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Collisionless magnetic reconnection dynamics with electron inertia and parallel electron compressibility

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Abstract. Collisionless magnetic reconnection dynamics is considered by including the effects of electron inertia as well as parallel electron compressibility. A fluid treatment is adopted for both electrons and ions. Collisionless plasma dynamics properties near a two-dimensional X-type magnetic neutral line in the steady state are explored. The effects of electron inertia and parallel electron compressibility on the hyperbolicity (or lack thereof) of the magnetic field lines in the neutral layer are discussed. A unified linear tearing-mode formulation incorporating both electron inertia and parallel electron compressibility is given. The parallel-electron-compressibility branch is shown to couple in general to the electron-inertia branch in the presence of resistivity. A sufficient condition for linear stability in the Lyapunov sense for steady states of this collisionless plasma system signifying current confinement is deduced. Bounds on the equilibrium current gradient are shown to constitute sufficient conditions for nonlinear stability in the Lyapunov sense for steady states via nonlinear bounds for a suitable perturbation norm.

1. Introduction

The high-temperature plasmas in space and fusion systems have been found to be collisionless, so considerable work has gone into collisionless reconnection processes (Coppi et al. 1964; Schindler 1974; Drake and Lee 1977; Galeev 1978; Wesson 1990; Drake and Kleva 1991; Porcelli 1991; Al-Salti and Shivamoggi 2003; Ottaviani and Porcelli 1993; Drake et al. 1994; Drake 1995; Shivamoggi 1997, 2001) among others. The reconnection rate was found to increase by an order of magnitude in the electron-inertia regime (Wesson 1990; Ottaviani and Porcelli 1993; Yamada et al. 2000).

In a low-β plasma (such as that in fusion systems), on length scales shorter than the ion gyro-radius ρ_s (as well as the ion skin depth d_i), the plasma dynamics is governed by the parallel electron compressibility associated with the electron-pressure-gradient term in the electron momentum balance equation (Aydemir 1992; Zakharov et al. 1993; Cafaro et al. 1998). The characteristic mode in this case is the kinetic Alfv́en wave (Drake 1995; Hesse and Winske 1998; Birn and Hesse 2001; Kuznetsova et al. 2001).

In this paper, we consider some aspects of collisionless plasma dynamics by including the effects of parallel electron compressibility. The electron-pressure term in generalized Ohm's law becomes important when the scale size of the dissipation region in resistive MHD is less than the ion gyro-radius ρ_s based on the electron

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temperature†. We adopt a fluid treatment for both electrons and ions (Cafaro et al. 1998). We explore collisionless plasma dynamics properties near a two-dimensional X-type magnetic neutral line in the steady state. We give a unified linear tearing mode formulation incorporating *both* electron inertia and parallel electron compressibility. We use a variational method due to Hazeltine and Ross (1978a) and Hazeltine and Strauss (1978) to develop an analytical dispersion relation for the tearing mode. We then deduce *sufficient* conditions for the *linear* and *nonlinear* stability in the *Lyapunov* sense for steady states of this collisionless plasma system via Arnol′d's (1978) stability method.

2. Governing equations

Consider a strongly magnetized, low-β plasma, where the dynamical length scale perpendicular to $\mathbf{B}$ is much smaller than that parallel to $\mathbf{B}$. We exclude compressional Alfvén waves. We consider plasma dynamics in the thin non-ideal layer near the magnetic neutral surface where the resistivity and electron-inertia/compressibility effects become important. We adopt a fluid treatment for both electrons and ions (Kleva et al. 1995; Cafaro et al. 1998). We assume the dynamics to preserve quasi-neutrality, so $n_{\mathrm{i}} \approx n_{\mathrm{e}} = n$, n being the number density of the particles and, the subscripts i and e being associated with ions and electrons, respectively. We take the electron response to be isothermal, which is appropriate in a high-temperature plasma where parallel thermal conduction rapidly carries energy along the magnetic field lines. The diamagnetic effects associated with the electron-pressure gradient are neglected which is valid if the time scale of the phenomena of interest is smaller than the electron diamagnetic frequency.

Let us take

$$\mathbf{B} = B_0\hat{\mathbf{i}}_z + \nabla\psi \times \hat{\mathbf{i}}_z \tag{1}$$

where $B_0\hat{\mathbf{i}}_z$ is the constant guide field, and let us assume B_0 is large ($|B_0| \gg |\nabla\psi|$) so that z is the ignorable coordinate. Consequently, the plasma flow is essentially incompressible (owing to the compressional Alfveń wave that now propagates much faster than any other wave in the system). Furthermore, magnetic curvature and ∇B effects are negligible and parallel/perpendicular particle dynamics become decoupled. Plasma dynamics in a strong magnetic field are described by the equations of *reduced MHD* (Kadomtsev and Pogutse 1974; Strauss 1976). Owing to the small electron mass, the polarization drift and finite Larmor radius effects for electrons are ignored. The electrons and the ions move with the $\mathbf{E} \times \mathbf{B}$ drift velocity in the x, y-plane (Ottaviani and Porcelli 1993),

$$\mathbf{v}_\perp = \nabla\tilde{\phi} \times \hat{\mathbf{i}}_z, \tag{2}$$

ϕ being the stream function that also plays the role of the electric potential.

