Math. Proc. Camb. Phil. Soc. (1997), **121**, 141
Printed in Great Britain

Separated Dirac operators and asymptotically constant linear systems

By V. ARNOLD
Institut für Mathematik, Vogelpothsweg 87, *D*-44227 *Dortmund*

H. KALF
Mathematisches Institut der Universität, Theresienstr. 39, *D*-80333 *München*

and A. SCHNEIDER
Institut für Mathematik, Vogelpothsweg 87, *D*-44227 *Dortmund*

(*Received* 12 *September* 1995, *revised* 8 *November* 1995)

Levinson's theorem with all its ramifications is a well-established tool in the spectral analysis of ordinary differential operators (see in particular §§3·10, 11 and 4·3, 4 of Eastham's book [**5**]). In this note we should like to draw attention to a result that describes the solutions of asymptotically constant linear systems under weaker assumptions less precisely than the Levinson theorem. This result can be called the Perron–Lettenmeyer–Hartman–Wintner theorem after the contributions of these authors in [**11**, **10**, **6**, **8**]. (Note that the Hartman–Wintner theorem that is discussed in [**5**, p. 17] is a substantially different version of this theorem.)

Let $J := [t_0, \infty)$. In [**6**] Hartman introduced the vector space

$$\mathfrak{h} := \left\{ f \in \mathscr{L}^1_{loc}(J) \,|\, \lim_{t\to\infty} \left[\sup_{s\geqslant t} (1+s-t)^{-1} \int_t^s |f(r)|\, dr \right] = 0 \right\}.$$

The linear version of the Perron–Lettenmeyer–Hartman–Wintner theorem can be described as follows.

Theorem 1. *Let A be a constant $n\times n$ matrix and $\lambda_1, \dots, \lambda_n$ its eigenvalues, counting multiplicities. If R is an $n\times n$ matrix function with entries in* $\mathfrak{h}$, *then*

$$\xi' = [A + R(t)]\,\xi \tag{1}$$

has a fundamental system $\xi_1, \dots, \xi_n$ of solutions with the property

$$\lim_{t\to\infty} \frac{1}{t} \log|\xi_j(t)| = Re\,\lambda_j \quad (j\in\{1,\dots,n\}). \tag{2}$$

Note that (2) involves the norm of the solutions and not, as in the Levinson theorem, the solution vectors themselves. On the other hand, the assumption of Theorem 1 is very mild as the following observation [**7**, p. 286] shows.

Remark 1. A function $f\in\mathscr{L}^1_{\mathrm{loc}}(J)$ lies in $\mathfrak{h}$, if it has one of the following two properties:

(a) $f\in\mathscr{L}^p(J)$ for some $1\leqslant p<\infty$;
(b) $f(t)\to 0$ as $t\to\infty$.

https://doi.org/10.1017/S0305004196001284 Published online by Cambridge University Press

In a number of instances additional information about the behaviour of the norm of the solutions is available.

Remark 2. If A can be diagonalized there is a $t_1 \geqslant t_0$ such that (1) has a fundamental system of solutions satisfying

$$\exp\left\{\int_{t_1}^{t} [Re\,\lambda_j - 2\,\|R(s)\|]\,ds\right\} \leqslant |\xi_j(t)| \leqslant 2\exp\left\{\int_{t_1}^{t} [Re\,\lambda_j + 2\,\|R(s)\|]\,ds\right\}$$

for $t \geqslant t_1$. ($\|R\|$ is the Euclidean matrix norm of R.)

The original proof by Hartman–Wintner is well presented in Coppel's book [**4**, pp. 97–99]. In Hartman's book [**7**, pp. 285, 291 f., 301 ff.] a result is established that is a great deal more general, using a topological principle of Ważewski. Both references require, and indeed use, the continuity of R. The general case can be dealt with, for example, by approximating $R \in \mathfrak{h}$ by continuous functions with compact support, but a direct proof that modifies the central lemma in [**7**, p. 285] is also possible. For a connected and detailed account we refer to [**1**].

In relativistic quantum mechanics a family of operators of the form

$$\sigma_3 + \sigma_2 \frac{1}{i}\frac{d}{dr} + \sigma_1\left(\frac{k}{r} + aV'(r)\right) + \sigma_0\, V(r), \quad 0 < r < \infty, \tag{3}$$

indexed by $k \in \mathbb{Z} \setminus \{0\}$ (the eigenvalues of the spin-orbit coupling operator) occurs when the Hamiltonian for a charged particle with non-zero rest mass, spin $\frac{1}{2}$ and magnetic moment $1+a$ is separated in spherical coordinates. V is the spherically symmetric potential function that describes the electric field; σ_1, σ_2, σ_3 are the Pauli matrices and σ_0 is the 2×2 unit matrix. Modification (3) of the Dirac operator ($a = 0$) was introduced by Pauli in 1941. (The number a was known to be non-zero, e.g. for the proton, but believed to be zero for the electron; for the electron a is in fact approximately $1{\cdot}159\,652\,.\,10^{-3}$.) In [**2**] Behncke made the remarkable observation that (3) is predominantly in the limit-point case (l.p.c.) at zero for $a \neq 0$ (for example, for all functions V that satisfy a power law). Since (3) is always in the l.p.c. at infinity (see, e.g. [**14**, Korollar 5·2]), this means that for such a and V the minimal Dirac operator that gives rise to (3) is essentially selfadjoint (see also [**12**, §5, 3·2]).

