Math. Proc. Camb. Phil. Soc. (1998), **124**, 15
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Semisimple Schur Algebras

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(*Received* 4 *October* 1996)

Abstract

Schur algebras are certain finite-dimensional algebras that completely control the polynomial representation theory of the general linear groups over an infinite field. Infinitesimal Schur algebras are truncated versions of the classical Schur algebras which control the polynomial representation theory of the Frobenius kernels of general linear groups. In this paper we use some elementary results on symmetric powers to classify the semisimple Schur algebras. We then classify the semisimple infinitesimal Schur algebras as well.

1. *Main Results*

Let k be an infinite field. The following result goes back to Schur; see, for example, Green [**Gr**, (2·6*e*)].

THEOREM 1. *If the characteristic of k is zero or strictly greater than d, then the Schur algebra $S_k(n,d)$ is semisimple.*

To prove this one may use the algebra isomorphism $S_k(n,d) \simeq \mathrm{End}_{k\Sigma_d}(E^{\otimes d})$ where $E = k^n$ is the vector space of n-dimensional column vectors upon which the symmetric group Σ_d acts by place-permutation amongst the tensor factors. In the cases mentioned in the theorem the group algebra $k\Sigma_d$ is semisimple. Hence $E^{\otimes d}$ is semisimple as $k\Sigma_d$-module and so $\mathrm{End}_{k\Sigma_d}(E^{\otimes d})$ is semisimple.

For a long time we believed in the truth of the converse of the above result, on the grounds that in the cases not covered by the theorem it is often very easy to construct non-split extensions, for instance by looking at the structure of symmetric powers of E. Then we discovered that in characteristic 2 there is an exception: $S_k(2,3)$ is semisimple. Our first result is that this is the one and only exception to the converse of Theorem 1.

† Authors partially supported by NSF grants DMS-9401576 and DMS-9500715

https://doi.org/10.1017/S0305004197002466 Published online by Cambridge University Press

Theorem 2. *Let p equal the characteristic of k. Then $S_k(n,d)$ is semisimple if and only if one of the following holds*:

(*a*) $p = 0$;
(*b*) $p > d$;
(*c*) $p = n = 2$ *and* $d = 3$.

It is remarkable that to prove this we need to know nothing more than the set of composition factors of symmetric powers of E.

Suppose now that k is perfect and $p > 0$. In [**DNP1**] certain subalgebras $S_k(n,d)_r$ of $S_k(n,d)$ were introduced. These subalgebras are known as infinitesimal Schur algebras and they completely control the polynomial representation theory of G_rT in the same way the ordinary Schur algebras control the polynomial representation theory of $G = \mathsf{GL}_n(k)$. Here G_rT denotes the scheme-theoretic pullback of the diagonal torus T under the rth iterate of the Frobenius morphism $G \to G$. Because of the way it is defined (truncating at height p^r away from the diagonal) it is clear that $S_k(n,d)_r$ coincides with $S_k(n,d)$ whenever $p^r > d$. Whenever this is the case and the corresponding ordinary Schur algebra is one of the semisimple ones then we have found an $S_k(n,d)_r$ which is also semisimple. Besides those trivial cases, the only other cases for which $S_k(n,d)_r$ is semisimple occur when $n = p = 2$ and $r = 1$ for d odd.

Theorem 3. *Let k be a perfect infinite field of characterstic $p > 0$. The infinitesimal Schur algebra $S_k(n,d)_r$ is semisimple if and only if one of the following holds*:

(*a*) $p \geqslant 2$, $n \geqslant 2$, $d < p$, $r \geqslant 1$;
(*b*) $p = 2$, $n = 2$, $d = 3$, $r \geqslant 1$;
(*c*) $p = 2$, $n = 2$, d *odd*, $r = 1$.