The set of equations governing this two-fluid MHD system is (Kleva et al. 1995; Cafaro et al. 1998)

$$\frac{\partial}{\partial t}(\psi - d_{\mathrm{e}}^2\nabla^2\psi) + [\psi - d_{\mathrm{e}}^2\nabla^2\psi, \phi] - \rho_s^2[\psi, \nabla^2\phi] = \tilde{\eta}\nabla^2\psi, \tag{3}$$

$$\frac{\partial}{\partial t}(\nabla^2\phi) + [\nabla^2\phi, \phi] + [\psi, \nabla^2\psi] = 0, \tag{4}$$

† This is made relevant by the fact that $\rho_s/d_{\mathrm{e}} > 1$ for the plasma regimes attained in fusion machines, such as JET (Porcelli 1991).

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where

$$\tilde{\eta} \equiv \eta c^2, \quad [f,g] \equiv \hat{\mathbf{i}}_z \cdot \nabla f \times \nabla g, \quad \phi \equiv \rho\tilde{\phi}, \quad \rho \equiv m_{\mathrm{i}} n.$$

The ions are taken to be cold ($T_{\mathrm{e}} \gg T_{\mathrm{i}}$). Equation (3) corresponds to the generalized Ohm's law while (4) is derived from the total momentum balance equation.

Equation (3) shows that the parallel electron compressibility ($\rho_s \neq 0$) has a strong effect on the reconnection process. However, the parallel electron compressibility by itself cannot drive the reconnection process in the absence of resistivity or electron inertia (Drake 1995).

3. Steady-state properties near an X-type neutral line

Consider collisionless plasma dynamics properties near a two-dimensional X-type magnetic neutral line in the steady state. Equations (3) and (4) now become

$$\begin{aligned}
&\phi_{01}[\psi_{10} - d_{\mathrm{e}}^2(\psi_{30} + \psi_{12})] - \phi_{10}[\psi_{01} - d_{\mathrm{e}}^2(\psi_{21} + \psi_{03})] \\
&\quad - \rho_s^2[\psi_{10}(\phi_{21} + \phi_{03}) - \psi_{01}(\phi_{30} + \phi_{12})] - cE = \tilde{\eta}(\psi_{20} + \psi_{02}), \qquad (5)\\
&\phi_{10}(\phi_{21} + \phi_{03}) - \phi_{01}(\phi_{30} + \phi_{12}) \\
&\quad = \psi_{10}(\psi_{21} + \psi_{03}) - \psi_{01}(\psi_{30} + \psi_{12}) - \nu(\phi_{40} + 2\phi_{22} + \phi_{04}), \qquad (6)
\end{aligned}$$

where

$$F_{mn} \equiv \frac{\partial^{m+n} F}{\partial x^m \, \partial y^n}, \qquad (7)$$

and we have now included in (4) viscous effects in the plasma which become important near the magnetic neutral line (Tsuda and Ugai 1977); ν is the viscosity coefficient.

Following Cowley (1975) and Shivamoggi (1985), let us expand the velocity and magnetic fields in a Taylor series about the neutral line taken to be at $x = 0, y = 0$. Equations (5) and (6) may then be used to derive relationships between the coefficients of the series. The latter are simply the partial derivatives of the fields at the neutral line. Thus, we write

$$\begin{aligned}
\psi &= \sum_m \sum_n \Psi_{2m,2n} \frac{x^{2m} y^{2n}}{(2m)!(2n)!}, \\
\phi &= \sum_m \sum_n \Phi_{2m+1,2n+1} \frac{x^{2m+1} y^{2n+1}}{(2m+1)!(2n+1)!}.
\end{aligned} \qquad (8)$$

Equation (8) reflects the fact that the origin in the x, y-plane is both the X-type neutral point and a stagnation point of the flow.

Using (8), Equation (5) gives, on evaluation at the origin,

$$\tilde{\eta}(\Psi_{20} + \Psi_{02}) = -cE. \qquad (9)$$

Let us next differentiate (5) and (6) with respect to x and y separately, and differentiate the resulting four equations, respectively, with respect to x, y, x and y.

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Furthermore, we note the expansion symmetry implied by (8) and then obtain

$$\tilde{\eta}(\psi_{40}+\psi_{22}) = 2\phi_{11}[\psi_{20} - d_e^2(\psi_{40}+\psi_{22})] - \rho_s^2[2\psi_{20}(\phi_{31}+\phi_{13})], \tag{10}$$

$$\tilde{\eta}(\psi_{22}+\psi_{04}) = -2\phi_{11}[\psi_{02} - d_e^2(\psi_{22}+\psi_{04})] + \rho_s^2[2\psi_{02}(\phi_{31}+\phi_{13})], \tag{11}$$

$$\psi_{20}(\psi_{22}+\psi_{04}) - \psi_{02}(\psi_{40}+\psi_{22}) = \nu(\phi_{51}+2\phi_{33}+\phi_{15}). \tag{12}$$

