The Review of Symbolic Logic
Volume 18, Number 2, June 2025

CUT-FREE SEQUENT CALCULI FOR THE PROVABILITY LOGIC D

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Abstract. We say that a Kripke model is a GL-model (Gödel and Löb model) if the accessibility relation $\prec$ is transitive and converse well-founded. We say that a Kripke model is a D-model if it is obtained by attaching infinitely many worlds $t_1, t_2, ...$, and t_ω to a world t_0 of a GL-model so that $t_0 \succ t_1 \succ t_2 \succ \cdots \succ t_\omega$. A non-normal modal logic **D**, which was studied by Beklemishev [3], is characterized as follows. A formula φ is a theorem of **D** if and only if φ is true at t_ω in any D-model. **D** is an intermediate logic between the provability logics **GL** and **S**. A Hilbert-style proof system for **D** is known, but there has been no sequent calculus. In this paper, we establish two sequent calculi for **D**, and show the cut-elimination theorem. We also introduce new Hilbert-style systems for **D** by interpreting the sequent calculi. Moreover, we show that D-models can be defined using an arbitrary limit ordinal as well as ω. Finally, we show a general result as follows. Let X and X^+ be arbitrary modal logics. If the relationship between semantics of X and semantics of X^+ is equal to that of **GL** and **D**, then X^+ can be axiomatized based on X in the same way as the new axiomatization of **D** based on **GL**.

§1. Introduction. We say that a Kripke model is a *GL-model* if the accessibility relation $\prec$ is transitive and converse well-founded. Then, a *D-model* is obtained by attaching infinitely many worlds $t_1, t_2, ...$ (called 'tail'), and t_ω (called 'bottom') to a world t_0 of a GL-model so that $t_0 \succ t_1 \succ t_2 \succ \cdots \succ t_\omega$ (Figure 1, Left). A non-normal modal logic **D**, which was studied by Beklemishev [3], is characterized as follows. A formula φ is a theorem of **D** if and only if φ is true at the bottom of any D-model.[1]

Received: October 22, 2023.
2020 *Mathematics Subject Classification*: Primary 03B45, 03F05, 03F45.
Key words and phrases: provability logic D, modal logic, cut-elimination, Kripke model.

[1] Following [3], we use the name **D**, while **D** denotes the modal logic of serial frames in most literature.

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doi:10.1017/S1755020325000036

https://doi.org/10.1017/S1755020325000036 Published online by Cambridge University Press

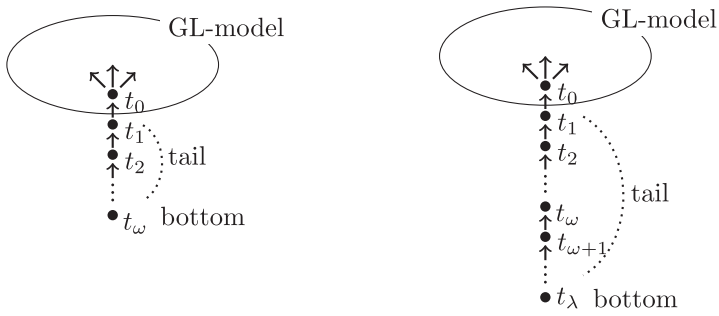


Figure 1. (Left) D-model by Beklemishev. (Right) New D-model.

D is a provability logic as follows. A formula φ is a theorem of **D** if and only if any φ^* is true in the standard model of arithmetic where φ^* is obtained from φ by interpreting the modal operator $\Box$ as the provability predicate of a c.e. extension of Peano Arithmetic that is Σ_1-sound but not sound. In this paper, we do not argue **D** from the perspective of provability logics, but we consider **D** as an interesting modal logic which is non-normal (not closed under the necessitation rule) and has simple Kripke-style semantics. We establish sequent calculi, cut-elimination, and new Hilbert-style axiomatizations for **D**; furthermore, we show that we can define D-models using an arbitrary limit ordinal λ as well as ω (Figure 1, Right).

D is closely related to the well-known provability logics **GL** and **S** (see, e.g., [5, 12] for the basic results on **GL** and **S**)[2]. A Hilbert-style proof system for **GL** is known as $\mathbf{K} + \Box(\Box\varphi\to\varphi)\to\Box\varphi$. A Hilbert-style proof system for **S** (we call this system $\mathbf{S}_\mathrm{H}$) is as follows. The axioms are all theorems of **GL** and all formulas $\Box\varphi\to\varphi$; and sole inference rule is modus ponens. Then a Hilbert-style proof system for **D** (we call this system $\mathbf{D}_\mathrm{H}$) is known to be obtained from $\mathbf{S}_\mathrm{H}$ by restricting φ in the axiom scheme $\Box\varphi\to\varphi$ to be $\bot$ or $\Box\psi\vee\Box\sigma$ (see [3]). Therefore **D** is an intermediate logic between **GL** and **S**.

As for sequent calculi, **GL** has been well studied (see, e.g., [8] and its references), and **S** has been studied in [2, 9, 10]. But there has been no sequent calculi for **D**. The sequent calculus for **S** in [2, 9, 10] was inspired by the Hilbert-style system $\mathbf{S}_\mathrm{H}$; so one may try to make a sequent calculus for **D** based on the system $\mathbf{D}_\mathrm{H}$. However, this attempt does not seem to work well because the axiom $\Box(\Box\psi\vee\Box\sigma)\to(\Box\psi\vee\Box\sigma)$ does not seem to be translatable into a rule of sequent calculi.

In this paper, we establish sequent calculi for **D** so that the completeness with respect to D-models naturally holds. A key idea of our calculi is the use of three kinds of sequents. While Kushida [10] used two kinds of sequents to make the calculus for **S**, we add one more kind. We call the three kinds 'GL-sequents', 'S-sequents', and 'D-sequents'; these respectively correspond to the truth at the GL-model, at the tail, and at the bottom in Figure 1. Moreover, as the names suggest, these correspond to the provability of **GL**, **S**, and **D**, respectively.

[2] Following [3, 6, 14], we use the name **S**, while it is called **GLS** in [5, 9, 10], G^* in [4], and G' in [12].

https://doi.org/10.1017/S1755020325000036 Published online by Cambridge University Press

Strictly speaking, we give two sequent calculi, called $\mathbf{D}^2_{\mathrm{seq}}$ and $\mathbf{D}^3_{\mathrm{seq}}$. The latter calculus $\mathbf{D}^3_{\mathrm{seq}}$ is cut-eliminable, and the former calculus $\mathbf{D}^2_{\mathrm{seq}}$ is not fully cut-eliminable but has the subformula property (we call this property 'analytic'). We show semantic cut-elimination for both calculi (i.e., completeness of cut-free $\mathbf{D}^3_{\mathrm{seq}}$ and analytic $\mathbf{D}^2_{\mathrm{seq}}$) and syntactic cut-elimination for $\mathbf{D}^3_{\mathrm{seq}}$. These semantic arguments are extensions of that for **S** in [9], and the syntactic cut-elimination is reduced to that for **S** by [10].

REMARK 1.1. *A proof in* $\mathbf{D}^3_{\mathrm{seq}}$ *has three layers with three kinds of sequents, and the syntactic cut-elimination for the bottom layer* (*with D-sequents*) *is reduced to that for the middle layer* (*with S-sequents*). *Similar arguments—a proof has layered structure and the cut-elimination for the lower layer is reduced to that for the upper layer—are found in* [11].

Then, two new Hilbert-style proof systems (we call these $\mathbf{D}^2_{\mathrm{H}}$ and $\mathbf{D}^3_{\mathrm{H}}$) for **D** are obtained by interpreting the sequent calculi $\mathbf{D}^2_{\mathrm{seq}}$ and $\mathbf{D}^3_{\mathrm{seq}}$, respectively. Here, not only the existing system $\mathbf{D}_{\mathrm{H}}$ but also the new systems $\mathbf{D}^2_{\mathrm{H}}$ and $\mathbf{D}^3_{\mathrm{H}}$ have 'all theorems of **GL**' as their axioms; so it is natural to argue a generalization as follows. Let L be an arbitrary modal logic, and let $\mathbf{D}_{\mathrm{H}}[L]$, $\mathbf{D}^2_{\mathrm{H}}[L]$, and $\mathbf{D}^3_{\mathrm{H}}[L]$ be the Hilbert-style systems obtained from $\mathbf{D}_{\mathrm{H}}$, $\mathbf{D}^2_{\mathrm{H}}$, and $\mathbf{D}^3_{\mathrm{H}}$, respectively, by replacing the axioms 'theorems of **GL**' with 'theorems of L'. On the other hand, let $\mathcal{F}$ be a class of Kripke frames, and let '$\mathbf{D}[\mathcal{F}]$-model' denote any Kripke model described as Figure 1 in which 'GL-model' is replaced with '$\mathcal{F}$-model'. Then we show the following:

> If L is characterized by $\mathcal{F}$ and if a certain condition is satisfied, then the following conditions are equivalent: (1) φ is true at the bottom of any $\mathbf{D}[\mathcal{F}]$-model. (2) φ is a theorem of $\mathbf{D}^2_{\mathrm{H}}[L]$. (3) φ is a theorem of $\mathbf{D}^3_{\mathrm{H}}[L]$.

This statement is just the completeness theorem of $\mathbf{D}^2_{\mathrm{H}}$ and $\mathbf{D}^3_{\mathrm{H}}$ if $L = \mathbf{GL}$ and $\mathcal{F}$ is the class of transitive and converse well-founded frames. It seems that the condition 'φ is a theorem of $\mathbf{D}_{\mathrm{H}}[L]$' is not equivalent to the above three conditions. This fact shows that the new proof systems—$\mathbf{D}^2_{\mathrm{H}}$, $\mathbf{D}^3_{\mathrm{H}}$, and the two sequent calculi—well reflect the essence of the modal logic **D**, and that the new systems are more natural than the existing system $\mathbf{D}_{\mathrm{H}}$.

The structure of this paper is as follows. In Section 2, we present results that are known or will be shown in later sections. In Section 3, we recall the sequent calculi for **GL** and **S**, and we introduce two sequent calculi $\mathbf{D}^2_{\mathrm{seq}}$ and $\mathbf{D}^3_{\mathrm{seq}}$. In Section 4, we give syntactic arguments on the sequent calculi. In Section 5, we show the semantic cut-elimination. In Section 6, we introduce Hilbert-style systems $\mathbf{D}^2_{\mathrm{H}}$ and $\mathbf{D}^3_{\mathrm{H}}$. In Section 7, we show the general result on $\mathbf{D}^2_{\mathrm{H}}[L]$ and $\mathbf{D}^3_{\mathrm{H}}[L]$.

§2. Preliminaries and results. Formulas are constructed from propositional variables, propositional constant $\bot$, logical operator $\rightarrow$, and modal operator $\Box$. The other operators are defined as abbreviations as usual. The letters $p, q, \ldots$ denote propositional variables and the letters $\varphi, \psi, \ldots$ denote formulas. The set of propositional variables is called **PropVar** and the set of formulas is called **Form**. Parentheses are omitted as, for example, $\Box\varphi\wedge\psi\rightarrow\Box\pi\vee\sigma = ((\Box\varphi)\wedge\psi)\rightarrow((\Box\pi)\vee\sigma)$.

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Hilbert-style proof systems $\mathbf{GL}_{\mathrm{H}}$, $\mathbf{S}_{\mathrm{H}}$, and $\mathbf{D}_{\mathrm{H}}$ are known as follows, where the subscript H denotes 'Hilbert-style'.

Axioms of $\mathbf{GL}_{\mathrm{H}}$: Tautologies, $\Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)$, and $\Box(\Box\varphi\to\varphi)\to\Box\varphi$.
Rules of $\mathbf{GL}_{\mathrm{H}}$: $\dfrac{\varphi\to\psi \quad \varphi}{\psi}$ (modus ponens) and $\dfrac{\varphi}{\Box\varphi}$ $(\Box)$.
Axioms of $\mathbf{S}_{\mathrm{H}}$: Theorems of $\mathbf{GL}_{\mathrm{H}}$ and $\Box\varphi\to\varphi$.
Rule of $\mathbf{S}_{\mathrm{H}}$: Modus ponens.
Axioms of $\mathbf{D}_{\mathrm{H}}$: Theorems of $\mathbf{GL}_{\mathrm{H}}$, $\neg\Box\bot$, and $\Box(\Box\varphi\vee\Box\psi)\to\Box\varphi\vee\Box\psi$.
Rule of $\mathbf{D}_{\mathrm{H}}$: Modus ponens.

