Glasgow Mathematical Journal (2026), 1–10
doi:10.1017/S0017089526101050

Glasgow Mathematical Journal

RESEARCH ARTICLE

Generalised Markov traces and Jucys–Murphy elements

Kostiantyn Tolmachov[1] and Heorhii Zhylinskyi[2]

[1]University of Hamburg, Germany
[2]Jagiellonian University, Poland
Corresponding author: Kostiantyn Tolmachov; Email: tolmak@khtos.com

Received: 6 November 2025 UTC; **Revised:** 6 May 2026 UTC; **Accepted:** 14 June 2026 UTC

Keywords: Markov traces; link invariants; Iwahori-Hecke algebras; HOMFLY-PT polynomial

2020 Mathematics Subject Classification: *Primary* - 20C08, 57K14, 57K18

Abstract
We give a simple construction of Markov traces for Iwahori–Hecke algebras associated with the crystallographic Coxeter groups of types A_n, B_n and D_n. In types B and D, it is new and generalises a known construction in type A employing symmetric polynomials in multiplicative Jucys–Murphy elements.

1. Introduction

A two-variable link invariant – the HOMFLY-PT polynomial – can be recovered using a family of trace functions on Iwahori–Hecke algebras in type A satisfying analogues of the conditions in Markov's theorem for the braid group (usually called 'Markov moves') [10]. The analogues of Markov moves can be formulated for Iwahori–Hecke algebras of other Coxeter groups, leading to a general notion of the Markov trace, see, for example, [3, 5, 7, 20]. The Markov traces were classified in type A, where there is, up to some change of scaling, a unique Markov trace, by Ocneanu and Jones [10], and in types B and D by Geck and Lambropoulou [3, 5]. In type B, Markov traces produce the invariants of the solid torus links [13, 5].

Of particular interest are specialisations of the general Markov traces in types B, D coming from geometry (or, alternatively, from the theory of Soergel bimodules). These were studied in [7] using Lusztig's Fourier transform, in [20, 21] using the equivariant Hecke category, and in [2] using the monodromic Hecke category. The latter two geometric constructions are both matched with the one coming from the Hochschild cohomology of Soergel bimodules, and, in type A, to the Soergel bimodule construction of Khovanov–Rozansky homology [12]. Note that the construction of trace-like functionals on the Hecke algebras through Hochschild cohomology of Soergel bimodules readily generalises to an arbitrary Coxeter group.

Iwahori–Hecke algebras in types A, B, D contain families of commuting operators whose properties are analogous to those of Jucys–Murphy elements in the group algebra of the symmetric group. Under a standard normalisation, the Markov trace in type A is polynomial in the variable a. Its individual coefficients turn out to be 'represented' (in the sense explained in the main text) by the elementary symmetric polynomials in multiplicative Jucys–Murphy elements. It was shown in [2] that this fact admits a categorification to the level of Khovanov–Rozansky homology. Moreover, in type A, multiplicative Jucys–Murphy elements themselves admit an interesting categorification, defining relative Serre functors with respect to the embedding of a smaller Hecke category into a larger one [4, 9]. Our investigation was motivated by a desire to see how far these patterns can be extended for other types.

In this paper, we give a new simple construction for Markov traces in types B, D using the theory of Jucys–Murphy elements in Iwahori–Hecke algebras of these types.

© The Author(s), 2026. Published by Cambridge University Press on behalf of Glasgow Mathematical Journal Trust. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

The paper is organised as follows. In Section 2, we collect the needed definitions about Coxeter groups and Iwahori–Hecke algebras. In Section 2.5, we define a family of commuting elements in Iwahori–Hecke algebras of type A, B, D coming from the extended affine Hecke algebra in type A. See [16] for the history and an alternative definition. We follow [17] in our treatment of these elements.

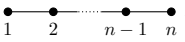
In Section 3, we first define the Markov traces and give, as a warm-up, a construction of a Markov trace defined by a uniform formula in all types A, B, D. See Theorem 3.1.4. In Section 3.2, we recall the results of Geck and Lambropoulou and give the formulas for the Markov traces in types B and D depending on a free parameter (Theorem 3.2.2 and Corollary 3.2.5, respectively). In Section 3.3, we specialise our results to obtain the geometric Markov traces.