We now use (8) and evaluate (10)–(12) at the origin:

$$(\tilde{\eta}+2d_e^2\Phi_{11})(\Psi_{40}+\Psi_{22}) = 2\Phi_{11}\Psi_{20} - 2\rho_s^2\Psi_{20}(\Phi_{31}+\Phi_{13}), \tag{13}$$

$$(\tilde{\eta}-2d_e^2\Phi_{11})(\Psi_{22}+\Psi_{04}) = -2\Phi_{11}\Psi_{02} + 2\rho_s^2\Psi_{02}(\Phi_{31}+\Phi_{13}), \tag{14}$$

$$\Psi_{20}(\Psi_{22}+\Psi_{04}) - \Psi_{02}(\Psi_{40}+\Psi_{22}) = \nu(\Phi_{51}+2\Phi_{33}+\Phi_{15}). \tag{15}$$

Using (13) and (14), Equation (15) gives

$$4[\Phi_{11} - \rho_s^2(\Phi_{31}+\Phi_{13})]\Psi_{20}\Psi_{02} = -\nu\tilde{\eta}\left(1 - \frac{4d_e^4\Phi_{11}^2}{\tilde{\eta}^2}\right)(\Phi_{51}+2\Phi_{33}+\Phi_{15}). \tag{16}$$

Equation (16) shows that, in general,

$$\Psi_{20} \neq 0, \quad \Psi_{02} \neq 0, \tag{17}$$

so the magnetic field to lowest order can be hyperbolic (Shivamoggi 1985). However, in the presence of electron-inertia effects ($d_e \neq 0$), a special case arises with

$$\frac{4d_e^4\Psi_{11}^2}{\tilde{\eta}^2} = 1, \tag{18}$$

for which

$$\Psi_{20} = 0 \quad \text{or} \quad \Psi_{02} = 0. \tag{19}$$

So, for this case, the magnetic field to lowest order cannot be hyperbolic but must form a neutral sheet (as with the inviscid MHD case (Cowley 1975)). On the other hand, in the presence of parallel electron compressibility ($\rho_s \neq 0$), another special case arises with

$$\Phi_{11} - \rho_s^2(\Phi_{31}+\Phi_{13}) = 0, \tag{20}$$

for which it is possible to have (17) again, hence restoring the hyperbolicity of the magnetic field.

4. Tearing modes

Let us write

$$\psi = \psi_0(x) + \tau_H\psi_1(x)e^{\gamma t}\cos ky, \quad \psi_0' = -B_*F,$$
$$\phi = \frac{1}{kB_*}\phi_1(x)e^{\gamma t}\sin ky, \tag{21}$$

where τ_H is the hydromagnetic evolution time scale,

$$\tau_H \equiv \frac{\sqrt{m_i n}}{kB_*}.$$

On linearizing in ψ_1 and ϕ_1, (3) and (4) give

$$\omega[\psi_1 - d_e^2(\psi_1'' - k^2\psi_1)] - (F - d_e^2F'')\phi_1 + \rho_s^2F(\phi_1'' - k^2\phi_1) = \epsilon(\psi_1'' - k^2\psi_1), \tag{22}$$

$$-\omega(\phi_1'' - k^2\phi_1) = F(\psi_1'' - k^2\psi_1) - F''\psi_1, \tag{23}$$

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where the primes denote differentiation with respect to x, the distances have been non-dimensionalized using a reference length L, and

$$\epsilon \equiv \frac{\tau_{\rm H}}{\tau_{\rm R}}, \quad \omega \equiv \gamma \tau_{\rm H}, \quad \tau_{\rm R} \equiv \frac{L^2}{\hat{\eta}}.$$

Near the magnetic neutral surface $x = 0$, where $F \approx x$ and the flow/field gradients become large, (22) and (23) may be approximated by

$$\omega(\psi_1 - d_{\rm e}^2 \psi_1'') - x\phi_1 + \rho_s^2 x \phi_1'' = \epsilon \psi_1'', \tag{24}$$

$$\omega \phi_1'' = -x \psi_1''. \tag{25}$$

We now use a variational method (Hazeltine and Ross 1978; Hazeltine and Strauss 1978) to develop an analytical dispersion relation for the tearing mode.

On Fourier transforming the variables, according to

$$Q(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} q(x)\, dx \tag{26}$$

we obtain from (24) and (25),

$$\omega \Psi_1 - i\Phi_1' - i\rho_s^2 (k^2 \Phi_1)' = -(\epsilon + \omega d_{\rm e}^2) k^2 \Psi_1, \tag{27}$$

$$-\omega k^2 \Phi_1 = i(k^2 \Psi_1)'. \tag{28}$$

Introducing

$$\mathcal{J}(k) \equiv k^2 \Psi_1, \tag{29}$$

(27) and (28) can be combined into the self-adjoint form

$$\left[\left(\frac{1}{\omega k^2} + \frac{\rho_s^2}{\omega}\right)\mathcal{J}'\right]' - \left(\epsilon + \frac{\omega}{k^2} + \omega d_{\rm e}^2\right)\mathcal{J} = 0. \tag{30}$$

Equation (30) can be obtained by extremizing the bilinear functional

$$\mathcal{S} = \int_{-\infty}^{\infty} \left[\left(\frac{1}{\omega k^2} + \frac{\rho_s^2}{\omega}\right)\mathcal{J}'^2 + \left(\epsilon + \frac{\omega}{k^2} + \omega d_{\rm e}^2\right)\mathcal{J}^2\right] dk. \tag{31}$$