The symbol $\vdash$ denotes provability as usual. We have $(\mathbf{GL}_{\mathrm{H}}\vdash\varphi)\Longrightarrow(\mathbf{D}_{\mathrm{H}}\vdash\varphi)\Longrightarrow(\mathbf{S}_{\mathrm{H}}\vdash\varphi)$, but the converse does not hold in general; for example, $\Box p\to p$ is provable in $\mathbf{S}_{\mathrm{H}}$ but not provable in $\mathbf{D}_{\mathrm{H}}$. Note that neither $\mathbf{D}_{\mathrm{H}}$ nor $\mathbf{S}_{\mathrm{H}}$ has the inference rule $(\Box)$. For example, we have $\mathbf{D}_{\mathrm{H}}\vdash\neg\Box\bot$ but $\mathbf{D}_{\mathrm{H}}\nvdash\Box\neg\Box\bot$.

A *Kripke model* $\langle W,\prec,V\rangle$ consists of a non-empty set W of *worlds*, an *accessibility relation* $\prec\subseteq W\times W$, and a *valuation* $V: W\times\mathbf{PropVar}\to\{\mathsf{true},\mathsf{false}\}$. The domain of V is extended to $W\times\mathbf{Form}$ by the following:

$V(w,\bot)=\mathsf{false}$.
$V(w,\varphi\to\psi)=\mathsf{true}\iff V(w,\varphi)=\mathsf{false}$ or $V(w,\psi)=\mathsf{true}$.
$V(w,\Box\varphi)=\mathsf{true}\iff(\forall w'\succ w)(V(w',\varphi)=\mathsf{true})$.

$\langle W,\prec\rangle$ is called the *Kripke frame of* this model, and $\langle W,\prec,V\rangle$ is called a Kripke model *based on* the frame $\langle W,\prec\rangle$. We say that a formula φ is *true at* a world w if $V(w,\varphi)=\mathsf{true}$.

A Kripke frame $\langle W,\prec\rangle$ is called a *GL-frame* if $\prec$ is converse well-founded (i.e., there is no infinitely ascending sequence $x_1\prec x_2\prec x_3\cdots$) and transitive. A Kripke model based on a GL-frame is called a *GL-model*. The completeness of $\mathbf{GL}_{\mathrm{H}}$ is well-known as below (see, e.g., [5]).

PROPOSITION 2.1 (Completeness of $\mathbf{GL}_{\mathrm{H}}$). *The following are equivalent*:

- $\mathbf{GL}_{\mathrm{H}}\vdash\varphi$.
- *φ is true at any world of any GL-model.*

To describe the semantics for $\mathbf{S}_{\mathrm{H}}$ and $\mathbf{D}_{\mathrm{H}}$, we need further definitions.

DEFINITION 2.2. *Let λ be a limit ordinal. We say that a frame $\langle W^+,\prec^+\rangle$ is a λ-extension of $\langle W,\prec\rangle$ if the following two conditions hold for some $t_0\in W$.*

$W^+=W\uplus\{t_\alpha\mid 0<\alpha\le\lambda\}$.
$\prec^+=\prec\cup\{(t_\alpha,t_\beta)\mid 0\le\beta<\alpha\le\lambda\}\cup\{(t_\alpha,x)\mid 0<\alpha\le\lambda \text{ and } (t_0,x)\in\prec\}$.

(*Refer to Figure 1, in which the ellipse is $\langle W,\prec\rangle$.*) *The infinite set $\{t_\alpha\mid 0<\alpha<\lambda\}$ is called the* tail *and the world t_λ is called the* bottom *of this frame. If a valuation V^+ coincides with V for any worlds in W, then we say that the Kripke model $M^+=\langle W^+,\prec^+,V^+\rangle$ is a* λ-extension *of $M=\langle W,\prec,V\rangle$. Moreover, M^+ is called a* constant *or a* strongly constant *λ-extension of M if the following condition holds, respectively.*

Constant: $(\forall p\in\mathbf{PropVar})(\forall\alpha,\beta<\lambda)(V^+(t_\alpha,p)=V^+(t_\beta,p))$.
Strongly constant: $(\forall p\in\mathbf{PropVar})(\forall\alpha,\beta\le\lambda)(V^+(t_\alpha,p)=V^+(t_\beta,p))$.

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We say that a formula φ is eventually always true in the tail of M^+ *if* $(\exists\alpha < \lambda)(\alpha < \forall\beta < \lambda)(V(t_\beta, \varphi) = \mathsf{true})$.

The following proposition is easy to be proved. This will be used implicitly in this paper.

PROPOSITION 2.3. *If $\langle W^+, \prec^+, V^+\rangle$ is a λ-extension of $\langle W, \prec, V\rangle$, then for any world $w \in W$ and any formula φ, we have $V^+(w, \varphi) = V(w, \varphi)$.*

The completeness of $\mathbf{S}_\mathrm{H}$ is described below, which will be proved in Section 5.

PROPOSITION 2.4 (Completeness of $\mathbf{S}_\mathrm{H}$). *Let λ be a limit ordinal. The following are equivalent*:

(s1) $\mathbf{S}_\mathrm{H} \vdash \varphi$.
(s2) *φ is true at the bottom of any strongly constant λ-extension of any GL-model.*
(s3) *φ is eventually always true in the tail of any (strongly) constant λ-extension of any GL-model.*
(s4) *φ is eventually always true in the tail of any λ-extension of any GL-model.*

REMARK 2.5. *The completeness of $\mathbf{S}_\mathrm{H}$ has been expressed in several different statements in* [3, 4, 6, 9, 14], *but all of them are essentially included in the $\lambda = \omega$ case of Proposition 2.4. In this paper, we extend it to arbitrary limit ordinals.*

REMARK 2.6. *The limit ordinal λ is arbitrarily fixed at the beginning of Proposition 2.4. On the other hand, we can consider propositions in which λ is bound at each sentence; for example, there are two variants of the condition $(s2)$ as follows*:

> $(s2^+)$ *φ is true at the bottom of any strongly constant λ'-extension of any GL-model, for* any *limit ordinal λ'.*
> $(s2^-)$ *φ is true at the bottom of any strongly constant λ'-extension of any GL-model, for* some *limit ordinal λ'.*

The conditions $(s2^+)$, $(s2^-)$, and $(s2)$ are equivalent for any λ, because we have the following implications.

$$(\mathrm{s2}^+) \implies (\mathrm{s2}^-) \overset{\textit{Prop } 2.4}{\Longrightarrow} (\mathrm{s1}) \overset{\textit{Prop } 2.4}{\Longrightarrow} (\mathrm{s2}^+).$$

So, from now on, we will not mention conditions like $(s2^+)$ or $(s2^-)$ (except for a condition in Theorem 7.7).

The completeness of $\mathbf{D}_\mathrm{H}$ is described below, which will be proved in Sections 5 and 6.

PROPOSITION 2.7 (Completeness of $\mathbf{D}_\mathrm{H}$). *Let λ be a limit ordinal. The following are equivalent*:

(d1) $\mathbf{D}_\mathrm{H} \vdash \varphi$.
(d2) *φ is true at the bottom of any constant λ-extension of any GL-model.*
(d3) *φ is true at the bottom of any λ-extension of any GL-model.*

REMARK 2.8. *Beklemishev* [3] *proved the equivalence between $(d1)$, $(d2)$ for $\lambda = \omega$, and another condition using 'accumulating root'. In Section 5, we will show the equivalence between $(d2)$ and $(d3)$. In Section 6, we will show the equivalence between $(d1)$ and others.*

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§3. Sequent calculi. We introduce sequent calculi. From now on, letters $\Gamma, \Delta, \ldots$ denote finite sets of formulas. As usual, the expression '$\Gamma, \varphi, \psi, \Delta$', for example, stands for the set $\Gamma \cup \{\varphi, \psi\} \cup \Delta$, and '$\Box\Gamma$' stands for the set $\{\Box\gamma \mid \gamma \in \Gamma\}$. $\mathrm{Sub}(\Gamma)$ denotes the set of all subformulas of formulas in Γ.

We use three different arrows $\Rightarrow$, $\Rrightarrow$, and $\Rightarrow\!\!\!\!\Rightarrow$ to define three kinds of sequents $\Gamma \Rightarrow \Delta$ (called *GL-sequent*), $\Gamma \Rrightarrow \Delta$ (called *S-sequent*), and $\Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$ (called *D-sequent*).

The following is the well-known sequent calculus for classical propositional logic; we call it $\mathbf{LK}_{\Rightarrow}$.

Initial sequents: $\varphi \Rightarrow \varphi$ and $\bot \Rightarrow$

Inference rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'}\ (\text{weakening}),\ \text{where}\ \Gamma \subseteq \Gamma'\ \text{and}\ \Delta \subseteq \Delta'.$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ (\to\text{left}) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (\text{cut}).$$

Similarly the sequent calculus $\mathbf{LK}_{\Rrightarrow}$ (or $\mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}$, respectively) consists of the above initial sequents and inference rules where all '$\Rightarrow$' are replaced with '$\Rrightarrow$' (or '$\Rightarrow\!\!\!\!\Rightarrow$', respectively). Next we introduce five rules (GL$\Box$), (GLtoS), (S$\Box$left), (GLtoD), and (StoD) as below

$$\frac{\Gamma, \Box\Gamma, \Box\varphi \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ (\text{GL}\Box)$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rrightarrow \Delta}\ (\text{GLtoS})(\text{used in } \mathbf{S}_{\text{seq}}, \mathbf{D}^3_{\text{seq}}) \qquad \frac{\varphi, \Gamma \Rrightarrow \Delta}{\Box\varphi, \Gamma \Rrightarrow \Delta}\ (\text{S}\Box\text{left})\ (\text{used in } \mathbf{S}_{\text{seq}}, \mathbf{D}^3_{\text{seq}})$$

$$\frac{\Gamma, \Box\Gamma \Rightarrow \Box\Delta}{\Box\Gamma \Rightarrow\!\!\!\!\Rightarrow \Box\Delta}\ (\text{GLtoD})\ (\text{used in } \mathbf{D}^2_{\text{seq}}) \qquad \frac{\Box\Gamma \Rrightarrow \Box\Delta}{\Box\Gamma \Rightarrow\!\!\!\!\Rightarrow \Box\Delta}\ (\text{StoD})\ (\text{used in } \mathbf{D}^3_{\text{seq}}).$$

Using these, we define four sequent calculi $\mathbf{GL}_{\text{seq}}$, $\mathbf{S}_{\text{seq}}$, $\mathbf{D}^2_{\text{seq}}$, and $\mathbf{D}^3_{\text{seq}}$ according to Table 1. $\mathbf{GL}_{\text{seq}}$ $(= \mathbf{LK}_{\Rightarrow} + (\text{GL}\Box))$ is a well-known sequent calculus for **GL**, and has been widely studied; see [8] and its references. $\mathbf{S}_{\text{seq}}$ $(= \mathbf{GL}_{\text{seq}} + (\text{GLtoS}) + (\text{S}\Box\text{left}) + \mathbf{LK}_{\Rrightarrow})$ is a sequent calculus for **S** using two kinds of sequents (GL- and S-sequents), and it (or similar calculi) has been studied in [2, 9, 10]. The two calculi $\mathbf{D}^2_{\text{seq}}$ and $\mathbf{D}^3_{\text{seq}}$ are new, and they are obtained as below

$$\mathbf{D}^2_{\text{seq}} = \mathbf{GL}_{\text{seq}} + (\text{GLtoD}) + \mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}.$$

$$\mathbf{D}^3_{\text{seq}} = \mathbf{S}_{\text{seq}} + (\text{StoD}) + \mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}.$$

The superscripts '2' and '3' denote the number of kinds of sequents; that is, $\mathbf{D}^2_{\text{seq}}$ uses two kinds (GL- and D-sequents) and $\mathbf{D}^3_{\text{seq}}$ uses three kinds of sequents.

Proofs in each calculus are called $\mathbf{GL}_{\text{seq}}$/$\mathbf{S}_{\text{seq}}$/$\mathbf{D}^2_{\text{seq}}$/$\mathbf{D}^3_{\text{seq}}$-proofs. In general, an $\mathbf{S}_{\text{seq}}$-proof has two-layered structure (the upper layer consists of GL-sequents and the lower layer consists of S-sequents). Similarly, a $\mathbf{D}^2_{\text{seq}}$-proof has two layers (upper: GL-sequents; lower: D-sequents), and a $\mathbf{D}^3_{\text{seq}}$-proof has three layers (top: GL-sequents;

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Table 1. *Four sequent calculi*

	$\mathbf{LK}_{\Rightarrow}$	$\mathbf{LK}_{\Rrightarrow}$	$\mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}$	(GL$\Box$)	(GLtoS)	(S$\Box$left)	(GLtoD)	(StoD)
$\mathbf{GL}_{\mathrm{seq}}$	✓			✓				
$\mathbf{S}_{\mathrm{seq}}$	✓	✓		✓	✓	✓		
$\mathbf{D}^2_{\mathrm{seq}}$	✓		✓	✓			✓	
$\mathbf{D}^3_{\mathrm{seq}}$	✓	✓	✓	✓	✓	✓		✓

middle: S-sequents; bottom: D-sequents). Note that only $\mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}$-rules can have D-sequents as assumptions. Therefore the bottom layer of a $\mathbf{D}^2_{\mathrm{seq}}/\mathbf{D}^3_{\mathrm{seq}}$-proof consists of only $\mathbf{LK}_{\Rrightarrow}$-rules.