We wish to show here that all the important limit-point and limit-circle criteria (we shall write l.c.c. for limit-circle case) that pertain to (3) immediately flow from Theorem 1. It suffices to investigate the number of square-integrable solutions of the system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y' + \begin{pmatrix} P_1(x) & P_3(x) \\ P_3(x) & P_2(x) \end{pmatrix} y = 0 \tag{4}$$

on the interval $I := (0, x_0]$ when $P_1, P_2, P_3 \in \mathscr{L}^1_{\mathrm{loc}}(I)$ are given real-valued functions. Let g be a bijection from I onto J such that g and g^{-1} are locally absolutely continuous. Then

$$\xi = y \circ g^{-1}$$

establishes a one-to-one correspondence between the solutions of (4) and those of

$$\xi' = \left\{(g^{-1})' \begin{pmatrix} P_3 \circ g^{-1} & P_2 \circ g^{-1} \\ -P_1 \circ g^{-1} & -P_3 \circ g^{-1} \end{pmatrix}\right\} \xi, \tag{5}$$

provided $y \in \mathscr{AC}_{\mathrm{loc}}(I)$ implies $\xi \in \mathscr{AC}_{\mathrm{loc}}(J)$ and conversely.

https://doi.org/10.1017/S0305004196001284 Published online by Cambridge University Press

The particularly simple choice

$$t = g(x) = -\log x \quad (x \in I) \tag{6}$$

yields the following result for Dirac systems that describe particles with a normal magnetic moment.

THEOREM 2. *Let* $c_1, c_2, k \in \mathbb{R}$ *and* λ *a branch of* $\sqrt{(k^2 - c_1 c_2)}$. *Suppose that* U_1, $U_2 : I \to \mathbb{R}$ *are functions such that* $t \mapsto e^{-t} U_j(e^{-t})$ $(t \in J;\ j \in \{1,2\})$ *lies in* $\mathfrak{h}$. *Then* (4) *with*

$$P_j(x) = \frac{c_j}{x} + U_j(x), \quad P_3(x) = \frac{k}{x} \quad (x \in I;\ j \in \{1,2\}) \tag{7}$$

has a fundamental system of solutions $y_\pm$ *with*

$$|y_\pm(x)| = x^{\pm Re\,\lambda + o(1)} \quad as \quad x \to 0. \tag{8}$$

Proof. Given (6), (7), equation (5) reads

$$\xi'(t) = \left\{ \begin{pmatrix} -k & -c_2 \\ c_1 & k \end{pmatrix} + \begin{pmatrix} 0 & -e^{-t} U_2(e^{-t}) \\ e^{-t} U_1(e^{-t}) & 0 \end{pmatrix} \right\} \xi(t).$$

The constant matrix has the eigenvalues $\pm\lambda$, so that the assertion is obvious from (2). ∎

With the help of Remark 1, Theorem 2 can be made more specific as follows.

COROLLARY 1. *Suppose either* (i) $x \mapsto x^{1-1/p} U_j(x)$ *lies in* $\mathscr{L}^p(I)$ *for some* $1 \leqslant p < \infty$ $(j \in \{1,2\})$ *or* (ii) $x U_j(x) \to 0$ *as* $x \to 0$ $(j \in \{1,2\})$. *Then* (4), (7) *is at zero in the l.p.c. if* $c_1 c_2 < k^2 - \frac{1}{4}$ *and in the l.c.c. if* $c_1 c_2 > k^2 - \frac{1}{4}$.

For $p = 1$ criterion (i) is due to Titchmarsh [**13**, p. 232 f.]. Criterion (ii) was given in [**9**] under the additional assumption $U_1 - U_2 \in \mathscr{L}^1(I)$, which is now revealed as redundant. On a technical level it is interesting to note that [**9**] combines Prüfer and Riccati techniques while Hartman–Wintner in their proof of Theorem 1 consider a Riccati inequality [**8**, §3; 4, p. 98]. Moreover, it should be noted that the l.c.c. is not explicitly mentioned in [**13**, **9**], but could easily be supplied, and that the continuity assumptions in these papers are not essential.