In order to obtain an analytical dispersion relation following Rayleigh and Ritz, we use a trial function for $\mathcal{J}$,

$$\mathcal{J} = e^{-\mu k^2/2}. \tag{32}$$

Using (32), Equation (31) becomes

$$\mathcal{S} = \sqrt{\pi}\left[\frac{\mu^{3/2}}{\omega} + (\epsilon + \omega d_{\rm e}^2)\mu^{-1/2} + \left(\frac{\rho_s^2}{2\omega} - 2\omega\right)\mu^{1/2}\right]. \tag{33}$$

The parameters μ and ω may then be determined by imposing the conditions

$$\mathcal{S} = 0, \tag{34}$$

$$\frac{\partial \mathcal{S}}{\partial \mu} = 0, \tag{35}$$

which lead to

$$\frac{\mu^2}{\omega} + \left(\frac{\rho_s^2}{2\omega} - 2\omega\right)\mu + (\epsilon + \omega d_{\rm e}^2) = 0, \tag{36a}$$

$$\frac{3}{\omega}\mu^2 + \left(\frac{\rho_s^2}{2\omega} - 2\omega\right)\mu - (\epsilon + \omega d_{\rm e}^2) = 0. \tag{36b}$$

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Eliminating μ, (36a) and (36b) lead to the dispersion relation

$$\omega^4 - \frac{1}{2}(\rho_s^2 + 2d_e^2)\omega^2 - \epsilon\omega + \frac{\rho_s^4}{16} = 0, \tag{37}$$

which may be rewritten as

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = \epsilon\omega \tag{38}$$

where

$$\omega_{1,2}^2 \equiv \tfrac{1}{4}[(\rho_s^2 + 2d_e^2) \pm 2d_e\sqrt{\rho_s^2 + d_e^2}]. \tag{39}$$

$\omega_{1,2}^2$ represent the electron-inertia/-compressibility branches, respectively, which become coupled in the presence of resistivity ($\epsilon \neq 0$).

The electron-inertia branch for weak electron compressibility ($\rho_s/d_e \ll 1$) is given by

$$\omega_1 \approx d_e\left(1 + \frac{\rho_s^2}{4d_e^2}\right). \tag{40}$$

The result $\omega_1 \approx d_e$, predicted by (40), is in agreement with that given by Coppi (1964). On the other hand, the electron-compressibility branch for weak electron inertia ($\rho_s/d_e \gg 1$) is given by

$$\omega_2 \approx \frac{1}{2}\rho_s\left(1 - \frac{d_e}{\rho_s}\right). \tag{41}$$

The result $\omega_2 \sim \rho_s$, predicted by (41), is in agreement with that given by Porcelli (1991).

It is to be noted that (40) and (41) are confirmed independently by the asymptotics of (24) and (25). For weak electron compressibility, we have from (24) and (25), for the electron-inertia branch,

$$-\omega d_e^2 \psi_1'' \approx x\phi_1,$$

$$\omega\phi_1'' \approx -x\psi_1''$$

which leads to $\omega^2 \sim d_e^2$. On the other hand, for weak electron inertia, we have from (24) and (25), for the electron-compressibility branch,

$$\omega\psi_1 + \rho_s^2 x\phi_1'' \approx 0,$$

$$\omega\phi_1'' \approx -x\psi_1''$$

which leads to $\omega^2 \sim \rho_s^2$.

For small resistivity, the resistive corrections to the collisionless branches are given by

$$\omega = \omega_{1,2} \pm \epsilon\frac{1}{2d_e}(\rho_s^2 + d_e^2)^{-1/2} + O(\epsilon^2). \tag{42}$$

On the other hand, for weak collisionless effects ($\epsilon \gg 1$), the collisionless correction to the resistive branch is given by

$$\omega = \epsilon^{1/3} + \frac{1}{6\epsilon^{1/3}}(\rho_s^2 + 2d_e^2) + O(\epsilon^{-1}). \tag{43}$$

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5. Stability in the Lyapunov sense

In the non-resistive limit, $\eta \to 0$, (3) and (4) become

$$\frac{\partial \psi_{\mathrm{e}}}{\partial t} + [\psi_{\mathrm{e}}, \phi] + \alpha^2 [\psi, \zeta] = 0, \tag{44}$$

$$\frac{\partial \zeta}{\partial t} + [\zeta, \phi] + [\psi, J] = 0, \tag{45}$$

where ψ_{e} is the generalized magnetic flux,

$$\psi_{\mathrm{e}} \equiv \psi + J,$$

J is the current density,

$$J \equiv -\nabla^2 \psi,$$

ζ is the vorticity,

$$\zeta = -\nabla^2 \phi,$$

and

$$\alpha^2 \equiv \frac{\rho_s^2}{d_{\mathrm{e}}^2}.$$

Here, we have scaled the distances with respect to the electron skin depth d_{e}.

