g^2 iu}$.

B. Derivation of the Characteristic Functions of Our Upper Bound of Calls when $r > q$

Since our upper bound follows

$$U^C = E_t\left[e^{\int_t^T -(y_u)du}\max\{(S_T - K), 0\}\right], \tag{A.7}$$

we define the characteristic function of the state-price density as

$$f(u) = E_t\left[e^{\int_t^T -(y_u)du}e^{iuX_T}\right] = \int_\Omega e^{\int_t^T -(y_u)du}e^{iuX_T}q(v)dv, \tag{A.8}$$

which is implicitly the time t price (with a discount rate $y = r - q$ rather than r) of a hypothetic claim which pays e^{iuX_T} at time T. We follow Bakshi and Madan (2000) to obtain the following analytical solution for our upper bound,

$$U^C = G(t,T)\Pi_1 - KM(t,T)\Pi_2, \tag{A.9}$$

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where $G(t,T)$ is the time t price of a future commitment to deliver e^{X_T} at time T (scaled-forward price at the rate y), $M(t,T)$ is the time t price (also with a discount rate y) of a discount bond maturing at time T, and Π_1 and Π_2 are the time t prices of two Arrow-Debreu securities,

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-iu\ln K} f_j(u)}{iu}\right] du, \quad j = 1,2. \tag{A.10}$$

The forward price, discount bond price, and the characteristic functions for Arrow-Debreu securities are all determined from $f(u)$ as

$$\begin{aligned} M(t,T) &= f(0), \\ G(t,T) &= f(-i), \\ f_1(u) &= \frac{1}{G(t,T)} \times f(u-i), \\ f_2(u) &= \frac{1}{M(t,T)} \times f(u). \end{aligned} \tag{A.11}$$

As before, one can derive the formula of $f(u)$ as

$$f(u) = \exp[C(t,T) + D(t,T)v_t + E(t,T)y_t + iuX_t], \tag{A.12}$$

where

$$\begin{aligned} C(t,T) = & -\frac{\alpha\beta}{\gamma^2}\left\{(d_D - b_D)(T-t) + 2\ln\left[1 + \left(\frac{d_D - b_D}{d_D + b_D}\right)e^{-d_D(T-t)}\right]\right. \\ & \left. - 2\ln\left(1 + \frac{d_D - b_D}{d_D + b_D}\right)\right\} \\ & - \frac{ab}{g^2}\left\{(d_E - b_E)(T-t) + 2\ln\left[1 + \left(\frac{d_E - b_E}{d_E + b_E}\right)e^{-d_E(T-t)}\right]\right. \\ & \left. - 2\ln\left(1 + \frac{d_E - b_E}{d_E + b_E}\right)\right\} \\ & + \lambda_J(T-t)\left[(1+\mu_J)^{iu} e^{(iu/2)(iu-1)\sigma_J^2} - 1\right] - \lambda_J iu\,\mu_J(T-t), \\ D(t,T) = & \frac{d_D}{\gamma^2}\left[\frac{\left(\frac{d_D - b_D}{d_D + b_D}\right)e^{-d_D(T-t)} - 1}{\left(\frac{d_D - b_D}{d_D + b_D}\right)e^{-d_D(T-t)} + 1} + \frac{b_D}{d_D}\right], \\ E(t,T) = & \frac{d_E}{g^2}\left[\frac{\left(\frac{d_E - b_E}{d_E + b_E}\right)e^{-d_E(T-t)} - 1}{\left(\frac{d_E - b_E}{d_E + b_E}\right)e^{-d_E(T-t)} + 1} + \frac{b_E}{d_E}\right], \end{aligned}$$

and $b_D = \alpha - \rho\gamma iu$, $d_D = \sqrt{[\alpha - \rho\gamma iu]^2 + \gamma^2(iu + u^2)}$, $b_E = a$, and $d_E = \sqrt{a^2 - 2g^2(iu-1)}$.

From the above equations, it can be verified that $G(t,T) = S_t$. This is not surprising since the growth rate (y) of the stock price equals the discount rate and, therefore, the scaled-forward price equals the spot price.

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