2. *Proof of Theorem* 2

Before embarking on the proof let us make a few preliminary remarks. First of all, let $G = \mathsf{GL}_n(k)$. One standard fact we will use is that the category of homogeneous polynomial G-modules of degree d is equivalent with the category of $S_k(n,d)$-modules. Another fact is that any integral power $\det^r$ of the determinant character $\det\colon G \to k^\times$ is another character on G and these powers give all the characters on G [**Jan**, II·1·21]. We shall denote the corresponding 1-dimensional G-modules by the same symbol $\det^r$. It is obvious that tensoring by det or by $\det^{-1}$ commutes with direct sums, so for any rational G-module V we have:

$$V \text{ is semisimple} \Leftrightarrow \det \otimes V \text{ is semisimple} \tag{1}$$

(as rational G-modules). Note that if V is an $S_k(n,d)$-module (i.e. a homogeneous polynomial G-module of degree d) then $\det \otimes V$ is an $S_k(n,d+n)$-module.

Now let B denote the Borel subgroup of lower triangular matrices in G. We will employ the notational conventions of [**Jan**, II·1·21] with regard to choice of root system, positive roots, etc. The simple $S_k(n,d)$-modules are parametrized by the partitions of d of not more than n parts and if λ is such a partition then we denote by $H^0(\lambda)$ the corresponding induced G-module $\operatorname{ind}_B^G k_\lambda$, where k_λ is the 1-dimensional B-module corresponding to λ. This is a homogeneous polynomial G-module of degree d (i.e. an $S_k(n,d)$-module) and it has simple socle isomorphic with the simple module corresponding to the partition λ. Moreover, any non-split extension between two simple $S_k(n,d)$-modules will embed in some $H^0(\lambda)$ (see [**Jan**, II·2·14]).

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We can now consider the proof of Theorem 2. Let us deal with the exception first; so suppose $p = 2$. The algebra $S_k(2, 3)$ is semisimple because the two induced modules in degree 3, $H^0(3, 0)$ and $H^0(2, 1)$, are both simple. Combining this with Theorem 1 we see that $S_k(n, d)$ is semisimple in all cases listed in Theorem 2.

To prove the converse we will show that $S_k(n, d)$ is not semisimple in the cases not listed in Theorem 2. For this it will suffice to know precisely which symmetric powers of E are semisimple.

THEOREM 4. *Let p be the characteristic of k. The symmetric power S^dE is semisimple if and only if one of the following conditions holds*:

(*a*) $p = 0$;

(*b*) $p > d$;

(*c*) $n = 2$ *and* $d = cp^e - 1$ *for* $e \geqslant 1$ *and* $0 < c \leqslant p - 1$.

That S^dE is semisimple in cases (*a*) and (*b*) follows immediately from Theorem 1. To see that it is semisimple in case (*c*) let us first observe that since S^dE is isomorphic as G-module to the induced module $H^0(d)$, it has a simple socle and hence is semisimple if and only if it is simple.

Now the submodule structure of S^dE is completely known by the main result of [**Dot1**] or [**Kr**]. Other references include [**Dot2**], [**Dot3**] and [**Wa**]. In our situation we do not need to know the submodule structure; it suffices to compute the number of composition factors and for that the older result [**Su**] is enough. The only fact we need from the above is that the composition factors of S^dE all have multiplicity 1 and are in one-to-one correspondence with the set of integral solutions $(a_0, a_1, \dots)$ of:

$$d = a_0 + a_1 p + a_2 p^2 + \cdots \tag{2}$$

subject to the condition

$$0 \leqslant a_i \leqslant n(p - 1) \quad \text{for all } i. \tag{3}$$

Let us refer to such solutions as *admissible*.

It is easy to see that in case (*c*) the only admissible solution of equation (2) is the p-adic expansion: $a_i = p - 1$ for all $i = 0, \dots, e - 1$; $a_e = c - 1$. This proves that S^dE is simple, hence semisimple, in case (*c*).

It remains to prove the converse. Suppose $0 < p \leqslant d$. Let

$$d = d_0 + d_1 p + d_2 p^2 + \cdots \tag{4}$$

be the p-adic expansion of d. Then d_i is greater than 0 for at least one index $i \geqslant 1$ and, as long as $n > 2$, the expansion

$$d = d_0 + \cdots + (p + d_{i-1})p^{i-1} + (d_i - 1)p^i + \cdots$$

is another admissible solution to equation (2). So under these conditions we see that S^dE has at least two composition factors and hence is not semisimple.