The system of equations (44) and (45) has the following Hamiltonian representation (Cafaro et al. 1998):

$$\mathscr{H}[\psi] = \frac{1}{2} \iint_v (\phi_{\mathrm{e}} \zeta + \psi_{\mathrm{e}} J)\, d\mathbf{x}, \tag{46}$$

where ϕ_{e} is the generalized flow flux,

$$\phi_{\mathrm{e}} \equiv \phi + \zeta.$$

Hamilton's equations for the canonical variables ψ_{e} and ζ (in the notation of Benjamin (1984)) are

$$\left[\frac{\partial \psi_{\mathrm{e}}}{\partial t}, \frac{\partial \zeta}{\partial t}\right] = \mathscr{J} \left[\frac{\delta \mathscr{H}}{\delta \psi_{\mathrm{e}}}, \frac{\delta \mathscr{H}}{\delta \zeta}\right]^{\mathrm{T}} \tag{47}$$

which are just (44) and (45). Here, $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ is a column vector, $\delta \mathscr{H}/\delta q$ is the variational derivative of $\mathscr{H}$ with respect to q, and $\mathscr{J}$ is the non-invertible symplectic operator given by

$$\mathscr{J} = \begin{pmatrix} \alpha^2 [\zeta, \cdot] & [\psi_{\mathrm{e}}, \cdot] \\ [\psi_{\mathrm{e}}, \cdot] & [\zeta, \cdot] \end{pmatrix}. \tag{48}$$

$v \subseteq \mathbb{R}^2$ is the appropriate two-dimensional domain on which the plasma flow and the magnetic field evolve.

Equations (44) and (45) show that the functionals

$$\mathscr{C}_{\pm}[\psi] = \iint_v F_{\pm}(\psi_{\mathrm{e}} \pm \alpha \zeta)\, d\mathbf{x} \tag{49}$$

for any functions $F_{\pm}$ are conserved (Cafaro et al. 1998). $\mathscr{C}_{\pm}$ are also Casimir invariants, because we have from (48) and (49) that

$$\mathscr{J} \left[\frac{\delta \mathscr{C}_{\pm}}{\delta \psi_{\mathrm{e}}}, \frac{\delta \mathscr{C}_{\pm}}{\delta \zeta}\right]^{\mathrm{T}} = \begin{pmatrix} \pm \alpha [\psi_{\mathrm{e}} \pm \alpha \zeta, F'_{\pm}(\psi_{\mathrm{e}} \pm \alpha \zeta)] \\ [\psi_{\mathrm{e}} \pm \alpha \zeta, F'_{\pm}(\psi_{\mathrm{e}} \pm \alpha \zeta)] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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5.1. Linear stability

The equilibrium state (denoted by an overbar) is a constrained conditional extremum of $\mathscr{H}$ and is, therefore, a stationary point of the functional

$$\mathscr{I}[\psi] = \mathscr{H}[\psi] + \mathscr{C}_+[\psi] + \mathscr{C}_-[\psi] \tag{50}$$

which becomes, on using (46) and (49),

$$\mathscr{I}[\psi] = \iint_v \left\{ \frac{1}{2}[(\nabla\psi)^2 + (\nabla^2\psi)^2 + (\nabla\phi)^2 + \alpha^2(\nabla^2\phi)^2] + F_+(\psi_e + \alpha\zeta) + F_-(\psi_e - \alpha\zeta) \right\} d\mathbf{x} \tag{51}$$

and is given by the vanishing of the first variation of $\mathscr{I}[\psi]$,

$$\delta\mathscr{I}[\psi] = \iint_v \left\{ [-\nabla^2\overline{\psi} + F'_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F'_-(\overline{\psi}_e - \alpha\overline{\zeta})]\delta\psi_e + [\overline{\phi} - \alpha\nabla^2\overline{\phi} + \alpha F'_+(\overline{\psi}_e + \alpha\overline{\zeta}) - \alpha F'_-(\overline{\psi}_e - \alpha\overline{\zeta})]\delta\zeta \right\} d\mathbf{x} = 0. \tag{52}$$

We have, from (52),

$$\nabla^2\overline{\psi} = F'_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F'_-(\overline{\psi}_e - \alpha\overline{\zeta}), \tag{53}$$

$$\overline{\phi} - \alpha\nabla^2\overline{\phi} = -\alpha F'_+(\overline{\psi}_e + \alpha\overline{\zeta}) + \alpha F'_-(\overline{\psi}_e - \alpha\overline{\zeta}). \tag{54}$$

Next, consider the second variation

$$\delta^2\mathscr{I}[\psi] = \mathscr{I}[\overline{\psi} + \delta\psi] - \mathscr{I}[\overline{\psi}]$$

$$= \iint_v \{(\nabla\delta\psi)^2 + (\nabla^2\delta\psi)^2 + (\nabla\delta\phi)^2 + \alpha^2(\nabla^2\delta\phi)^2 + F''_+(\overline{\psi}_e + \alpha\overline{\zeta})\cdot(\delta\psi_e + \alpha\delta\zeta)^2 + F''_-(\overline{\psi}_e - \alpha\overline{\zeta})\cdot(\delta\psi_e - \alpha\delta\zeta)^2\}\, d\mathbf{x}, \tag{55a}$$