To deal with Dirac systems that describe particles with an anomalous magnetic moment, the transformation of variables (6) has to be changed but slightly.

THEOREM 3. *Let* $c_1, c_2, c_3 \in \mathbb{R}$ *and* λ *a branch of* $\sqrt{(c_3^2 - c_1 c_2)}$. *Suppose that* $0 < q \in \mathscr{L}^1_{\mathrm{loc}}(I)$ *and*

$$Q(x) := \int_x^{x_0} q \to \infty \quad as \quad x \to 0. \tag{9}$$

If $v_1, v_2, v_3 : I \to \mathbb{R}$ *are such that*

$$v_j \circ Q^{-1} \in \mathfrak{h} \quad (j \in \{1,2,3\}), \tag{10}$$

then (4) *with*

$$P_j = q(c_j + v_j) \quad (j \in \{1,2,3\})$$

has a fundamental system of solutions $y_\pm$ *with*

$$|y_\pm(x)| = \exp\{[\pm Re\,\lambda + o(1)]\, Q(x)\} \quad \text{as} \quad x \to 0. \tag{11}$$

https://doi.org/10.1017/S0305004196001284 Published online by Cambridge University Press

Proof. The assumptions are arranged in such a way that Theorem 1 can be applied to (5) with $g = Q$, i.e. to

$$\xi'(t) = \left\{\begin{pmatrix} -c_3 & -c_2 \\ c_1 & c_3 \end{pmatrix} + \begin{pmatrix} -v_3 & -v_2 \\ v_1 & v_3 \end{pmatrix} \circ Q^{-1}(t)\right\} \xi(t).$$

(Note that Q^{-1} is not only monotonic, but also locally absolutely continuous, since Q' is a.e. non-zero.) ∎

With part (b) of Remark 1 the following two important results can be derived, the first excluding, but the second including Coulomb interactions.

Corollary 2. *Let $a, k \in \mathbb{R}$. If $V \in \mathscr{AC}_{\mathrm{loc}}(I)$ satisfies*

$$\lim_{x \to 0} xV'(x) = s \tag{12}$$

for some $s \in \mathbb{R}$, then (4) *with*

$$P_j(x) = V(x), \quad P_3(x) = \frac{k}{x} + aV'(x) \quad (x \in I;\, j \in \{1, 2\}) \tag{13}$$

is at zero in the l.p.c. if $|k + sa| > \frac{1}{2}$ and in the l.c.c. if $|k + sa| < \frac{1}{2}$.

Proof. Condition (12) clearly implies $\lim_{x \to 0} xV(x) = 0$. The assumptions of Theorem 3 are therefore satisfied if we put

$$c_1 = c_2 = 0, \quad c_3 = k + sa,$$

and for $x \in (0, 1]$

$$q(x) = \frac{1}{x}, \quad v_1(x) = v_2(x) = xV(x), \quad v_3(x) = a[xV'(x) - s].$$

Corollary 3. *Let $a, k \in \mathbb{R}$, $a \neq 0$. Suppose that $V \in \mathscr{AC}_{\mathrm{loc}}(I)$ is a function with the following two properties:*

(i) *sign V' is constant near zero;*

(ii) $\lim_{x \to 0} x|V'(x)| = \lim_{x \to 0} \left|\frac{V'(x)}{V(x)}\right| = \infty.$

Then (4), (13) *has a fundamental system of solutions $y_\pm$ with*

$$|y_\pm(x)| = e^{\pm V(x)[|a| + o(1)]} \quad \text{as} \quad x \to 0. \tag{14}$$

In particular there is l.p.c. at zero. (Observe the striking difference between (8) *and* (14).)

Proof. Corollary 3 is the special case of Theorem 3 where

$$c_1 = c_2 = 0, \quad c_3 = a \operatorname{sign} V'$$

and for $x \in (0, x_0]$

$$q(x) = |V'(x)|, \quad v_1(x) = v_2(x) = \frac{V(x)}{|V'(x)|}, \quad v_3(x) = \frac{k}{x|V'(x)|}.$$ ∎

Corollary 2 and the limit-point criterion in Corollary 3 were first given by Behncke [**2**]. They could also be derived from [**9**, theorem 3], but it is certainly more convenient to proceed as above.

https://doi.org/10.1017/S0305004196001284 Published online by Cambridge University Press

Remark 3. Let q be as in Theorem 3. In view of Remark 1 condition (10) certainly holds good if $v_3 \in \mathscr{L}^1_{\mathrm{loc}}(I)$ satisfies $\lim_{x\to 0} v_3(x) = 0$ and if there is a decomposition $v_j = u_j + w_j$ such that $qu_j \in \mathscr{L}^1(I)$ and $w_j \in \mathscr{L}^1_{\mathrm{loc}}(I)$ has the property $\lim_{x\to 0} w_j(x) = 0$ $(j \in \{1,2\})$. It follows from (11) that there is l.p.c. at zero, for example, if $c_1 c_2 \neq c_3^2$ and $\lim_{x\to 0} xq(x) = \infty$. (This latter condition of course implies (9).) This shows that [**3**, theorem 1] (read 'integrable' instead of 'locally integrable' in condition (iii) of this theorem) also subordinates itself to Theorem 3.