Now only the case $n = 2$ remains. If $0 < p \leqslant d$ then in the p-adic expansion (4) of d there is at least one index $i \geqslant 1$ for which the coefficient $d_i > 0$. So long as $d_{i-1} < p - 1$ we can construct another admissible solution:

$$d = d_0 + \cdots + (p + d_{i-1})p^{i-1} + (d_i - 1)p^i + \cdots$$

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of equation (2). Hence S^dE is not semisimple unless the coefficients in the p-adic expansion satisfy: $d_j > 0$ implies $d_{j-1} = p-1$ for all $j \geqslant 1$. Assuming d satisfies this condition and letting m be the largest index j for which $d_j > 0$ we see immediately that the p-adic expansion of d must have the form:

$$d = (p-1) + (p-1)p + \cdots + (p-1)p^{m-1} + bp^m$$

for some $1 \leqslant b \leqslant p-1$. It follows that $d = cp^e - 1$ for some $e \geqslant 1$ and $1 \leqslant c \leqslant p-1$. (We have $b = c$ and $m = e$ in all but one case.) This completes the proof of Theorem 4.

Now we can finish the proof of Theorem 2. By Theorem 4 the only possible exceptions to the converse of Theorem 1 occur when $n = 2$ and d has the form $cp^e - 1$ for $0 < c \leqslant p-1$ and $e \geqslant 1$. Suppose that $p > 2$. Then the dual Weyl module $H^0(d-1, 1)$ is isomorphic to $\det \otimes S^{d-2}E$ and hence (by (1) and Theorem 4) is not semisimple.

This brings us down to the case $n = p = 2$ with d as above. Then necessarily $c = 1$. The case $e = 1$ is not exceptional. By the argument given at the beginning of this section, the case $e = 2$ is an exception to the converse of Theorem 1. Assume then that $e > 2$. Then by Theorem 4 the module S^dE is semisimple but $H^0(d-1, 1) \simeq \det \otimes S^{d-2}E$ is not. This shows that the converse of Theorem 1 holds in all cases except the case $n = p = 2$, $d = 3$. This completes the proof of Theorem 2.

3. *Proof of Theorem* 3

We begin with some general observations. As in [**DNP3**, 3·1], we regard $A = S_k(n, d)$ as a subalgebra of $A' = S_k(n', d)$ for $n' \geqslant n$. Then, by [**DNP3**, 3·2], given any A-module N the corresponding induced A'-module

$$\operatorname{ind}_A^{A'} N = A' \otimes_A N \tag{5}$$

has a direct summand isomorphic to N when regarded as an A-module. This fact leads to the following result:

$$\text{if } S_k(n, d)_r \text{ is not semisimple then neither is } S_k(n', d)_r, \tag{6}$$

for any $n' \geqslant n$.

To prove this, let

$$0 \to M_1 \xrightarrow{\sigma_1} M \xrightarrow{\sigma_2} M_2 \to 0$$

be a non-split short exact sequence of A-modules. Then N is a direct summand of $\operatorname{ind}_A^{A'} N$ upon restriction to A. This property is natural with respect to the induction functor so there exist maps i, i_2, p, p_2 such that $p_2 \circ i_2 = \mathrm{id}_{M_2}$ and $p \circ i = \mathrm{id}_M$, making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{\sigma_1} & M & \xrightarrow{\sigma_2} & M_2 & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle p} & & \downarrow{\scriptstyle i_2} & & \\
0 & \longrightarrow & K & \longrightarrow & \operatorname{ind}_A^{A'} M & \xrightarrow{\sigma_2^*} & \operatorname{ind}_A^{A'} M_2 & \longrightarrow & 0
\end{array}$$

where the map σ_2^* is the mapping induced from σ_2 and K is its kernel.

Suppose the exact sequence in the bottom row splits. Then there exists a map

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Ψ: $\mathrm{ind}_A^{A'} M_2 \to \mathrm{ind}_A^{A'} M$ such that $\sigma_2^* \circ \Psi = \mathrm{id}$. Define an A-module homomorphism $\overline{\Psi}$ by $\overline{\Psi} = p \circ \Psi \circ i_2$. From the commutativity of the diagram it follows that

$$\sigma_2 \circ \overline{\Psi} = \sigma_2 \circ p \circ \Psi \circ i_2 = p_2 \circ \sigma_2^* \circ \Psi \circ i_2 = p_2 \circ i_2 = \mathrm{id}_{M_2},$$

implying that the short exact sequence in the top row splits, a contradition. This completes the proof.