which may be rewritten as

$$\delta^2\mathscr{I}[\overline{\psi}] = \iint_v \left\{ (\nabla\delta\psi)^2 + (\nabla^2\delta\psi)^2 + (\nabla\delta\phi)^2 + \frac{[F''_+(\overline{\psi}_e + \alpha\overline{\zeta})\cdot(\delta\psi_e + \alpha\delta\zeta) + F''_-(\overline{\psi}_e - \alpha\overline{\zeta})\cdot(\delta\psi_e - \alpha\delta\zeta)]^2}{F''_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F''_-(\overline{\psi}_e - \alpha\overline{\zeta})} + \left[\frac{1}{4} + \frac{F''_+(\overline{\psi}_e + \alpha\overline{\zeta})\cdot F''_-(\overline{\psi}_e - \alpha\overline{\zeta})}{F''_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F''_-(\overline{\psi}_e - \alpha\overline{\zeta})}\right] 4\alpha^2(\nabla^2\delta\phi)^2 \right\} d\mathbf{x}. \tag{55b}$$

A sufficient condition for linear stability in the Lyapunov sense (Arnol′d 1978) is that $\delta^2\mathscr{I}$ is positive (or negative) definite for all suitably smooth perturbations $\delta\psi$ and $\delta\phi$. This requires, from (55), that

$$F''_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F''_-(\overline{\psi}_e - \alpha\overline{\zeta}) \geqslant 0 \tag{56a}$$

and

$$\frac{1}{4} + \frac{F''_+(\overline{\psi}_e + \alpha\overline{\zeta})\cdot F''_-(\overline{\psi}_e - \alpha\overline{\zeta})}{F''_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F''_-(\overline{\psi}_e - \alpha\overline{\zeta})} \geqslant 0, \quad \text{for all } \mathbf{x} \in v. \tag{56b}$$

Equation (56) leads to the following analogue of the Arnol′d first stability theorem.

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THEOREM 1. *The steady state* $(\overline{\psi}_e, \overline{\zeta})$ *is* linearly *stable in the* Lyapunov *sense with respect to the perturbation norm*

$$\|(\delta\psi_e, \delta\zeta)\| = \{\delta^2 \mathscr{I}[\overline{\psi}]\}^{1/2} \tag{57}$$

if the Casimir functions $F_\pm(\psi_e \pm \alpha\zeta)$, *introduced in (53), satisfy*

$$F''_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F''_-(\overline{\psi}_e - \alpha\overline{\zeta}) \geqslant 0, \tag{56a}$$

$$F''_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F''_-(\overline{\psi}_e - \alpha\overline{\zeta}) \geqslant -4F''_+(\overline{\psi}_e + \alpha\overline{\zeta}) \cdot F''_-(\overline{\psi}_e - \alpha\overline{\zeta}), \quad \textit{for all } \mathbf{x} \in v. \tag{56b}$$

It may be noted that, in cases where $F''_+(\overline{\psi}_e + \alpha\overline{\zeta})$ and $F''_-(\overline{\psi}_e - \alpha\overline{\zeta})$ are of the same sign, (56a) yields the stronger condition, while, in cases where $F''_+(\overline{\psi}_e + \alpha\overline{\zeta})$ and $F''_-(\overline{\psi}_e - \alpha\overline{\zeta})$ are of the opposite sign, (56b) yields the stronger condition.

On using (53), Equation (56a) implies that

$$\frac{\partial \overline{J}}{\partial \overline{\psi}} \leqslant 0, \quad \text{for all } \mathbf{x} \in v, \tag{58}$$

which signifies current confinement and is satisfied by the usual confined plasma systems.

5.2. Nonlinear stability

For infinite-dimensional systems, the positive (or negative) definiteness of the second variation of the functional $\mathscr{I}$ does not imply *nonlinear* stability (unlike finite-dimensional Hamiltonian systems). Nonlinear stability theory seeks to establish stability of a steady state by showing that it is a local extremum of energy with respect to *finite*-amplitude, Casimir-preserving perturbations. To address nonlinear stability, according to the Arnol′d stability method, one needs to examine the sign of the *exact* invariant,

$$\Delta\mathscr{I}[\psi] \equiv \mathscr{I}[\psi] - \mathscr{I}[\overline{\psi}], \tag{59}$$

to which the second variation $\frac{1}{2}\delta^2\mathscr{I}$ is only the small-amplitude approximation. If this invariant is positive (or negative) definite, then it can be used to define a perturbation norm, and the steady state is nonlinearly stable in this norm. For this purpose, additional convexity conditions need to be imposed on the functional $\mathscr{I}[\psi]$ in order to insure that the functional $\mathscr{I}[\psi]$ is convex in a small but finite neighborhood of the steady state $(\overline{\psi}_e, \overline{\zeta})$.

Let us rewrite the functional $\Delta\mathscr{I}[\psi]$ as follows:

$$\Delta\mathscr{I}[\hat{\psi}] \equiv \mathscr{I}[\overline{\psi} + \hat{\psi}] - \mathscr{I}[\overline{\psi}], \tag{60}$$

where $\hat{\psi}$ is the finite-amplitude perturbation imposed on the steady state $\overline{\psi}$.