As usual, the term 'cut-free' means 'without using the cut rule'. The following examples are cut-free $\mathbf{D}^2_{\mathrm{seq}}/\mathbf{D}^3_{\mathrm{seq}}$-proofs, where 'w.' stands for 'weakening', and ($\vee$left), ($\vee$right), and ($\neg$right) are suitable combinations of rules in **LK** (since $\vee$ and $\neg$ are abbreviations).

EXAMPLE 3.1. *Cut-free* $\mathbf{D}^2_{\mathrm{seq}} \vdash \Rightarrow\!\!\!\!\Rightarrow \Box(\Box\varphi\vee\Box\psi)\to\Box\varphi\vee\Box\psi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\varphi \Rightarrow \Box\varphi \quad \Box\psi \Rightarrow \Box\psi}{\Box\varphi\vee\Box\psi \Rightarrow \Box\varphi, \Box\psi}\,(\vee\text{left})}{\Box\varphi\vee\Box\psi, \Box(\Box\varphi\vee\Box\psi) \Rightarrow \Box\varphi, \Box\psi}\,(\text{w.})}{\Box(\Box\varphi\vee\Box\psi) \Rrightarrow \Box\varphi, \Box\psi}\,(\text{GLtoD})}{\Box(\Box\varphi\vee\Box\psi) \Rrightarrow \Box\varphi\vee\Box\psi}\,(\vee\text{right})}{\Rrightarrow \Box(\Box\varphi\vee\Box\psi)\to\Box\varphi\vee\Box\psi}\,(\to\text{right}).$$

EXAMPLE 3.2. *Cut-free* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Rightarrow\!\!\!\!\Rightarrow \Box(\Box\varphi\vee\Box\psi)\to\Box\varphi\vee\Box\psi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\varphi \Rrightarrow \Box\varphi \quad \Box\psi \Rrightarrow \Box\psi}{\Box\varphi \vee \Box\psi \Rrightarrow \Box\varphi, \Box\psi}\,(\vee\text{left})}{\Box(\Box\varphi \vee \Box\psi) \Rrightarrow \Box\varphi, \Box\psi}\,(\text{S}\Box\text{left})}{\Box(\Box\varphi \vee \Box\psi) \Rrightarrow \Box\varphi, \Box\psi}\,(\text{StoD})}{\Box(\Box\varphi \vee \Box\psi) \Rrightarrow \Box\varphi \vee \Box\psi}\,(\vee\text{right})}{\Rrightarrow \Box(\Box\varphi \vee \Box\psi) \to \Box\varphi \vee \Box\psi}\,(\to \text{right}).$$

EXAMPLE 3.3. *Cut-free* $\mathbf{D}^2_{\mathrm{seq}} \vdash \Rightarrow\!\!\!\!\Rightarrow \neg\Box\bot$.

$$\dfrac{\dfrac{\dfrac{\bot \Rightarrow}{\bot, \Box\bot \Rightarrow}\,(\text{w.})}{\Box\bot \Rrightarrow}\,(\text{GLtoD})}{\Rrightarrow \neg\Box\bot}\,(\neg\text{right}).$$

The correspondence between sequent calculi and Hilbert-style systems is the same as in classical logic, as below.

PROPOSITION 3.4. *The following are equivalent*:

- $\mathbf{GL}_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta$.
- $\mathbf{GL}_{\mathrm{H}} \vdash \bigwedge\Gamma \to \bigvee\Delta$.

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PROPOSITION 3.5. *The following are equivalent*:

- $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rrightarrow \Delta$.
- $\mathbf{S}_{\mathrm{H}} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

PROPOSITION 3.6. *The following are equivalent*:

- $\mathbf{D}^2_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$.
- $\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$.
- $\mathbf{D}_{\mathrm{H}} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

Propositions 3.4 and 3.5 are well-known and easy to prove. Proposition 3.6 will be shown in Section 6.

REMARK 3.7. $\mathbf{D}^3_{\mathrm{seq}}$ *is obtained from* $\mathbf{S}_{\mathrm{seq}}$ *by adding some rules, while the logic* $\mathbf{D}$ *is strictly weaker than* $\mathbf{S}$.

In the rest of this section, we mention the calculus for $\mathbf{S}$ by Kushida[10], which we call $\mathbf{S}_{\mathrm{Kushida}}$. Written in our notation, the calculus $\mathbf{S}_{\mathrm{Kushida}}$ consists of $\mathbf{LK}_{\Rightarrow}$, $\mathbf{LK}_{\Rrightarrow}$, and the four rules below

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rrightarrow \Delta}\ (\text{Kushida-GLtoS}\Box\text{left}) \qquad \frac{\Box\Gamma, \Delta, \Box\varphi \Rightarrow \varphi}{\Box\Gamma, \Box\Delta \Rightarrow \Box\varphi}\ (\text{Kushida-GL}\Box)$$

$$\frac{\varphi, \Gamma \Rrightarrow \Delta}{\Box\varphi, \Gamma \Rrightarrow \Delta}\ (\text{S}\Box\text{left}) \qquad \frac{\Box\Gamma, \Delta, \Box\varphi \Rightarrow \varphi}{\Box\Gamma, \Box\Delta \Rrightarrow \Box\varphi}\ (\text{Kushida-GLtoS}\Box).$$

Consequently, $\mathbf{S}_{\mathrm{Kushida}}$ is obtained from $\mathbf{S}_{\mathrm{seq}}$ by replacing the two rules $\{(\mathrm{GL}\Box), (\mathrm{GLtoS})\}$ with the three rules $\{(\text{Kushida-GLtoS}\Box\text{left}), (\text{Kushida-GL}\Box), (\text{Kushida-GLtoS}\Box)\}$. Note that sequents in [10] consist of *multisets* of formulas, while our sequents consist of *sets* of formulas. However, this difference is not critical because of the existence of the contraction-rules in [10]; so $\Gamma, \Delta, \ldots$ here denote sets of formulas.

The following lemma can be easily proved by induction on $\mathbf{S}_{\mathrm{Kushida}}$-proofs.

LEMMA 3.8 (The rule (GLtoS) is cut-free admissible in $\mathbf{S}_{\mathrm{Kushida}}$). *If* $\mathbf{S}_{\mathrm{Kushida}} \vdash \Gamma \Rightarrow \Delta$, *then* $\mathbf{S}_{\mathrm{Kushida}} \vdash \Gamma \Rrightarrow \Delta$. *If cut-free* $\mathbf{S}_{\mathrm{Kushida}} \vdash \Gamma \Rightarrow \Delta$, *then cut-free* $\mathbf{S}_{\mathrm{Kushida}} \vdash \Gamma \Rrightarrow \Delta$.

Then we have the following theorem.

THEOREM 3.9 ($\mathbf{S}_{\mathrm{Kushida}}$ and $\mathbf{S}_{\mathrm{seq}}$ are equivalent). *Let* $\blacktriangleright \in \{\Rightarrow, \Rrightarrow\}$. $\mathbf{S}_{\mathrm{Kushida}} \vdash \Gamma \blacktriangleright \Delta$ *if and only if* $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \blacktriangleright \Delta$. *Cut-free* $\mathbf{S}_{\mathrm{Kushida}} \vdash \Gamma \blacktriangleright \Delta$ *if and only if cut-free* $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \blacktriangleright \Delta$.

Proof. Each rule in $\mathbf{S}_{\mathrm{Kushida}}$ is cut-free derivable in $\mathbf{S}_{\mathrm{seq}}$; for example, the rule $(\text{Kushida-GLtoS}\Box)$ is shown below

$$\dfrac{\dfrac{\dfrac{\Box\Gamma, \Delta, \Box\varphi \Rightarrow \varphi}{\Box\Gamma, \Box\Delta, \Gamma, \Delta, \Box\varphi \Rightarrow \varphi}\ (\text{w.})}{\Box\Gamma, \Box\Delta \Rightarrow \Box\varphi}\ (\mathrm{GL}\Box)}{\Box\Gamma, \Box\Delta \Rrightarrow \Box\varphi}\ (\mathrm{GLtoS}).$$

On the other hand, each rule in $\mathbf{S}_{\mathrm{seq}}$ is cut-free admissible in $\mathbf{S}_{\mathrm{Kushida}}$ (the rule (GLtoS) is shown by the previous lemma, and the rule $(\mathrm{GL}\Box)$ is trivial). Hence, this Theorem 3.9 is easily shown by induction. □

https://doi.org/10.1017/S1755020325000036 Published online by Cambridge University Press

§4. Syntactic arguments. In this section, we show some basic properties of our four calculi, which can be shown by syntactic method (i.e., induction on proofs).

While the logics **S** and **D** are proper extensions of **GL**, the sequent calculi $\mathbf{S}_{\mathrm{seq}}$, $\mathbf{D}^2_{\mathrm{seq}}$, and $\mathbf{D}^3_{\mathrm{seq}}$ are conservative extensions of $\mathbf{GL}_{\mathrm{seq}}$ with respect to GL-sequents. Moreover, $\mathbf{D}^3_{\mathrm{seq}}$ is a conservative extension of $\mathbf{S}_{\mathrm{seq}}$ with respect to S-sequents. These are shown by the following theorems.

THEOREM 4.1 (Conservativity of provability of GL-sequents). *The following four conditions are equivalent*: $\mathbf{GL}_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta$; $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta$; $\mathbf{D}^2_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta$; *and* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta$.

THEOREM 4.2 (Conservativity of provability of S-sequents). *The following two conditions are equivalent*: $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rrightarrow \Delta$ *and* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \Rrightarrow \Delta$.

Theorems 4.1 and 4.2 are trivial because any $\mathbf{S}_{\mathrm{seq}}/\mathbf{D}^2_{\mathrm{seq}}/\mathbf{D}^3_{\mathrm{seq}}$-proofs of GL-sequents consist of only the rules of $\mathbf{GL}_{\mathrm{seq}}$, and any $\mathbf{D}^3_{\mathrm{seq}}$-proofs of S-sequents consist of only the rules of $\mathbf{S}_{\mathrm{seq}}$. These two theorems and their cut-free versions (e.g., cut-free $\mathbf{GL}_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta$ if and only if cut-free $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta$) will be used implicitly from now on.

The fact 'both the logics **S** and **D** are extensions of **GL**' is expressed by the following.

THEOREM 4.3. *If* $\mathbf{GL}_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta$, *then we have* $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rrightarrow \Delta$, $\mathbf{D}^2_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$, *and* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$.

Proof. Suppose there is a $\mathbf{GL}_{\mathrm{seq}}$-proof P of $\Gamma \Rightarrow \Delta$. $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rrightarrow \Delta$ is trivial because of the rule (GLtoS). On the other hand, $\mathbf{D}^2_{\mathrm{seq}}/\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$ is shown by induction on P. If the last inference rule of P is (GL$\Box$), then $\Gamma \Rightarrow \Delta$ is of the form $\Box\Pi \Rightarrow \Box\varphi$, and we get $\mathbf{D}^2_{\mathrm{seq}}$- and $\mathbf{D}^3_{\mathrm{seq}}$-proofs as below without using the induction hypothesis.

$$\dfrac{\dfrac{\dfrac{\vdots\, P}{\Box\Pi \Rightarrow \Box\varphi}}{\Pi, \Box\Pi \Rightarrow \Box\varphi}\,(\mathrm{w.})}{\Box\Pi \Rightarrow\!\!\!\!\Rightarrow \Box\varphi}\,(\mathrm{GLtoD}) \qquad \dfrac{\dfrac{\dfrac{\vdots\, P}{\Box\Pi \Rightarrow \Box\varphi}}{\Box\Pi \Rrightarrow \Box\varphi}\,(\mathrm{GLtoS})}{\Box\Pi \Rightarrow\!\!\!\!\Rightarrow \Box\varphi}\,(\mathrm{StoD}).$$

In other words, a $\mathbf{GL}_{\mathrm{seq}}$-proof

$$\dfrac{\dfrac{\vdots}{\Pi, \Box\Pi, \Box\varphi \Rightarrow \varphi}}{\Box\Pi \Rightarrow \Box\varphi}\,(\mathrm{GL}\Box)$$
$$\vdots\ \mathbf{LK}_{\Rightarrow}\text{-rules}$$
$$\Gamma \Rightarrow \Delta$$

is translated into the following $\mathbf{D}^2_{\mathrm{seq}}$- and $\mathbf{D}^3_{\mathrm{seq}}$-proofs.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Pi, \Box\Pi, \Box\varphi \Rightarrow \varphi}}{\Box\Pi \Rightarrow \Box\varphi}\,(\mathrm{GL}\Box)}{\Pi, \Box\Pi \Rightarrow \Box\varphi}\,(\mathrm{w.})}{\Box\Pi \Rightarrow\!\!\!\!\Rightarrow \Box\varphi}\,(\mathrm{GLtoD})}{\vdots\ \mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}\text{-rules}}$$
$$\Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta \qquad\qquad \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Pi, \Box\Pi, \Box\varphi \Rightarrow \varphi}}{\Box\Pi \Rightarrow \Box\varphi}\,(\mathrm{GL}\Box)}{\Box\Pi \Rrightarrow \Box\varphi}\,(\mathrm{GLtoS})}{\Box\Pi \Rightarrow\!\!\!\!\Rightarrow \Box\varphi}\,(\mathrm{StoD})$$
$$\vdots\ \mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}\text{-rules}$$
$$\Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta.$$

□

https://doi.org/10.1017/S1755020325000036 Published online by Cambridge University Press

The fact 'logic **S** is an extension of **D**' is expressed by the following theorem.