We shall abuse notation and denote by det the restriction to G_rT of the determinant representation. For any rational G_rT-module V we have:

$$V \text{ is semisimple} \Leftrightarrow \det \otimes V \text{ is semisimple} \tag{7}$$

(as rational G_rT-modules). Note that if V is an $S_k(n,d)_r$-module (i.e. a homogeneous polynomial G_rT-module of degree d) then $\det \otimes V$ is an $S_k(n,d+n)_r$-module. From this we see immediately that:

$$\text{if } S_k(n,d)_r \text{ is not semisimple then neither is } S_k(n,d+n)_r. \tag{8}$$

We now turn to the proof of Theorem 3. First we show that the algebras listed in the theorem are semisimple. In the cases listed in parts (a) and (b) of the theorem we have the equality $S_k(n,d)_r = S_k(n,d)$, and hence in those cases the algebras are semisimple by Theorem 2. This brings us to the cases listed in part (c) of the theorem.

For $p = 2$, $S_k(2,d)_1$ has dimension $4d$. If d is odd, then there are d simple modules each having dimension two, by the proof of [**DNP2**, theorem 2·6]. The dimension of $S_k(2,d)_1$ in this case is the sum of the squares of the dimensions of the simple modules. Hence, $S_k(2,d)_1$ is semisimple for d odd.

It only remains to show that $S_k(n,d)_r$ is not semisimple for the cases not listed in Theorem 3. Suppose first that $p \geqslant 3$. If $r \geqslant 2$ then $S_k(2,d)_r = S_k(2,d)$ for $d = p, p+1$ and hence is not semisimple by Theorem 2. If $r = 1$, then $S_k(2,d)_1$ for $d = p, p+1$ is not semisimple by [**DNP2**, proposition 3·2]. By (8) it follows that $S_k(2,d)_r$ is not semisimple for $d \geqslant p$, $r \geqslant 1$ and by (6) it follows that $S_k(n,d)_r$ is not semisimple for $n \geqslant 2$, $d > p$ and $r \geqslant 1$.

This brings us down to the case $p = 2$. It is easy to see that $S_k(3,d)_r$ is not semisimple in degrees $d = 2,3,4$ for $r \geqslant 3$ since in these cases we have equality $S_k(3,d)_r = S_k(3,d)$ and we can appeal to Theorem 2. $S_k(3,d)_2$ is not semisimple in degrees $d = 2,3$ for the same reason, and $S_k(3,4)_2$ can be seen to be non-semisimple by direct computation or by appealing to the fact that it has infinite representation type [**DNP3**, theorem 1·1]. Finally, $S_k(3,d)_1$ in degrees $d = 2,3,4$ are not semisimple, again by direct computation or by [**DNP3**, theorem 1·1]. By (8) and (6) the non-semisimplicity of $S_k(n,d)_r$ now follows for $n \geqslant 3$, $d \geqslant 2$, $r \geqslant 1$.

Now only the case $p = 2$, $n = 2$ remains. Since $S_k(2,2)_r = S_k(2,2)$ for $r \geqslant 2$ and $S_k(2,d)_r = S_k(2,d)$ for $d = 4,5$ and $r \geqslant 3$, it follows by Theorem 2 that these algebras are not semisimple. Moreover, $S_k(2,4)_2$ and $S_k(2,5)_2$ are not semisimple, by direct calculation or by [**DNP3**, theorem 1·1]. By (8) and (6) the non-semisimplicity of $S_k(2,d)_r$ now follows for the case $r \geqslant 2$ if $d = 2$ or $d \geqslant 4$.

This brings us to the final case. Let $p = 2$. In this case the algebra $S_k(2,2)_1$ is 8-dimensional and has three one-dimensional simple modules. (See the end of [**DNP2**, theorem 2·6].) Hence $S_k(2,2)_1$ is not semisimple. It follows by (8) that $S_k(2,d)_1$ is not semisimple for any even d. This completes the proof of Theorem 3.

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