Using (53) and (54), Equation (60) becomes

$$\begin{aligned}
\Delta\mathscr{I}[\hat{\psi}] = \frac{1}{2}\iint_v [(\nabla\hat{\psi})^2 + (\nabla^2\hat{\psi})^2 + (\nabla\hat{\phi})^2 + \alpha^2(\nabla^2\hat{\phi})^2]\, d\mathbf{x} \\
- \iint_v [F'_+(\overline{\psi}_e + \alpha\overline{\zeta}) \cdot (\hat{\psi}_e + \alpha\hat{\zeta}) + F'_-(\overline{\psi}_e - \alpha\overline{\zeta}) \cdot (\hat{\psi}_e - \alpha\hat{\zeta})]\, d\mathbf{x} \\
+ \iint_v [F_+(\overline{\psi}_e + \alpha\overline{\zeta} + \hat{\psi}_e + \alpha\hat{\zeta}) - F_+(\overline{\psi}_e + \alpha\overline{\zeta}) \\
+ F_-(\overline{\psi}_e - \alpha\overline{\zeta} + \hat{\psi}_e - \alpha\hat{\zeta}) - F_-(\overline{\psi}_e - \alpha\overline{\zeta})]\, d\mathbf{x}.
\end{aligned} \tag{61}$$

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Observe that, if (61) is Taylor expanded about the steady state $(\overline{\psi}_e, \overline{\zeta})$, the leading-order term is just the second variation $\frac{1}{2}\delta^2\mathscr{I}$, given by (55).

Suppose that the Casimir functions $F_\pm$ satisfy the convexity condition

$$0 < \alpha_1 < F_+''(\psi_e + \alpha\zeta) + F_-''(\psi_e - \alpha\zeta) < \beta_1 < \infty, \quad \text{for all } (\psi_e \pm \alpha\zeta), \tag{62}$$

where α_1 and β_1 are real numbers. Equation (62) implies that

$$\begin{aligned}
&\frac{1}{2\alpha_1}\Bigg\{[F_+''(\overline{\psi}_e + \alpha\overline{\zeta})\cdot(\hat{\psi}_e + \alpha\hat{\zeta}) + F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\cdot(\hat{\psi}_e - \alpha\hat{\zeta})]^2 \\
&\quad + \left[\frac{1}{4}\{F_+''(\overline{\psi}_e + \alpha\overline{\zeta}) + F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\} + F_+''(\overline{\psi}_e + \alpha\overline{\zeta})\cdot F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\right] \\
&\quad \times 4\alpha^2(\nabla^2\hat{\phi})^2\Bigg\} \\
&> \Bigg\{\frac{1}{2}\alpha^2(\nabla^2\hat{\phi})^2 + F_+(\overline{\psi}_e + \alpha\overline{\zeta} + \hat{\psi}_e + \alpha\hat{\zeta}) + F_-(\overline{\psi}_e - \alpha\overline{\zeta} + \hat{\psi}_e - \alpha\hat{\zeta}) \\
&\quad - F_+(\overline{\psi}_e + \alpha\overline{\zeta}) - F_-(\overline{\psi}_e - \alpha\overline{\zeta}) - F_+'(\overline{\psi}_e + \alpha\overline{\zeta})\cdot(\hat{\psi}_e + \alpha\hat{\zeta}) \\
&\quad - F_-'(\overline{\psi}_e - \alpha\overline{\zeta})\cdot(\hat{\psi}_e - \alpha\hat{\zeta})\Bigg\} \\
&> \frac{1}{2\beta_1}\Bigg\{[F_+''(\overline{\psi}_e + \alpha\overline{\zeta})\cdot(\hat{\psi}_e + \alpha\hat{\zeta}) + F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\cdot(\hat{\psi}_e - \alpha\hat{\zeta})]^2 \\
&\quad + \left[\frac{1}{4}\{F_+''(\overline{\psi}_e + \alpha\overline{\zeta}) + F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\} + F_+''(\overline{\psi}_e + \alpha\overline{\zeta})\cdot F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\right] \\
&\quad \times 4\alpha^2(\nabla^2\hat{\phi})^2\Bigg\}.
\end{aligned} \tag{63}$$

Using (63), we have from (61),

$$\begin{aligned}
&\frac{1}{2}\iint_v\Bigg[(\nabla\hat{\psi})^2 + (\nabla^2\hat{\psi})^2 + (\nabla\hat{\phi})^2 + \frac{1}{\beta_1}\Bigg\{[F_+''(\overline{\psi}_e + \alpha\overline{\zeta})\cdot(\hat{\psi}_e + \alpha\hat{\zeta}) \\
&\quad + F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\cdot(\hat{\psi}_e - \alpha\hat{\zeta})]^2 + \left[\frac{1}{4}\{F_+''(\overline{\psi}_e + \alpha\overline{\zeta}) + F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\}\right. \\
&\quad \left.+ F_+''(\overline{\psi}_e + \alpha\overline{\zeta})\cdot F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\right]4\alpha^2(\nabla^2\hat{\phi})^2\Bigg\}\Bigg]d\mathbf{x} \\
&< \Delta\mathscr{I}[\hat{\psi}] < \frac{1}{2}\iint_v\Bigg[(\nabla\hat{\psi})^2 + (\nabla^2\hat{\psi})^2 + (\nabla\hat{\phi})^2 + \frac{1}{\alpha_1}\Bigg\{[F_+''(\overline{\psi}_e + \alpha\overline{\zeta})\cdot(\hat{\psi}_e + \alpha\hat{\zeta}) \\
&\quad + F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\cdot(\hat{\psi}_e - \alpha\hat{\zeta})]^2 + \left[\frac{1}{4}\{F_+''(\overline{\psi}_e + \alpha\overline{\zeta}) + F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\}\right. \\
&\quad \left.+ F_+''(\overline{\psi}_e + \alpha\overline{\zeta})\cdot F_-''(\overline{\psi}_e - \alpha\overline{\zeta})\right]4\alpha^2(\nabla^2\hat{\phi})^2\Bigg\}\Bigg]d\mathbf{x}.
\end{aligned} \tag{64}$$