THEOREM 4.4. *If* $\mathbf{D}^2_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$, *then* $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rrightarrow \Delta$. *If* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$, *then* $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rrightarrow \Delta$.

Proof. Roughly speaking, we translate the $\mathbf{D}^2_{\mathrm{seq}}$- and $\mathbf{D}^3_{\mathrm{seq}}$-proofs

$$\dfrac{\dfrac{\vdots}{\Psi, \Box\Psi \Rightarrow \Box\Phi}}{\Box\Psi \Rightarrow\!\!\!\!\Rightarrow \Box\Phi}\ (\mathrm{GLtoD}) \qquad \dfrac{\dfrac{\vdots}{\Box\Psi \Rrightarrow \Box\Phi}}{\Box\Psi \Rightarrow\!\!\!\!\Rightarrow \Box\Phi}\ (\mathrm{StoD})$$

$$\begin{array}{c} \vdots\ \mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}\text{ – rules} \\ \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta \end{array} \qquad \begin{array}{c} \vdots\ \mathbf{LK}_{\Rightarrow\!\!\!\!\Rightarrow}\text{ – rules} \\ \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta \end{array}$$

into the following $\mathbf{S}_{\mathrm{seq}}$-proofs respectively.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Psi, \Box\Psi \Rightarrow \Box\Phi}}{\Psi, \Box\Psi \Rrightarrow \Box\Phi}\ (\mathrm{GLtoS})}{\Box\Psi \Rrightarrow \Box\Phi}\ (\mathrm{S}\Box\mathrm{left}) \qquad \dfrac{\vdots}{\Box\Psi \Rrightarrow \Box\Phi}$$

$$\begin{array}{c} \vdots\ \mathbf{LK}_{\Rightarrow}\text{-rules} \\ \Gamma \Rrightarrow \Delta \end{array} \qquad \begin{array}{c} \vdots\ \mathbf{LK}_{\Rightarrow}\text{-rules} \\ \Gamma \Rrightarrow \Delta. \end{array}$$

To be precise, we show this Theorem 4.4 by induction on the $\mathbf{D}^2_{\mathrm{seq}}$- and $\mathbf{D}^3_{\mathrm{seq}}$-proofs of $\Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$. □

The following theorem is expected to be hold as a matter of course.

THEOREM 4.5 (Equivalence between $\mathbf{D}^2_{\mathrm{seq}}$ and $\mathbf{D}^3_{\mathrm{seq}}$). $\mathbf{D}^2_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$ *if and only if* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$.

The only-if part is easily shown by induction on $\mathbf{D}^2_{\mathrm{seq}}$-proofs of $\Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$; we use the rules (GLtoS), (S□left), and (StoD) if the last inference rule is (GLtoD). On the other hand, the if-part is not trivial because of the existence of S-sequents in $\mathbf{D}^3_{\mathrm{seq}}$-proofs. We need some lemmas as below. In the following, $\mathrm{ref}(\Sigma)$ will denote the set $\{\Box\sigma \to \sigma \mid \sigma \in \Sigma\}$ (the name 'ref' comes from 'reflection').

LEMMA 4.6. *If* $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \Rrightarrow \Delta$, *then there is a finite set* Σ *of formulas such that* $\mathbf{GL}_{\mathrm{seq}} \vdash \mathrm{ref}(\Sigma), \Gamma \Rightarrow \Delta$.

Proof. By induction on the $\mathbf{S}_{\mathrm{seq}}$-proof P of $\Gamma \Rrightarrow \Delta$. If P is

$$\dfrac{\dfrac{\vdots\ P'}{\varphi, \Pi \Rrightarrow \Delta}}{\Box\varphi, \Pi \Rrightarrow \Delta}\ (\mathrm{S}\Box\mathrm{left})$$

then we have

$$\dfrac{\Box\varphi \Rightarrow \Box\varphi \quad \mathrm{ref}(\Sigma'), \varphi, \Pi \Rightarrow \Delta}{\Box\varphi \to \varphi, \mathrm{ref}(\Sigma'), \Box\varphi, \Pi \Rightarrow \Delta.}\ (\mathrm{w.})(\to\mathrm{left}) \qquad \begin{array}{c} \vdots\ \text{i.h. for}\ P' \\ \end{array}$$

□

LEMMA 4.7. *If* $\mathbf{GL}_{\mathrm{seq}} \vdash \mathrm{ref}(\Sigma), \Box\Psi \Rightarrow \Box\Phi$, *then* $\mathbf{D}^2_{\mathrm{seq}} \vdash \Box\Psi \Rightarrow\!\!\!\!\Rightarrow \Box\Phi$.

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Proof. It is easy to show that $\mathbf{GL}_{\mathrm{seq}}$ has the following property, which we call $(\to\mathrm{left}^{-1})$: If $\mathbf{GL}_{\mathrm{seq}} \vdash \varphi\to\psi, \Gamma \Rightarrow \Delta$, then $\mathbf{GL}_{\mathrm{seq}} \vdash \Gamma \Rightarrow \Delta, \varphi$ and $\mathbf{GL}_{\mathrm{seq}} \vdash \psi, \Gamma \Rightarrow \Delta$. Now assume $\mathbf{GL}_{\mathrm{seq}} \vdash \mathrm{ref}(\Sigma), \Box\Psi \Rightarrow \Box\Phi$ where $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$. For any subset $\Sigma' \subseteq \Sigma$, we have $\mathbf{D}^2_{\mathrm{seq}} \vdash \Box\Sigma', \Box\Psi \Rrightarrow \Box\Phi, \Box(\Sigma \setminus \Sigma')$ as below.

$$\mathbf{GL}_{\mathrm{seq}} \vdash \Box\sigma_1\to\sigma_1, \dots, \Box\sigma_n\to\sigma_n, \Box\Psi \Rightarrow \Box\Phi \qquad \text{(Assumption of the Lemma)}$$

$$\mathbf{GL}_{\mathrm{seq}} \vdash \Sigma', \Box\Psi \Rightarrow \Box\Phi, \Box(\Sigma \setminus \Sigma') \qquad \text{(By } n\text{-times application of } (\to\mathrm{left}^{-1})\text{)}$$

$$\mathbf{D}^2_{\mathrm{seq}} \vdash \Box\Sigma', \Box\Psi \Rrightarrow \Box\Phi, \Box(\Sigma \setminus \Sigma'). \qquad \text{(By weakening and (GLtoD))}$$

Then, by using the cut rule to the 2^n D-sequents, we have $\mathbf{D}^2_{\mathrm{seq}} \vdash \Box\Psi \Rrightarrow \Box\Phi$. For example, if $n = 2$, the proof is below.

$$\dfrac{\dfrac{\Box\Psi \Rrightarrow \Box\Phi, \Box\sigma_1, \Box\sigma_2 \quad \Box\sigma_2, \Box\Psi \Rrightarrow \Box\Phi, \Box\sigma_1}{\Box\Psi \Rrightarrow \Box\Phi, \Box\sigma_1}\ (\mathrm{cut}) \qquad \dfrac{\Box\sigma_1, \Box\Psi \Rrightarrow \Box\Phi, \Box\sigma_2 \quad \Box\sigma_1, \Box\sigma_2, \Box\Psi \Rrightarrow \Box\Phi}{\Box\sigma_1, \Box\Psi \Rrightarrow \Box\Phi}\ (\mathrm{cut})}{\Box\Psi \Rrightarrow \Box\Phi}\ (\mathrm{cut})$$

□

Proof of if-part of Theorem 4.5. By induction on $\mathbf{D}^3_{\mathrm{seq}}$-proof P of $\Gamma \Rrightarrow \Delta$. If P is of the form

$$\dfrac{\begin{array}{c}\vdots\ P'\\ \Box\Psi \Rightarrow \Box\Phi\end{array}}{\Box\Psi \Rrightarrow \Box\Phi}\ (\mathrm{StoD})$$

then we have $\mathbf{D}^2_{\mathrm{seq}} \vdash \Box\Psi \Rrightarrow \Box\Phi$ by Lemmas 4.6 and 4.7. □

Two calculi $\mathbf{D}^2_{\mathrm{seq}}$ and $\mathbf{D}^3_{\mathrm{seq}}$ are equivalent as above; however, they are not equivalent with respect to cut-eliminability.

THEOREM 4.8 (Cut-elimination for $\mathbf{D}^3_{\mathrm{seq}}$). *If* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Psi \Rrightarrow \Phi$, *then cut-free* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Psi \Rrightarrow \Phi$.

Proof. Combining cut-free versions of Theorems 4.1 and 4.2, Theorem 3.9, and the syntactic cut-elimination for $\mathbf{S}_{\mathrm{Kushida}}$ (shown in [10]), we have cut-free admissibility of the cut rules for GL- and S-sequents in $\mathbf{D}^3_{\mathrm{seq}}$: that is, we have the following:

$(\mathrm{cut}_{\Rightarrow})$ If both $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Pi \Rightarrow \Sigma$ are cut-free provable in $\mathbf{D}^3_{\mathrm{seq}}$, then so is $\Gamma, \Pi \Rightarrow \Delta, \Sigma$.
$(\mathrm{cut}_{\Rrightarrow})$ If both $\Gamma \Rrightarrow \Delta, \varphi$ and $\varphi, \Pi \Rrightarrow \Sigma$ are cut-free provable in $\mathbf{D}^3_{\mathrm{seq}}$, then so is $\Gamma, \Pi \Rrightarrow \Delta, \Sigma$.

Then we show this property for D-sequents; that is, we eliminate

$$\dfrac{\Gamma \Rrightarrow \Delta, \varphi \quad \varphi, \Pi \Rrightarrow \Sigma}{\Gamma, \Pi \Rrightarrow \Delta, \Sigma}\ (\mathrm{cut}_{\Rrightarrow}).$$

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As usual, this is shown by double induction on the length of the cut-formula φ and sum of the length of the left and right upper proofs. In the case of

$$\dfrac{\dfrac{\dfrac{\vdots\, P}{\Box\Gamma \Rrightarrow \Box\Delta, \Box\varphi}\,(\text{StoD})}{\Box\Gamma \Rightarrow\!\!\!\!\Rightarrow \Box\Delta, \Box\varphi} \qquad \dfrac{\dfrac{\vdots\, Q}{\Box\varphi, \Box\Pi \Rrightarrow \Box\Sigma}\,(\text{StoD})}{\Box\varphi, \Box\Pi \Rightarrow\!\!\!\!\Rightarrow \Box\Sigma}}{\Box\Gamma, \Box\Pi \Rightarrow\!\!\!\!\Rightarrow \Box\Delta, \Box\Sigma}\,(\text{cut}_{\Rightarrow\!\!\!\!\Rightarrow})$$

we have the following:

$$\dfrac{\dfrac{\vdots\, P \qquad \vdots\, Q}{\Box\Gamma \Rrightarrow \Box\Delta, \Box\varphi \quad \Box\varphi, \Box\Pi \Rrightarrow \Box\Sigma}\,(\text{cut}_{\Rrightarrow})}{\dfrac{\Box\Gamma, \Box\Pi \Rrightarrow \Box\Delta, \Box\Sigma}{\Box\Gamma, \Box\Pi \Rightarrow\!\!\!\!\Rightarrow \Box\Delta, \Box\Sigma}\,(\text{StoD}).}$$

The other cases are easily done by the standard method. □

Note that the above proof is an *indirect* cut-elimination in the sense that it relies on cut-admissibility of other systems which is obtained by proof translations in previous theorems. A semantical proof of it will be given in the next section.

THEOREM 4.9 (Failure of cut-elimination for $\mathbf{D}^2_{\text{seq}}$). *There is a D-sequent that is provable but not cut-free provable in* $\mathbf{D}^2_{\text{seq}}$.