Remark 4. Let $c_1, c_2, k \in \mathbb{R}$ be numbers with $c_1 c_2 < k^2$, $\lambda := \sqrt{(k^2 - c_1 c_2)}$ and $0 < q \in \mathscr{L}^1_{\mathrm{loc}}(I) \setminus \mathscr{L}^1(I)$. Let $U_1, U_2, U_3 : I \to \mathbb{R}$, $\|U\|_1 := |U_1| + |U_2| + 2|U_3|$,

$$M := \frac{1}{q}\begin{pmatrix} -U_3 & -U_2 \\ U_1 & U_3 \end{pmatrix} \quad \text{and} \quad Q(x) := \int_x^{x_0} q \quad (x \in I).$$

If $M \circ Q^{-1}$ lies in $\mathfrak{h}$, then (4) with

$$P_j = c_j q + U_j \quad (j \in \{1,2\}), \quad P_3 = kq + U_3$$

is at zero

$$\text{in the l.p.c. if } \exp\left\{2\int_x^{x_0} [\lambda q(r) - 2\|U(r)\|_1]\, dr\right\} \notin \mathscr{L}^1(I)$$

$$\text{and in the l.c.c. if } \exp\left\{2\int_x^{x_0} [\lambda q(r) + 2\|U(r)\|_1]\, dr\right\} \in \mathscr{L}^1(I).$$

This follows immediately from Remark 2 since

$$\int_0^{Q(x)} \|M(Q^{-1}(s))\|\, ds \leqslant \int_0^{Q(x)} \|M(Q^{-1}(s))\|_1\, ds \leqslant \int_x^{x_0} \|U(r)\|_1\, dr \quad (x \in I).$$

In comparing this result with lemma 3 in [**2**] it should be noted that the latter is affected by an arithmetical error in [**2**, equation (10)]. The oversight was first corrected (in a manner different from ours) by W. Coy in 1980 in an unpublished state examination thesis that was supervised by J. Weidmann.

Acknowledgement. We are grateful to the referee whose remarks improved the presentation of our results.

REFERENCES

[**1**] V. Arnold. *Bestimmung von Defektindizes mit Hilfe asymptotisch konstanter Differentialgleichungssysteme*. Dissertation, Universität Dortmund (1995).
[**2**] H. Behncke. The Dirac equation with an anomalous magnetic moment. *Math. Z.* **174** (1980), 213–225.
[**3**] H. Behncke. The Dirac equation with an anomalous magnetic moment II; in Proc. 7th Conf. on Ordinary and Partial Differential Equations, Dundee 1982, Lecture Notes in Math. vol. **964** (Springer-Verlag, 1982), 77–85.
[**4**] W. A. Coppel. *Stability and asymptotic behaviour of differential equations* (D. C. Heath, 1965).
[**5**] M. S. P. Eastham. *The asymptotic solution of linear differential systems, applications of the Levinson theorem* (Clarendon Press, 1989).
[**6**] Ph. Hartman. Unrestricted solution fields of almost separable differential equations. *Trans. Amer. Math. Soc.* **63** (1948), 560–580.
[**7**] Ph. Hartman. *Ordinary differential equations* (Wiley, 1964).
[**8**] Ph. Hartman and A. Wintner. Asymptotic integration of linear differential equations. *Amer. J. Math.* **77** (1955), 45–86, 932.
[**9**] H. Kalf. A limit-point criterion for separated Dirac operators and a little known result on Riccati's equation. *Math. Z.* **129** (1972), 75–82.

https://doi.org/10.1017/S0305004196001284 Published online by Cambridge University Press

[**10**] F. Lettenmeyer. Über das asymptotische Verhalten der Lösungen von Differentialgleichungen und Differentialgleichungssystemen. *Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber.* (1929), 201–252.
[**11**] O. Perron. Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen. *Math. Z.* **29** (1929), 129–160.
[**12**] B. Thaller. *The Dirac equation* (Springer-Verlag, 1992).
[**13**] E. C. Titchmarsh. On the nature of the spectrum in problems of relativistic quantum mechanics. *Quart. J. Math. Oxford* (2) **12** (1961), 227–240.
[**14**] J. Weidmann. Oszillationsmethoden für Systeme gewöhnlicher Differentialgleichungen. *Math. Z.* **119** (1971), 349–373.

https://doi.org/10.1017/S0305004196001284 Published online by Cambridge University Press