Equation (64) shows that the convexity condition (62) is *sufficient* to establish the positive (or negative) definiteness of the *exact* invariant $\Delta\mathscr{I}[\hat{\psi}]$.

If one defines a perturbation norm by

$$\|(\hat{\psi}_e, \hat{\zeta})\| = \{\widetilde{\Delta\mathscr{I}}[\hat{\psi}]\}^{1/2}, \tag{65}$$

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where

$$\widetilde{\Delta\mathscr{I}}[\hat{\psi}] \equiv \frac{1}{2}\iint_v \Bigg[(\nabla\hat{\psi})^2 + (\nabla^2\hat{\psi})^2 + (\nabla\hat{\phi})^2 + \frac{1}{\beta_1}\Bigg\{[F''_+(\overline{\psi}_e + \alpha\overline{\zeta})\cdot(\hat{\psi}_e + \alpha\hat{\zeta}) + F''_-(\overline{\psi}_e - \alpha\zeta)\cdot(\hat{\psi}_e - \alpha\hat{\zeta})]^2 + \Bigg[\frac{1}{4}\{F''_+(\overline{\psi}_e + \alpha\overline{\zeta}) + F''_-(\overline{\psi}_e - \alpha\overline{\zeta})\} + F''_+(\overline{\psi}_e + \alpha\overline{\zeta})\cdot F''_-(\overline{\psi}_e - \alpha\overline{\zeta})\Bigg]4\alpha^2(\nabla^2\hat{\phi})^2\Bigg\}\Bigg]d\mathbf{x},$$

then (64) shows that this norm is bounded from above. We thus have the following Nonlinear stability theorem.

THEOREM 2. *Suppose that the Casimir functions* $F_\pm(\psi_e \pm \alpha\zeta)$*, introduced in (49), satisfy the following convexity condition:*

$$0 < \alpha_1 < F''_+(\psi_e + \alpha\zeta) + F''_-(\psi_e - \alpha\zeta) < \beta_1 < \infty, \quad \textit{for all } (\psi_e \pm \alpha\zeta), \tag{62}$$

for some real constants α_1 *and* β_1*. Then, the steady state* $(\overline{\psi}_e, \overline{\zeta})$*, determined by* $F_\pm(\psi_e \pm \alpha\zeta)$ *through relations (53) and (54), is* nonlinearly *stable in the* Lyapunov *sense with respect to the perturbation norm*

$$\|(\hat{\psi}_e, \hat{\zeta})\| = \{\widetilde{\Delta\mathscr{I}}[\hat{\psi}]\}^{1/2}. \tag{65}$$

It is to be noted that the global existence of smooth solutions of (3) and (4) is not known so that the above result is valid only for times as long as smooth (C^1) solutions exist.

It may be noted that, on using (53), Equation (62) implies

$$0 < \alpha_1 < \left(-\frac{\partial\overline{J}}{\partial\overline{\psi}}\right) < \beta_1 < \infty. \tag{66}$$

Equation (66) implies, in turn, explicit bounds on the equilibrium current gradient which signify conditions stronger than (58) that are required for linear stability. For nonlinear stability, the equilibrium current gradient not only needs to be negative (as for linear stability) but it must also be bounded from below.

6. Discussion

In this paper, we have considered some aspects of collisionless magnetic reconnection dynamics by including the effects of electron inertia as well as parallel electron compressibility. Collisionless plasma dynamics properties near a two-dimensional X-type magnetic neutral line in the steady state are explored. The effects of electron inertia and parallel electron compressibility on the hyperbolicity (or lack thereof) of the magnetic field in the neutral layer are discussed. A unified formulation for collisionless linear tearing modes has been given incorporating the effects of both electron inertia and parallel electron compressibility. The parallel electron compressibility generates a new branch of the collisionless tearing mode. This new branch is shown in general to couple to the electron-inertia branch in the presence of resistivity. A sufficient condition for linear stability in the Lyapunov sense for steady states of this collisionless plasma system signifying current confinement has been deduced. Explicit convexity conditions for the Casimir functions, and hence

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bounds on the equilibrium current gradient, are shown to constitute sufficient conditions for nonlinear stability in the Lyapunov sense for steady states via nonlinear bounds for a suitable perturbation norm. For nonlinear stability, the equilibrium current gradient not only needs to be negative (as for linear stability) but it must also be bounded from below.

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