Proof. We have a $\mathbf{D}^2_{\text{seq}}$-proof:

$$\dfrac{\dfrac{\dfrac{\Box\Box p \Rightarrow \Box\Box p}{\Box\Box\Box p \Rightarrow\!\!\!\!\Rightarrow \Box\Box p}\,(\text{w.})(\text{GLtoD}) \qquad \dfrac{\Box p \Rightarrow \Box p}{\Box\Box p \Rightarrow\!\!\!\!\Rightarrow \Box p}\,(\text{w.})(\text{GLtoD})}{\Box\Box\Box p \Rightarrow\!\!\!\!\Rightarrow \Box p}\,(\text{cut})$$

This last sequent is not cut-free provable because otherwise the last inference should be one of the following rules:

$$\dfrac{\Box\Box p, \Box\Box\Box p \Rightarrow \Box p}{\Box\Box\Box p \Rightarrow\!\!\!\!\Rightarrow \Box p}\,(\text{GLtoD})$$

$$\dfrac{\Rightarrow\!\!\!\!\Rightarrow \Box p}{\Box\Box\Box p \Rightarrow\!\!\!\!\Rightarrow \Box p}\,(\text{w.}) \qquad \dfrac{\Box\Box\Box p \Rightarrow\!\!\!\!\Rightarrow}{\Box\Box\Box p \Rightarrow\!\!\!\!\Rightarrow \Box p}\,(\text{w.}) \qquad \dfrac{\Rightarrow\!\!\!\!\Rightarrow}{\Box\Box\Box p \Rightarrow\!\!\!\!\Rightarrow \Box p}\,(\text{w.}).$$

But none of these upper sequents are provable. The fact $\mathbf{GL}_{\text{seq}} \nvdash \Box\Box p, \Box\Box\Box p \Rightarrow \Box p$ can be shown using the soundness of $\mathbf{GL}_{\text{seq}}$ (i.e., combination of Propositions 2.1 and 3.4) and a GL-model $\langle \{x, y\}, \prec, V \rangle$ where $\prec = \{(x, y)\}$ and $V(y, p) = \mathsf{false}$ (thus, $V(x, \Box p) = \mathsf{false}$ and $V(x, \Box\Box p) = V(x, \Box\Box\Box p) = \mathsf{true}$). The other sequents $(\Rightarrow\!\!\!\!\Rightarrow \Box p)$, $(\Box\Box\Box p \Rightarrow\!\!\!\!\Rightarrow)$, and $(\Rightarrow\!\!\!\!\Rightarrow)$ are easily shown to be cut-free unprovable. □

In the next section, we will show a weaker version of cut-elimination for $\mathbf{D}^2_{\text{seq}}$—we can eliminate all but the cuts that have subformula property—by a semantical method.

§5. Cut-free completeness. In this section, we prove cut-free completeness of the sequent calculi. As shown above, $\mathbf{D}^2_{\text{seq}}$ does not have the full cut-elimination property; so, for $\mathbf{D}^2_{\text{seq}}$, we show the completeness of 'analytic $\mathbf{D}^2_{\text{seq}}$' which is defined below.

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DEFINITION 5.1. Analytic $\mathbf{D}^2_{\mathrm{seq}}$ *is a restriction of* $\mathbf{D}^2_{\mathrm{seq}}$ *where cut is allowed only in the form*

$$\frac{\Gamma \Rrightarrow \Delta, \Box\varphi \quad \Box\varphi, \Pi \Rrightarrow \Sigma}{\Gamma, \Pi \Rrightarrow \Delta, \Sigma.} \ (cut_{\Rrightarrow}) \ \textit{Proviso} : \Box\varphi \in \mathrm{Sub}(\Gamma, \Delta, \Pi, \Sigma).$$

The following are key notions for proving the completeness.

DEFINITION 5.2. *Let* $\blacktriangleright \in \{\Rightarrow, \Rightarrow\!\!\!\!\!\!\Rightarrow, \Rrightarrow\}$.

- *A GL/S/D-sequent* $\Gamma \blacktriangleright \Delta$ *is* $\rightarrow$-saturated $\overset{def}{\Longleftrightarrow}$ $(\forall \varphi, \psi \in \boldsymbol{Form})((\varphi \rightarrow \psi \in \Gamma \Longrightarrow \varphi \in \Delta \text{ or } \psi \in \Gamma)$ *and* $(\varphi \rightarrow \psi \in \Delta \Longrightarrow \varphi \in \Gamma \text{ and } \psi \in \Delta))$.
- *A GL/S/D-sequent* $\Gamma \blacktriangleright \Delta$ *is* $\Box$-left-saturated $\overset{def}{\Longleftrightarrow}$ $(\forall \varphi \in \boldsymbol{Form})(\Box\varphi \in \Gamma \Longrightarrow \varphi \in \Gamma)$.
- *A GL-sequent* $\Psi^+ \Rightarrow \Phi^+$ *is a* $\mathbf{GL}_{\mathrm{seq}}$-saturation of $\Psi \Rightarrow \Phi$ $\overset{def}{\Longleftrightarrow}$ $\Psi \subseteq \Psi^+ \subseteq Sub(\Psi, \Phi)$; $\Phi \subseteq \Phi^+ \subseteq Sub(\Psi, \Phi)$; $\Psi^+ \Rightarrow \Phi^+$ *is* $\rightarrow$*-saturated*; *and cut-free* $\mathbf{GL}_{\mathrm{seq}} \nvdash \Psi^+ \Rightarrow \Phi^+$.
- *An S-sequent* $\Psi^+ \Rightarrow\!\!\!\!\!\!\Rightarrow \Phi^+$ *is an* $\mathbf{S}_{\mathrm{seq}}$-saturation of $\Psi \Rightarrow\!\!\!\!\!\!\Rightarrow \Phi$ $\overset{def}{\Longleftrightarrow}$ $\Psi \subseteq \Psi^+ \subseteq Sub(\Psi, \Phi)$; $\Phi \subseteq \Phi^+ \subseteq Sub(\Psi, \Phi)$; $\Psi^+ \Rightarrow\!\!\!\!\!\!\Rightarrow \Phi^+$ *is* $\rightarrow$*-saturated and* $\Box$*-left-saturated*; *and cut-free* $\mathbf{S}_{\mathrm{seq}} \nvdash \Psi^+ \Rightarrow\!\!\!\!\!\!\Rightarrow \Phi^+$.
- *A D-sequent* $\Psi^+ \Rrightarrow \Phi^+$ *is a* $\mathbf{D}^2_{\mathrm{seq}}$-saturation of $\Psi \Rrightarrow \Phi$ $\overset{def}{\Longleftrightarrow}$ $\Psi \subseteq \Psi^+ \subseteq Sub(\Psi, \Phi)$; $\Phi \subseteq \Phi^+ \subseteq Sub(\Psi, \Phi)$; $\Psi^+ \Rrightarrow \Phi^+$ *is* $\rightarrow$*-saturated*; $\Box\varphi \in \Psi^+ \cup \Phi^+$ *for any* $\Box\varphi \in Sub(\Psi, \Phi)$; *and analytic* $\mathbf{D}^2_{\mathrm{seq}} \nvdash \Psi^+ \Rrightarrow \Phi^+$.
- *A D-sequent* $\Psi^+ \Rrightarrow \Phi^+$ *is a* $\mathbf{D}^3_{\mathrm{seq}}$-saturation of $\Psi \Rrightarrow \Phi$ $\overset{def}{\Longleftrightarrow}$ $\Psi \subseteq \Psi^+ \subseteq Sub(\Psi, \Phi)$; $\Phi \subseteq \Phi^+ \subseteq Sub(\Psi, \Phi)$; $\Psi^+ \Rrightarrow \Phi^+$ *is* $\rightarrow$*-saturated*; *and cut-free* $\mathbf{D}^3_{\mathrm{seq}} \nvdash \Psi^+ \Rrightarrow \Phi^+$.

The following lemma is a standard tool for the semantical cut-eliminations.

LEMMA 5.3.

(1) *If cut-free* $\mathbf{GL}_{\mathrm{seq}} \nvdash \Psi \Rightarrow \Phi$, *then there is a* $\mathbf{GL}_{\mathrm{seq}}$*-saturation of it*.
(2) *If cut-free* $\mathbf{S}_{\mathrm{seq}} \nvdash \Psi \Rightarrow\!\!\!\!\!\!\Rightarrow \Phi$, *then there is an* $\mathbf{S}_{\mathrm{seq}}$*-saturation of it*.
(3) *If analytic* $\mathbf{D}^2_{\mathrm{seq}} \nvdash \Psi \Rrightarrow \Phi$, *then there is a* $\mathbf{D}^2_{\mathrm{seq}}$*-saturation of it*.
(4) *If cut-free* $\mathbf{D}^3_{\mathrm{seq}} \nvdash \Psi \Rrightarrow \Phi$, *then there is a* $\mathbf{D}^3_{\mathrm{seq}}$*-saturation of it*.

Proof. A proof of (1) is well-known. We add proper formulas to Ψ and Φ step by step (to be precise, if $\varphi \rightarrow \psi$ is in the left-hand side, we do either adding φ to the right or adding ψ to the left; if $\varphi \rightarrow \psi$ is in the right-hand side, we add φ to the left and ψ to the right) while preserving cut-free unprovability, and eventually we obtain an $\rightarrow$-saturated sequent. The same proof can be done for (4). For the proofs of (2) and (3), we combine the following step with the above procedure. (For 2): If $\Box\varphi$ is in the left-hand side, then we add φ there. (For 3): If $\Box\varphi \in \mathrm{Sub}(\Psi, \Phi)$, then we add $\Box\varphi$ to either the left-hand or right-hand side while preserving analytic unprovability. □

THEOREM 5.4 (Soundness and cut-free completeness of $\mathbf{GL}_{\mathrm{seq}}$). *For any GL-sequent* $\Psi \Rightarrow \Phi$, *the following three conditions are equivalent*:

(1) $\mathbf{GL}_{\mathrm{seq}} \vdash \Psi \Rightarrow \Phi$.
(2) *Cut-free* $\mathbf{GL}_{\mathrm{seq}} \vdash \Psi \Rightarrow \Phi$.
(3) $\bigwedge \Psi \rightarrow \bigvee \Phi$ *is true at any world of any GL-model*.

Theorem 5.4 is well-known; see, e.g.,[1] for the proof.

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Before showing the theorem for $\mathbf{S}_{\text{seq}}$, we state a proposition.

PROPOSITION 5.5. *If* $\mathbf{GL}_{\text{seq}} \vdash \Psi \Rrightarrow \Phi$, *then* $\bigwedge \Psi \to \bigvee \Phi$ *is eventually always true in the tail of any* λ*-extension of any GL-model, for any limit ordinal* λ.

This proposition is trivial because a λ-extension of a GL-model is also a GL-model. The reason for stating such a trivial proposition is as follows. We will generalize the results of this section in Section 7, where the generalized proposition is not trivial but considered as a hypothesis.

THEOREM 5.6 (Soundness and cut-free completeness of $\mathbf{S}_{\text{seq}}$). *Let* λ *be a limit ordinal. For any S-sequent* $\Psi \Rrightarrow \Phi$, *the following six conditions are equivalent.*

(1) $\mathbf{S}_{\text{seq}} \vdash \Psi \Rrightarrow \Phi$.
(2) *Cut-free* $\mathbf{S}_{\text{seq}} \vdash \Psi \Rrightarrow \Phi$.
(3) $\bigwedge \Psi \to \bigvee \Phi$ *is true at the bottom of any strongly constant* λ*-extension of any GL-model.*
(4) $\bigwedge \Psi \to \bigvee \Phi$ *is eventually always true in the tail of any strongly constant* λ*-extension of any GL-model.*
(5) $\bigwedge \Psi \to \bigvee \Phi$ *is eventually always true in the tail of any constant* λ*-extension of any GL-model.*
(6) $\bigwedge \Psi \to \bigvee \Phi$ *is eventually always true in the tail of any* λ*-extension of any GL-model.*

Proof. Note that the implications '(6) $\Rightarrow$ (5) $\Rightarrow$ (4)' and '(2) $\Rightarrow$ (1)' are trivial.

(Proof of (1) $\Rightarrow$ (6)) By induction on the $\mathbf{S}_{\text{seq}}$-proof of $\Psi \Rrightarrow \Phi$. When the last inference rule is (GLtoS), we use Proposition 5.5. When the last inference rule is (S$\Box$left), we use the fact that $\Box\varphi \to \varphi$ *is eventually always true in the tail* $\{t_\alpha \mid 0 < \alpha < \lambda\}$ *of any* λ*-extension of any Kripke model.* (Proof of this fact: If φ is true at all the worlds in the tail, then so is $\Box\varphi \to \varphi$. If φ is false at t_α for some α, then $\Box\varphi \to \varphi$ is true at t_β for any β such that $\alpha < \beta < \lambda$.)

(Proof of (4) $\Rightarrow$ (3)) Let $M = \langle W, \prec, V\rangle$ be a strongly constant λ-extension of a Kripke-model, t_λ be the bottom, and $\{t_\alpha \mid 0 < \alpha < \lambda\}$ be the tail of M. For any formula φ, we consider four conditions below. (λT): $V(t_\lambda, \varphi) = \mathsf{true}$. ($\lambda$F): $V(t_\lambda, \varphi) = \mathsf{false}$. (ET): $(\exists\alpha < \lambda)(\alpha < \forall\beta < \lambda)(V(t_\beta, \varphi) = \mathsf{true})$. (EF): $(\exists\alpha < \lambda)(\alpha < \forall\beta < \lambda)(V(t_\beta, \varphi) = \mathsf{false})$. By induction on φ, we can show two implications '(λT) $\Rightarrow$ (ET)' and '(λF) $\Rightarrow$ (EF)'. Therefore we have '(ET) $\Rightarrow$ (λT)' because the conditions (ET) and (EF) are exclusive, and the condition (λT) is the negation of (λF).