2. Iwahori–Hecke algebras and multiplicative Jucys–Murphy elements

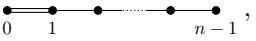
2.1. Coxeter groups

Let $(W, S), S = \{s_1, \dots, s_n\}$, be a Coxeter system. Thus, W is a group with a presentation $(s_0, \dots, s_n | (s_i s_j)^{m_{ij}} = e)$, where $m_{ij} \in \mathbb{Z}^+ \cup \{\infty\}$, with $m_{ii} = 1, m_{ji} = m_{ij} \geq 2$ if $i \neq j$, and $m_{ij} = \infty$ means that there is no relation included for the pair s_i, s_j. Such a presentation is read off the Coxeter diagram of (W, S) in a standard way: such a diagram is a non-oriented graph with S as a set of vertices, and $m_{ij} - 2$ edges between the vertices s_i and s_j. Let $l : W \to \mathbb{Z}_{\geq 0}$ be the Bruhat length function with respect to S.

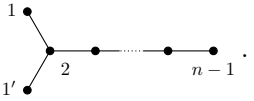
We will be primarily interested in Coxeter groups that are Weyl groups of classical types $A, B = C, D$. To fix the numbering of the generators, their Coxeter diagrams are presented, with a vertex corresponding to $s_i \in S$ marked by i. The Coxeter diagram of type A_n is



of type $B_n = C_n$ is



and of type D_n is



When n is small, these are interpreted in a standard way as follows: the Coxeter system $A_1 = B_1 = D_1$ has a single vertex, $D_2 = A_1 \times A_1$, $D_3 = A_3$. We also adopt the convention $A_0 = B_0 = D_0$ corresponding to the empty Coxeter graph and the trivial Coxeter group.

Many of the results and definitions in this paper are valid in all types A, B, D, with definitions for type A_{n-1} naturally fitting with definitions in types B_n, D_n. In this case, we will write 'type X_n, where X is A_{-1}, B or D' (or something in a similar vein).

2.2. Iwahori–Hecke algebras

Let $\mathcal{A} = \mathbb{Q}(v)$. The Iwahori–Hecke algebra $\mathcal{H}_v(W, S) = \mathcal{H}(W, S)$ associated with a Coxeter system (W, S) is a unital $\mathcal{A}$-algebra with generators t_s, $s \in S$, subject to the relations:

(i) $\langle t_{s_i}, t_{s_j} \rangle_{m_{ij}} = \langle t_{s_j}, t_{s_i} \rangle_{m_{ij}}$, where $\langle x, y \rangle_m$ is an alternating product $xyx \dots$ of x and y with m factors.
(ii) $(t_s - v)(t_s + v^{-1}) = 0$.

When (W, S) is of type B_n, we will also be interested in the Iwahori–Hecke algebra $\mathcal{H}_{v,v_0}(W, S)$ with unequal parameters $v, v_0 \in \mathcal{A}$. This is a unital algebra over the ring $\mathcal{A}$, with the relation (ii) replaced by the relations:

(ii′) $(t_{s_i} - v)(t_{s_i} + v^{-1}) = 0$ for $i > 0$,
(ii′) $(t_{s_0} - v_0)(t_{s_0} + v_0^{-1}) = 0$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

When (W, S) is of type X_n with $X \in \{A, B = C, D\}$, we will write $W(X_n)$ and $\mathcal{H}(X_n)$ for W and $\mathcal{H}(W, S)$, or $\mathcal{H}_{v,v_0}(B_n)$ if we need to emphasise that we use unequal parameters. In each of these cases, we will write t_i for t_{s_i}.

Let $w = s_{i_1} \dots s_{i_r}$, $w \in W$, $s_{i_k} \in S$ be a reduced expression. Then the element $t_w = t_{s_{i_1}} \dots t_{s_{i_r}}$ in $\mathcal{H}(W, S)$ is independent of the choice of a reduced expression, and $\mathcal{H}(W, S)$ is a free $\mathcal{A}$-module with a basis t_w, $w \in W$. The elements t_w are invertible, and the collection t_w^{-1}, $w \in W$, gives another basis of $\mathcal{H}(W, S)$ over $\mathcal{A}$.