(Proof of (3) $\Rightarrow$ (2)) We show the contraposition. Suppose cut-free $\mathbf{S}_{\text{seq}} \nvdash \Psi \Rrightarrow \Phi$. We will construct a strongly constant λ-extension of a GL-model such that $\bigwedge \Psi \to \bigvee \Phi$ is false at the bottom. First we apply Lemma 5.3(2) to $\Psi \Rrightarrow \Phi$, and we get an $\mathbf{S}_{\text{seq}}$-saturation $\Psi^+ \Rrightarrow \Phi^+$. The GL-sequent $\Psi^+ \Rightarrow \Phi^+$ is not cut-free provable because of the (GLtoS)-rule, and this sequent is not cut-free provable also in $\mathbf{GL}_{\text{seq}}$. Then we get a GL-model $\langle W, \prec, V\rangle$ such that $\bigwedge \Psi^+ \to \bigvee \Phi^+$ is false at a world t_0, by the cut-free completeness of $\mathbf{GL}_{\text{seq}}$ (Th.5.4). Thus, we have the following: (A) If $p \in \Psi^+$, then $V(t_0, p) = \mathsf{true}$. (B) If $p \in \Phi^+$, then $V(t_0, p) = \mathsf{false}$. (C) If $\Box\psi \in \Psi^+$, then $V(x, \psi) = \mathsf{true}$ for any $x \in W$ such that $t_0 \prec x$. (D) If $\Box\psi \in \Phi^+$, then $V(x, \psi) = \mathsf{false}$ for some $x \in W$ such that $t_0 \prec x$. Then we define a strongly constant λ-extension $\langle W^+, \prec^+, V^+\rangle$ of $\langle W, \prec, V\rangle$ where $\{t_\alpha \mid 0 < \alpha < \lambda\}$ is the tail, t_λ is the bottom, and $V^+(t_\alpha, p) = V(t_0, p)$ for any $\alpha \leq \lambda$ and p. (In Figure 2 (left), the essence of $\langle W^+, \prec^+, V^+\rangle$ is illustrated: each world is a sequent, and the goal is to show that each formula in the

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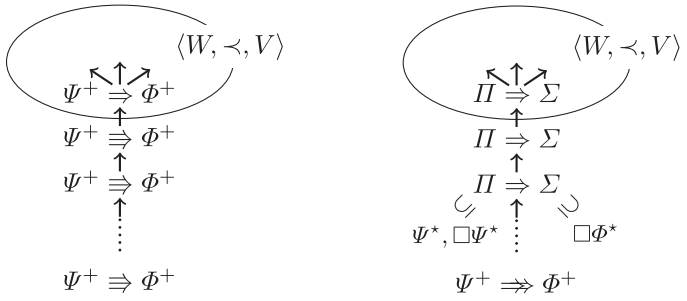


Figure 2. $\langle W^+, \prec^+, V^+\rangle$ in Theorem 5.6 (left) and in Theorem 5.7 (right).

left (or right) hand side is true (or false, resp.) there.) Now, for any formula φ and any ordinal number $\alpha \leq \lambda$, we prove the following two properties by induction on φ. (L) *If* $\varphi \in \Psi^+$, *then* $V^+(t_\alpha, \varphi) = \mathsf{true}$. (R) *If* $\varphi \in \Phi^+$, *then* $V^+(t_\alpha, \varphi) = \mathsf{false}$. In the case of $\varphi = p$, we use the facts (A) and (B). In the case of $\varphi = \Box\psi$, we use $\Box$-left-saturatedness, the induction hypothesis, and the fact (C) to show the property (L); and we use the fact (D) to show the property (R). Thus we have $V^+(t_\lambda, \bigwedge \Psi \to \bigvee \Phi) = \mathsf{false}$. □

THEOREM 5.7 (Soundness and analytic completeness of $\mathbf{D}^2_{\mathrm{seq}}$). *Let* λ *be a limit ordinal. For any D-sequent* $\Psi \Rrightarrow \Phi$, *the following four conditions are equivalent*:

(1) $\mathbf{D}^2_{\mathrm{seq}} \vdash \Psi \Rrightarrow \Phi$.
(2) *Analytic* $\mathbf{D}^2_{\mathrm{seq}} \vdash \Psi \Rrightarrow \Phi$.
(3) $\bigwedge \Psi \to \bigvee \Phi$ *is true at the bottom of any constant* λ*-extension of any GL-model.*
(4) $\bigwedge \Psi \to \bigvee \Phi$ *is true at the bottom of any* λ*-extension of any GL-model.*

Proof. The implications '(2) ⇒ (1)' and '(4) ⇒ (3)' are trivial.

(Proof of (1) ⇒ (4)) Let $M^+ = \langle W^+, \prec^+, V^+\rangle$ be a λ-extension of a GL-model. We prove, by induction on the $\mathbf{D}^2_{\mathrm{seq}}$-proof of $\Psi \Rrightarrow \Phi$, that $\bigwedge \Psi \to \bigvee \Phi$ is true at the bottom t_λ of M^+. Here we show the case that $(\Psi \Rrightarrow \Phi) = (\Box\Psi' \Rrightarrow \Box\Phi')$ and the last inference rule is (GLtoD); that is, we have the hypothesis (†) $\mathbf{GL}_{\mathrm{seq}} \vdash \Psi', \Box\Psi' \Rightarrow \Box\Phi'$. We assume $\bigwedge \Box\Psi'$ is true at t_λ; then, the goal is to show $\bigvee \Box\Phi'$ is true at t_λ. By the assumption and the definition of $\prec^+$, we have both $\bigwedge \Psi'$ and $\bigwedge \Box\Psi'$ are true at any world in the tail. Then the hypothesis (†) and Proposition 5.5 imply that $\bigvee \Box\Phi'$ is eventually always true in the tail $\{t_\alpha \mid 0 < \alpha < \lambda\}$. Therefore, there must be a formula $\Box\varphi \in \Box\Phi'$ such that the condition (‡) $(\forall\alpha < \lambda)(\alpha < \exists\beta < \lambda)(V^+(t_\beta, \Box\varphi) = \mathsf{true})$ holds because otherwise every $\Box\varphi \in \Box\Phi'$ satisfies $(\exists\alpha < \lambda)(\alpha < \forall\beta < \lambda)(V^+(t_\beta, \Box\varphi) = \mathsf{false})$, and this contradicts the fact that $\bigvee \Box\Phi'$ is eventually always true. This formula $\Box\varphi$ is also true at the bottom t_λ because the condition (‡) implies $(\forall\alpha < \lambda)[(V^+(t_\alpha, \varphi) = \mathsf{true})$ and $(\forall x \succ^+ t_\alpha)(V^+(x, \varphi) = \mathsf{true})]$. Thus, we have $V^+(t_\lambda, \bigvee \Box\Phi') = \mathsf{true}$.

(Proof of (3) ⇒ (2)) We show the contraposition. Suppose analytic $\mathbf{D}^2_{\mathrm{seq}} \nvdash \Psi \Rrightarrow \Phi$. We will construct a constant λ-extension of a GL-model such that $\bigwedge \Psi \to \bigvee \Phi$ is false at the bottom. First we apply Lemma 5.3(3) to $\Psi \Rrightarrow \Phi$, and we get a $\mathbf{D}^2_{\mathrm{seq}}$-saturation $\Psi^+ \Rrightarrow \Phi^+$. Let $\Psi^\star = \{\psi \mid \Box\psi \in \Psi^+\}$ and $\Phi^\star = \{\varphi \mid \Box\varphi \in \Phi^+\}$. The GL-sequent $\Psi^\star, \Box\Psi^\star \Rightarrow \Box\Phi^\star$ is not cut-free provable in $\mathbf{GL}_{\mathrm{seq}}$ because otherwise analytic $\mathbf{D}^2_{\mathrm{seq}} \vdash \Psi^+ \Rrightarrow \Phi^+$ by the rules (GLtoD) and weakening. Then applying Lemma 5.3(1),

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we get a $\mathbf{GL}_{\mathrm{seq}}$-saturation $\Pi \Rightarrow \Sigma$ of $\Psi^\star, \Box\Psi^\star \Rightarrow \Box\Phi^\star$. We show that $\Pi \Rightarrow \Sigma$ is $\Box$-left-saturated:

> Suppose $\Box\pi \in \Pi$. Then $\Box\pi \in \mathrm{Sub}(\Psi, \Phi)$ (because $\Pi \Rightarrow \Sigma$ is a $\mathbf{GL}_{\mathrm{seq}}$-saturation of $\Psi^\star, \Box\Psi^\star \Rightarrow \Box\Phi^\star$ which consists of the elements of $\mathrm{Sub}(\Psi, \Phi)$), and we have either $\Box\pi \in \Psi^+$ or $\Box\pi \in \Phi^+$ (because $\Psi^+ \Rrightarrow \Phi^+$ is a $\mathbf{D}^2_{\mathrm{seq}}$-saturation of $\Psi \Rrightarrow \Phi$). But the latter condition $\Box\pi \in \Phi^+$ does not hold because, if $\Box\pi \in \Phi^+$, then $\Box\pi \in \Box\Phi^\star$ and $\Pi \Rightarrow \Sigma$ is cut-free provable from the initial sequent $\Box\pi \Rightarrow \Box\pi$ by the weakening rule in $\mathbf{GL}_{\mathrm{seq}}$. Therefore the former condition $\Box\pi \in \Psi^+$ holds, and we have $\pi \in \Psi^\star \subseteq \Pi$.

The remaining part is similar to the proof of '(3) $\Rightarrow$ (2)' of Theorem 5.6. We get a GL-model $\langle W, \prec, V\rangle$ such that $\bigwedge \Pi \to \bigvee \Sigma$ is false at a world t_0, by the cut-free completeness of $\mathbf{GL}_{\mathrm{seq}}$. Then we define a constant λ-extension of $\langle W^+, \prec^+, V^+\rangle$ of $\langle W, \prec, V\rangle$ where $\{t_\alpha \mid 0 < \alpha < \lambda\}$ is the tail, t_λ is the bottom, $V^+(t_\alpha, p) = V(t_0, p)$ for any $\alpha < \lambda$, and $V^+(t_\lambda, p) = \mathsf{true} \iff p \in \Psi^+$. (In Figure 2 (right), the essence of $\langle W^+, \prec^+, V^+\rangle$ is illustrated.) We consider four properties below, where $\alpha < \lambda$. (L) *If* $\varphi \in \Pi$, *then* $V^+(t_\alpha, \varphi) = \mathsf{true}$. (R) *If* $\varphi \in \Sigma$, *then* $V^+(t_\alpha, \varphi) = \mathsf{false}$. (L$\lambda$) *If* $\varphi \in \Psi^+$, *then* $V^+(t_\lambda, \varphi) = \mathsf{true}$. (R$\lambda$) *If* $\varphi \in \Phi^+$, *then* $V^+(t_\lambda, \varphi) = \mathsf{false}$. The properties (L) and (R) are proved in the same way as that in Theorem 5.6 for any ordinal number $\alpha < \lambda$. Note that $\Pi \Rightarrow \Sigma$ is $\Box$-left-saturated as shown above (while $\Psi^+ \Rrightarrow \Phi^+$ may not). Finally, we prove the properties (Lλ) and (Rλ) by induction on φ. In the case of $\varphi = \Box\psi \in \Psi^+$, for example, the proof is done as follows.

$$\Box\psi \in \Psi^+ \overset{\text{def. of } \Pi}{\Longrightarrow} \psi \in \Pi \overset{\text{(L)}}{\Longrightarrow} V^+(t_\alpha, \psi) = \mathsf{true} \text{ for any } \alpha < \lambda. \tag{$\dagger$}$$

$$\Box\psi \in \Psi^+ \overset{\text{def. of } \Pi}{\Longrightarrow} \Box\psi \in \Pi \overset{\text{(L)}}{\Longrightarrow} V^+(t_0, \Box\psi) = \mathsf{true} \Longrightarrow V^+(x, \psi) = \mathsf{true} \text{ for any } x \in W \text{ such that } t_0 \prec x. \tag{$\ddagger$}$$

$$\Box\psi \in \Psi^+ \overset{(\dagger)(\ddagger)}{\Longrightarrow} V^+(t_\lambda, \Box\psi) = \mathsf{true}.$$

□

THEOREM 5.8 (Soundness and cut-free completeness of $\mathbf{D}^3_{\mathrm{seq}}$). *Let λ be a limit ordinal. For any D-sequent $\Psi \Rrightarrow \Phi$, the following four conditions are equivalent*:

(1) $\mathbf{D}^3_{\mathrm{seq}} \vdash \Psi \Rrightarrow \Phi$.
(2) *Cut-free* $\mathbf{D}^3_{\mathrm{seq}} \vdash \Psi \Rrightarrow \Phi$.
(3) $\bigwedge \Psi \to \bigvee \Phi$ *is true at the bottom of any constant λ-extension of any GL-model.*
(4) $\bigwedge \Psi \to \bigvee \Phi$ *is true at the bottom of any λ-extension of any GL-model.*

Proof. Proof can be done in a similar way to Theorem 5.7. Points of differences between two proofs are below. In the proof of '(1) $\Rightarrow$ (4)', we use '(1) $\Rightarrow$ (6)' of Theorem 5.6 instead of Proposition 5.5. In the proof of '(3) $\Rightarrow$ (2)', we use a $\mathbf{D}^3_{\mathrm{seq}}$-saturation $\Psi^+ \Rrightarrow \Phi^+$ of $\Psi \Rrightarrow \Phi$ (by Lemma 5.3(4)) to define the valuation V^+ at the bottom; and we use an $\mathbf{S}_{\mathrm{seq}}$-saturation $\Pi \Rrightarrow \Sigma$ of $\Box\Psi^\star \Rrightarrow \Box\Phi^\star$ (and the GL-sequent $\Pi \Rightarrow \Sigma$ which is not cut-free provable in $\mathbf{GL}_{\mathrm{seq}}$) to define the valuation V^+ at the tail. □

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§6. Hilbert-style systems for D. Corresponding to $\mathbf{D}^2_{\mathrm{seq}}$ and $\mathbf{D}^3_{\mathrm{seq}}$, we introduce new Hilbert-style proof systems, called $\mathbf{D}^2_{\mathrm{H}}$ and $\mathbf{D}^3_{\mathrm{H}}$:

Axioms of $\mathbf{D}^2_{\mathrm{H}}$: Tautologies, and formulas of the form $\bigwedge \Box\Gamma \to \bigvee \Box\Delta$ such that $\bigwedge(\Gamma, \Box\Gamma) \to \bigvee \Box\Delta$ is a theorem of $\mathbf{GL}_{\mathrm{H}}$.
Rule of $\mathbf{D}^2_{\mathrm{H}}$: Modus ponens.
Axioms of $\mathbf{D}^3_{\mathrm{H}}$: Tautologies, and formulas of the form $\bigwedge \Box\Gamma \to \bigvee \Box\Delta$ which is a theorem of $\mathbf{S}_{\mathrm{H}}$.
Rule of $\mathbf{D}^3_{\mathrm{H}}$: Modus ponens.

Similarly to Propositions 3.4 and 3.5, we have the following.

PROPOSITION 6.1. $\mathbf{D}^2_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$ *if and only if* $\mathbf{D}^2_{\mathrm{H}} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

PROPOSITION 6.2. $\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \Rightarrow\!\!\!\!\Rightarrow \Delta$ *if and only if* $\mathbf{D}^3_{\mathrm{H}} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

These are proved by induction. To be exact, for the if-parts, we show '$\mathbf{D}^i_{\mathrm{seq}} \vdash (\Rightarrow\!\!\!\!\Rightarrow \varphi)$ if $\mathbf{D}^i_{\mathrm{H}} \vdash \varphi$' by induction on the $\mathbf{D}^i_{\mathrm{H}}$-proof of φ. Note that Propositions 3.4 and 3.5 and Theorems 4.1, 4.2, and 4.3 are used in the proofs.

Now the logic **D** has three Hilbert-style systems $\mathbf{D}_{\mathrm{H}}$, $\mathbf{D}^2_{\mathrm{H}}$ and $\mathbf{D}^3_{\mathrm{H}}$; and the equivalence of these is shown by combining Proposition 2.7 (for $\lambda = \omega$), Theorems 5.7 and 5.8 (for $\lambda = \omega$), and Propositions 6.1 and 6.2. But it is a natural question whether we can show this equivalence syntactically (since the above combination uses soundness and completeness). Syntactic proof of equivalence between $\mathbf{D}^2_{\mathrm{H}}$ and $\mathbf{D}^3_{\mathrm{H}}$ can be obtained by Theorem 4.5 and Propositions 6.1 and 6.2. In the following, we show a syntactic proof of equivalence between $\mathbf{D}^2_{\mathrm{H}}$ and $\mathbf{D}_{\mathrm{H}}$; consequently the equivalence between the three systems is shown syntactically.

THEOREM 6.3. $\mathbf{D}^2_{\mathrm{H}} \vdash \varphi$ *if and only if* $\mathbf{D}_{\mathrm{H}} \vdash \varphi$.

Proof. (Proof of only-if-part) We show the following by induction on $\mathbf{D}^2_{\mathrm{H}}$-proof of φ: *If* $\mathbf{D}^2_{\mathrm{H}} \vdash \varphi$, *then* $\mathbf{D}_{\mathrm{H}} \vdash \varphi$. If φ is the axiom $\bigwedge \Box\Gamma \to \bigvee \Box\Delta$, an outline of the proof is as follows.

(1) $\mathbf{GL}_{\mathrm{H}} \vdash \bigwedge(\Gamma, \Box\Gamma) \to \bigvee \Box\Delta$ (By definition of the axiom)

(2) $\mathbf{GL}_{\mathrm{H}} \vdash \bigwedge(\Box\Gamma, \Box\Box\Gamma) \to \Box \bigvee \Box\Delta$ (By (1) and inference in $\mathbf{GL}_{\mathrm{H}}$)

(3) $\mathbf{GL}_{\mathrm{H}} \vdash \bigwedge \Box\Gamma \to \Box \bigvee \Box\Delta$ (By (2) and the fact $\mathbf{GL}_{\mathrm{H}} \vdash \Box\gamma \to \Box\Box\gamma$)

(4) $\mathbf{D}_{\mathrm{H}} \vdash \Box \bigvee \Box\Delta \to \bigvee \Box\Delta$ (Lemma 2.7.1 in [3])

(5) $\mathbf{D}_{\mathrm{H}} \vdash \bigwedge \Box\Gamma \to \bigvee \Box\Delta$. (By (3), (4), and definition of the axioms of $\mathbf{D}_{\mathrm{H}}$.)

(Proof of if-part) First we show the following claim by induction on the $\mathbf{GL}_{\mathrm{H}}$-proof of φ: *If* $\mathbf{GL}_{\mathrm{H}} \vdash \varphi$, *then* $\mathbf{D}^2_{\mathrm{H}} \vdash \varphi$. A point is that the axioms $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ and the conclusion of $(\Box)$-rule are respectively equivalent (over classical propositional logic) to formulas of the form $\bigwedge \Box\Gamma \to \bigvee \Box\Delta$ such that $\bigwedge(\Gamma, \Box\Gamma) \to \bigvee \Box\Delta$ is a theorem of $\mathbf{GL}_{\mathrm{H}}$. For example, $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is equivalent to $\bigwedge \Box(\varphi \to \psi, \varphi) \to \Box\psi$, where $\bigwedge(\varphi \to \psi, \varphi, \Box(\varphi \to \psi, \varphi)) \to \Box\psi$ is a theorem of $\mathbf{GL}_{\mathrm{H}}$. Then we show the following claim by induction on the $\mathbf{D}_{\mathrm{H}}$-proof of φ: *If*

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$\mathbf{D}_{\mathrm{H}} \vdash \varphi$, *then* $\mathbf{D}^2_{\mathrm{H}} \vdash \varphi$. For the axioms $\neg\Box\bot$ and $\Box(\Box\varphi\vee\Box\psi)\to\Box\varphi\vee\Box\psi$, proofs are done corresponding to the $\mathbf{D}^2_{\mathrm{seq}}$-proofs in Examples 3.3 and 3.1. □

§7. Generalization. In this section, we generalize the completeness of $\mathbf{S}_{\mathrm{H}}$, $\mathbf{D}^2_{\mathrm{H}}$, and $\mathbf{D}^3_{\mathrm{H}}$.

In the following, L denotes an arbitrary set of formulas. We define five sets of formulas depending on L.

DEFINITION 7.1 ($\mathrm{MP}[\cdot], \mathbf{S}[\cdot], \mathbf{D}[\cdot], \mathbf{D}^2[\cdot], \mathbf{D}^3[\cdot]$).

$\mathrm{MP}[L]$ *is the smallest set of formulas that contains* L *and is closed under modus ponens*;
$\mathbf{S}[L] = \mathrm{MP}[L \cup \{\Box\varphi\to\varphi \mid \varphi \in \mathbf{Form}\}]$;
$\mathbf{D}[L] = \mathrm{MP}[L \cup \{\neg\Box\bot\} \cup \{\Box(\Box\varphi\vee\Box\psi)\to\Box\varphi\vee\Box\psi \mid \varphi, \psi \in \mathbf{Form}\}]$;
$\mathbf{D}^2[L] = \mathrm{MP}[\mathbf{Taut} \cup \{\bigwedge\Box\Gamma\to\bigvee\Box\Delta \mid \bigwedge(\Gamma,\Box\Gamma)\to\bigvee\Box\Delta \in L\}]$;
and
$\mathbf{D}^3[L] = \mathrm{MP}[\mathbf{Taut} \cup \{\bigwedge\Box\Gamma\to\bigvee\Box\Delta \mid \bigwedge\Box\Gamma\to\bigvee\Box\Delta \in \mathbf{S}[L]\}]$;

where **Taut** *is the set of tautologies.*

For example, $\mathbf{S}[\mathbf{GL}]$, $\mathbf{D}[\mathbf{GL}]$, $\mathbf{D}^2[\mathbf{GL}]$, and $\mathbf{D}^3[\mathbf{GL}]$ are the sets of theorems of $\mathbf{S}_{\mathrm{H}}$, $\mathbf{D}_{\mathrm{H}}$, $\mathbf{D}^2_{\mathrm{H}}$, and $\mathbf{D}^3_{\mathrm{H}}$, respectively, if $\mathbf{GL}$ is the set of theorems of $\mathbf{GL}_{\mathrm{H}}$.

DEFINITION 7.2. *Let* $\mathcal{F}$ *be a class of Kripke frames and* L *be a set of formulas. We say that a Kripke model is an* $\mathcal{F}$-model *if it is based on a frame in* $\mathcal{F}$. *We say that* L *is* characterized by $\mathcal{F}$ *if the following two conditions are equivalent for any formula* φ.

- $\varphi \in L$.
- φ *is true at any world of any* $\mathcal{F}$*-model.*

Moreover, we say that L *is* λ-tail-sound for $\mathcal{F}$ *if any formula in* L *is eventually always true in the tail of any* λ*-extension of any* $\mathcal{F}$*-model, where* λ *is a limit ordinal.*

For example, $\mathbf{GL}$ is characterized by and λ-tail-sound for the class of GL-frames.
The following are the generalization of the completeness of $\mathbf{S}_{\mathrm{H}}$, $\mathbf{D}^2_{\mathrm{H}}$, and $\mathbf{D}^3_{\mathrm{H}}$.

THEOREM 7.3. *Let* λ *be a limit ordinal. If* L *is characterized by* $\mathcal{F}$ *and* λ*-tail-sound for* $\mathcal{F}$, *then the following five conditions are equivalent*:

(1) $\varphi \in \mathbf{S}[L]$.
(2) φ *is true at the bottom of any strongly constant* λ*-extension of any* $\mathcal{F}$*-model.*
(3) φ *is eventually always true in the tail of any strongly constant* λ*-extension of any* $\mathcal{F}$*-model.*
(4) φ *is eventually always true in the tail of any constant* λ*-extension of any* $\mathcal{F}$*-model.*
(5) φ *is eventually always true in the tail of any* λ*-extension of any* $\mathcal{F}$*-model.*

THEOREM 7.4. *Let* λ *be a limit ordinal. If* L *is characterized by* $\mathcal{F}$ *and* λ*-tail-sound for* $\mathcal{F}$, *then the following four conditions are equivalent*:

(1) $\varphi \in \mathbf{D}^2[L]$.
(2) $\varphi \in \mathbf{D}^3[L]$.
(3) φ *is true at the bottom of any constant* λ*-extension of any* $\mathcal{F}$*-model.*
(4) φ *is true at the bottom of any* λ*-extension of any* $\mathcal{F}$*-model.*

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Table 2. *Rewriting for generalization*

Terms in Section 5	Replaced with
GL-model	$\mathcal{F}$-model
(cut-free) $\mathbf{GL}_{\mathrm{seq}} \vdash \Gamma \blacktriangleright \Delta$	$\bigwedge \Gamma \to \bigvee \Delta \in L$
(cut-free) $\mathbf{S}_{\mathrm{seq}} \vdash \Gamma \blacktriangleright \Delta$	$\bigwedge \Gamma \to \bigvee \Delta \in \mathbf{S}[L]$
(analytic) $\mathbf{D}^2_{\mathrm{seq}} \vdash \Gamma \blacktriangleright \Delta$	$\bigwedge \Gamma \to \bigvee \Delta \in \mathbf{D}^2[L]$
(cut-free) $\mathbf{D}^3_{\mathrm{seq}} \vdash \Gamma \blacktriangleright \Delta$	$\bigwedge \Gamma \to \bigvee \Delta \in \mathbf{D}^3[L]$
Theorem 5.4, Proposition 5.5	assumption of the theorem

Proof of Theorems 7.3 and 7.4. Suppose that L is characterized by $\mathcal{F}$. If X is L, $\mathbf{S}[L]$, $\mathbf{D}^2[L]$, or $\mathbf{D}^3[L]$, then X contains all tautologies, and X is closed under tautological consequence; therefore, X can simulate **LK** as below.

- (Initial sequents) $\varphi \to \varphi \in X$.
- (Inference rules) X is closed under the following rules:

$$\frac{\bigwedge \Gamma \to \bigvee \Delta}{\bigwedge \Gamma' \to \bigvee \Delta'} \text{ (weakening), where } \Gamma \subseteq \Gamma' \text{ and } \Delta \subseteq \Delta',$$

$$\frac{\bigwedge \Gamma \to \bigvee (\Delta, \varphi) \quad \bigwedge (\psi, \Gamma) \to \bigvee \Delta}{\bigwedge (\varphi \to \psi, \Gamma) \to \bigvee \Delta} (\to\text{left}), \quad \frac{\bigwedge (\varphi, \Gamma) \to \bigvee (\Delta, \psi)}{\bigwedge \Gamma \to \bigvee (\Delta, \varphi \to \psi)} (\to\text{right}),$$

$$\frac{\bigwedge \Gamma \to \bigvee (\Delta, \varphi) \quad \bigwedge (\varphi, \Pi) \to \bigvee \Sigma}{\bigwedge (\Gamma, \Pi) \to \bigvee (\Delta, \Sigma)} \text{ (cut)}.$$

Using these, we generalize the arguments of Section 5. Roughly speaking, this is done by replacing terms according to Table 2. In the following, we show a detailed proof of Theorem 7.3.

First we redefine '$\mathbf{S}_{\mathrm{seq}}$-saturation':

> An S-sequent $\Psi^+ \Rrightarrow \Phi^+$ is an $\mathbf{S}_{\mathrm{seq}}$-*saturation of* $\Psi \Rrightarrow \Phi$ $\overset{\text{def}}{\iff}$ $\Psi \subseteq \Psi^+ \subseteq Sub(\Psi, \Phi)$; $\Phi \subseteq \Phi^+ \subseteq Sub(\Psi, \Phi)$; $\Psi^+ \Rrightarrow \Phi^+$ is $\to$-saturated and $\Box$-left-saturated; and $\bigwedge \Psi^+ \to \bigvee \Phi^+ \notin \mathbf{S}[L]$.

Then we show the generalization of Lemma 5.3(2):

> *If* $\bigwedge \Psi \to \bigvee \Phi \notin \mathbf{S}[L]$, *then there is an* $\mathbf{S}_{\mathrm{seq}}$-*saturation of* $\Psi \Rrightarrow \Phi$.

This is shown by simulating the proof of Lemma 5.3(2). Note that the rule (S$\Box$left) is available in $\mathbf{S}[L]$ as below

$$\frac{\bigwedge (\varphi, \Gamma) \to \bigvee \Delta}{\bigwedge (\Box\varphi, \Gamma) \to \bigvee \Delta} \text{ (S}\Box\text{left)}$$

because $\Box\varphi \to \varphi \in \mathbf{S}[L]$. Then we prove the generalization of Theorem 5.6.

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The following five conditions are equivalent:

(1) $\bigwedge \Psi \to \bigvee \Phi \in \mathbf{S}[L]$.
(2) $\bigwedge \Psi \to \bigvee \Phi$ is true at the bottom of any strongly constant λ-extension of any $\mathcal{F}$-model.
(3) $\bigwedge \Psi \to \bigvee \Phi$ is eventually always true in the tail of any strongly constant λ-extension of any $\mathcal{F}$-model.
(4) $\bigwedge \Psi \to \bigvee \Phi$ is eventually always true in the tail of any constant λ-extension of any $\mathcal{F}$-model.
(5) $\bigwedge \Psi \to \bigvee \Phi$ is eventually always true in the tail of any λ-extension of any $\mathcal{F}$-model.

Note that $\varphi \in \mathbf{S}[L]$ if and only if $\bigwedge \emptyset \to \bigvee \{\varphi\} \in \mathbf{S}[L]$; therefore, showing the above equivalence is sufficient for Theorem 7.3.

(Proof of (1) $\Rightarrow$ (5)) We show that each element of $\mathbf{S}[L]$ is eventually always true in the tail of any λ-extension of any $\mathcal{F}$-model. This is easy because of the definition of $\mathbf{S}[L]$, the assumption that L is λ-tail-sound for $\mathcal{F}$, and the fact that $\Box\varphi \to \varphi$ is eventually always true (this was shown in the proof of '(1) $\Rightarrow$ (6)' of Theorem 5.6).

(Proof of (5) $\Rightarrow$ (4) $\Rightarrow$ (3)) Trivial.

(Proof of (3) $\Rightarrow$ (2)) Shown in the proof of '(4) $\Rightarrow$ (3)' of Theorem 5.6.

(Proof of (2) $\Rightarrow$ (1)) We generalize the proof of '(3) $\Rightarrow$ (2)' of Theorem 5.6. Suppose $\bigwedge \Psi \to \bigvee \Phi \notin \mathbf{S}[L]$. By the 'generalized Lemma 5.3(2)', we get an $\mathbf{S}_{\mathrm{seq}}$-saturation $\Psi^+ \Rrightarrow \Phi^+$ of $\Psi \Rrightarrow \Phi$. By the definition of $\mathbf{S}_{\mathrm{seq}}$-saturation, we have $\bigwedge \Psi^+ \to \bigvee \Phi^+ \notin \mathbf{S}[L]$; therefore $\bigwedge \Psi^+ \to \bigvee \Phi^+ \notin L$ because of the definition of $\mathbf{S}[L]$. Hence, there is an $\mathcal{F}$-model $\langle W, \prec, V\rangle$ such that $\bigwedge \Psi^+ \to \bigvee \Phi^+$ is false at a world t_0 because of the assumption that L is characterized by $\mathcal{F}$. The rest of the proof is the same as in the proof of '(3) $\Rightarrow$ (2)' of Theorem 5.6.

This completes the proof of Theorem 7.3. Similarly, Theorem 7.4 can be proved by generalizing the proofs of Theorems 5.7 and 5.8. □

REMARK 7.5. *In the proof of Theorem* 6.3, *we use the following particular properties of* **GL**: $\Box\varphi \to \Box\Box\varphi \in \mathbf{GL}$, *and* $\mathbf{GL} \subseteq \mathbf{D}^2[\mathbf{GL}]$ (*due to the property of the axioms of* **GL** *specified in the proof of Theorem* 6.3). *Therefore, in general, the condition '*$\varphi \in \mathbf{D}[L]$*' seems nonequivalent to the conditions* (1)–(4) *of Theorem* 7.4.

In the rest of this section, we show an application of Theorem 7.4 to the extensions of **GL** and **D** with linear order models.

Let λ be a limit ordinal, and M be $\langle W, \prec, V\rangle$. We define 'finite linear GL-model', 'linear GL-model', and 'λ-GL-model' as below.

M is a *finite linear GL-model* $\overset{\text{def}}{\Longleftrightarrow} \langle W, \prec\rangle \simeq \langle \{\alpha \mid \alpha \leq n\}, >\rangle$ for some natural number n. (We call $\langle \{\alpha \mid \alpha \leq n\}, >\rangle$ a *finite linear GL-frame*.)

M is a *linear GL-model* $\overset{\text{def}}{\Longleftrightarrow} \langle W, \prec\rangle \simeq \langle \{\alpha \mid \alpha < \beta\}, >\rangle$ for some ordinal number $\beta > 0$.

M is a *λ-GL-model* $\overset{\text{def}}{\Longleftrightarrow} \langle W, \prec\rangle \simeq \langle \{\alpha \mid \alpha \leq \lambda\}, >\rangle$.

Then we define logics $\mathbf{LinGL}^{\mathrm{F}}$, **LinGL**, $\mathbf{LinD}^{\lambda}$, and **LinD** as below.

$\mathbf{LinGL}^{\mathrm{F}} = \{\varphi \mid \varphi$ is true at any world of any finite linear GL-model.$\}$
$\mathbf{LinGL} = \{\varphi \mid \varphi$ is true at any world of any linear GL-model.$\}$

https://doi.org/10.1017/S1755020325000036 Published online by Cambridge University Press

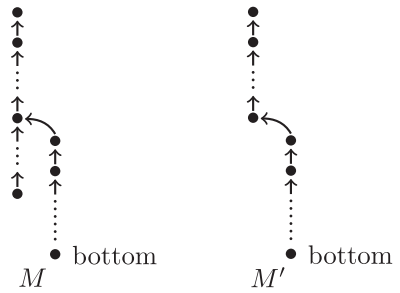


Figure 3. A λ-extension of a finite linear GL-model and its submodel.

$$\mathbf{LinD}^{\lambda} = \{\varphi \mid \varphi \text{ is true at the bottom of any } \lambda\text{-GL-model.}\}$$
$$\mathbf{LinD} = \{\varphi \mid \varphi \text{ is true at the bottom of any } \lambda'\text{-GL-model, for any limit ordinal } \lambda'.\}$$

THEOREM 7.6. $\mathbf{LinGL}^{\mathrm{F}} = \mathbf{LinGL}$. (*Therefore*, $\mathbf{LinGL}$ *is characterized by the class of finite linear GL-frames.*)

Proof. $\mathbf{LinGL}^{\mathrm{F}}$ is known to be axiomatized by adding an axiom scheme $\Box(\Box\varphi\to\psi)\vee\Box(\psi\wedge\Box\psi\to\varphi)$ to $\mathbf{GL}_{\mathrm{H}}$ ([7, sec. 25]). We call this system $\mathbf{LinGL}_{\mathrm{H}}$. The soundness of $\mathbf{LinGL}_{\mathrm{H}}$ with respect to $\mathbf{LinGL}$ is easily shown. Therefore $\mathbf{LinGL}^{\mathrm{F}} \subseteq \mathbf{LinGL}$ is shown as below: $\varphi \in \mathbf{LinGL}^{\mathrm{F}} \Longrightarrow \mathbf{LinGL}_{\mathrm{H}} \vdash \varphi \Longrightarrow \varphi \in \mathbf{LinGL}$. The converse inclusion $\mathbf{LinGL}^{\mathrm{F}} \supseteq \mathbf{LinGL}$ is trivial by the definition. □

THEOREM 7.7. *Let λ be a limit ordinal. The following conditions are equivalent*:

(1) $\varphi \in \mathbf{D}^2[\mathbf{LinGL}]$.
(2) $\varphi \in \mathbf{D}^3[\mathbf{LinGL}]$.
(3) *φ is true at the bottom of any λ-extension of any finite linear GL-model.*
(4) *φ is true at the bottom of any λ'-extension of any finite linear GL-model, for any limit ordinal λ'.*
(5) $\varphi \in \mathbf{LinD}^{\lambda}$.
(6) $\varphi \in \mathbf{LinD}$.

Proof. If M is a λ-extension of a finite linear GL-model, and if M' is a submodel of M as in Figure 3, then M' is a λ-GL-model and the truth value of each formula at each world is inherited from M. Using this, we can show the following three facts. (i) $\mathbf{LinGL}$ is λ-tail-sound for the class of finite linear GL-frames. (ii) The conditions (3) and (5) are equivalent. (iii) The conditions (4) and (6) are equivalent. Then, the fact (i) and Theorems 7.6 and 7.4 imply the equivalence between the conditions (1), (2), and (3). Finally, '(3) $\Longleftrightarrow$ (4)' is shown similarly to Remark 2.6. □

REMARK 7.8. *Also the condition '$\varphi \in \mathbf{D}[\mathbf{LinGL}]$' is equivalent to the conditions in Theorem 7.7 because we can show* $\mathbf{D}^2[\mathbf{LinGL}] = \mathbf{D}[\mathbf{LinGL}]$ *in the same way as Theorem 6.3, using the Hilbert-style system* $\mathbf{LinGL}_{\mathrm{H}}$ (*mentioned in the proof of Theorem 7.6*). *A cut-free sequent calculus for* $\mathbf{LinGL}$ *was discussed in* [13].

Acknowledgments. The authors would like to thank the referees for their valuable comments.

Funding. This work was supported by JSPS KAKENHI Grant Numbers JP20K03712 and JP23K03200.

https://doi.org/10.1017/S1755020325000036 Published online by Cambridge University Press

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