2.3. Trace functional and a pairing on $\mathcal{H}(W, S)$

Let $x \mapsto \overline{x}$ be the Kazhdan–Lusztig involution on $\mathcal{H}(W, S)$. It is a $\mathbb{Q}$-algebra involution, defined by the assignments $\overline{v} = v^{-1}$, $\overline{t_w} = t_{w^{-1}}^{-1}$. Let $\omega : \mathcal{H}(W, S)$ stand for the $\mathcal{A}$-linear algebra anti-involution defined by $\omega(t_w) = t_{w^{-1}}$.

For $h \in \mathcal{H}(W, S), x \in W$, let h_x stand for the t_x^{-1}-coefficient of h in the basis t_w^{-1}, where w runs through W. Let $\tau : \mathcal{H}(W, S) \to \mathcal{A}$ be a functional defined by $\tau(h) = h_e$, where $e \in W$ is the unit. For $h, h' \in \mathcal{H}(W, S)$, define a pairing $\langle h, h' \rangle = \tau(\omega(\overline{h})h')$. We have the following standard proposition, see, for example, [8, Chapter 8].

Proposition 2.3.1. *The functional τ and the pairing $\langle\ ,\ \rangle$ satisfy the following properties:*

(a) The functional τ is an $\mathcal{A}$-linear trace on $\mathcal{H}(W, S)$, that is, $\tau(hh') = \tau(h'h)$.
(b) The pairing $\langle\ ,\ \rangle$ is $\mathcal{A}$-antilinear in the first argument and $\mathcal{A}$-linear in the second argument and satisfies the orthogonality relation $\langle t_{w_1}, t_{w_2}^{-1} \rangle = \delta_{w_1, w_2^{-1}}$.
(c) We have $\langle h_1 h_2, h_3 \rangle = \langle h_1, h_3 \omega(\overline{h_2}) \rangle = \langle h_2, \omega(\overline{h_1}) h_3 \rangle$ for any $h_1, h_2, h_3 \in \mathcal{H}(W, S)$.

We will use the following straightforward corollary of this proposition repeatedly.

Corollary 2.3.2. *Let (W', S') be a parabolic subgroup of W. Assume that $x, y \in \mathcal{H}(W', S') \subset \mathcal{H}(W, S)$, and assume that $z \in \operatorname{span}_{\mathcal{A}} \langle t_w, w \in W \backslash W' \rangle$ (resp. $z \in \operatorname{span}_{\mathcal{A}} \langle t_w^{-1}, w \in W \backslash W' \rangle$). Then $\langle xz, y \rangle = \langle zx, y \rangle = 0$ (resp. $\langle x, yz \rangle = \langle x, zy \rangle = 0$).*

2.4. 'Full twist' and Jucys–Murphy elements

Assume that (W, S) is of finite type. Let w_0 stand for the longest element in W with respect to the Bruhat length. Let $\mathbb{S}(W, S) = t_{w_0}^{-2} \in \mathcal{H}(W, S)$. In classical types, we will write $\mathbb{S}(X_n) \in \mathcal{H}(X_n)$, when X is A, B or D, or simply $\mathbb{S}$ when the type is clear from the context.

The following proposition is also standard:

Proposition 2.4.1. *The element $\mathbb{S}(W, S)$ is central in $\mathcal{H}(W, S)$.*

We will need the following 'Serre property' of the operator of multiplication by $\mathbb{S}(W, S)$.

Proposition 2.4.2. *For any $h_1, h_2 \in \mathcal{H}(W, S)$, we have the identity*

$$\langle h_1, \mathbb{S}(W, S) h_2 \rangle = \overline{\langle h_2, h_1 \rangle}.$$

Proof. By Proposition 2.3.1 (b), it is enough to prove the statement for $h_1 = t_{w_1}^{-1}$, $h_2 = t_{w_2}$ for arbitrary $w_1, w_2 \in W$. We have, by Proposition 2.3.1 (c),

$$\begin{aligned} \langle t_{w_1}^{-1}, \mathbb{S}(W, S) t_{w_2} \rangle &= \langle t_{w_0} t_{w_1}^{-1}, t_{w_0}^{-1} t_{w_2} \rangle = \langle t_{w_0 w_1^{-1}}, t_{w_2^{-1} w_0}^{-1} \rangle \\ &= \delta_{w_0 w_1^{-1}, w_0 w_2} = \delta_{w_1, w_2^{-1}} = \overline{\langle t_{w_2}, t_{w_1}^{-1} \rangle}. \end{aligned}$$

This completes the proof. □

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

We have the natural embedding $\iota_n : \mathcal{H}(X_n) \to \mathcal{H}(X_{n+1})$, where X is A, B or D, induced by the embedding of the sets of labels of the Coxeter diagram. Using these embeddings, an element of $\mathcal{H}(X_n)$ can be considered as an element of $\mathcal{H}(X_m)$ for $m > n$.

Definition 2.4.3. *For $n \geq 2$, let $J_n^X = \mathbb{S}(X_n)^{-1} \cdot \mathbb{S}(X_{n-1})$ when X is A_{-1}, B or D. We let $J_1^A = 1, J_1^B = t_0^2$, $J_1^D = 1$. We call these elements Jucys–Murphy elements of finite Hecke algebras in classical types.*

We have the following corollary of Proposition 2.4.1.

Corollary 2.4.4. *For X equal to A_{-1}, B or D, elements J_k^X commute with each other when considered lying in the same Iwahori–Hecke algebra via the embedding described above. The element J_n^X lies in the centraliser of $\mathcal{H}(X_{n-1})$ in $\mathcal{H}(X_n)$.*

Remark 2.4.5. *The elements J_k^A are sometimes called multiplicative Jucys–Murphy elements in the Iwahori–Hecke algebra of type A. There seems to be no agreement in the literature on which elements to call the Jucys–Murphy elements in other types. We chose our convention because under it the definition of the Jucys–Murphy element is uniform in all types, but elements j_k^X for $X = B, D$ defined below arguably also deserve the name.*

2.5. Some central elements in the classical Iwahori–Hecke algebras

In this section, we follow [17]. Let $\mathcal{H}^{\mathrm{aff}}(A_{n-1})$ be the affine Hecke algebra associated with the root datum of GL_n. It is a unital algebra over $\mathcal{A}$ generated by the elements $t_1, \ldots, t_{n-1}$ and $\Theta_1^{\pm 1}$, where $t_1, \ldots, t_{n-1}$ satisfy the relations of the Iwahori–Hecke algebra of type A_{n-1} and, additionally, the following relations hold:

(iii) $t_1\Theta_1 t_1 \Theta_1 = \Theta_1 t_1 \Theta_1 t_1$,
(iv) $t_i\Theta_1 = \Theta_1 t_i$ for $i > 1$.

Inductively define $\Theta_i = t_{i-1}\Theta_{i-1}t_{i-1} \in \mathcal{H}^{\mathrm{aff}}(A_{n-1})$, $i = 2, \ldots, n$. It is well known that $\Theta_i^{\pm 1}$ generate a commutative subalgebra of $\mathcal{H}^{\mathrm{aff}}(A_{n-1})$, called the Bernstein subalgebra. We have the following particular case of a classical theorem, proved in [14, Theorem 8.1] for all affine types.

Theorem 2.5.1. *The centre of $\mathcal{H}^{\mathrm{aff}}(A_{n-1})$ consists of the symmetric Laurent polynomials in $\Theta_1, \ldots, \Theta_n$.*

Fix $n \geq 2$. Note that

(i) The Hecke algebra $\mathcal{H}(A_{n-1})$ is the quotient of the affine Hecke algebra $\mathcal{H}^{\mathrm{aff}}(A_{n-1})$ by the relation $\Theta_1 = 1$.
(ii) The Hecke algebra $\mathcal{H}_{v,v_0}(B_n)$ is the quotient of $\mathcal{H}^{\mathrm{aff}}(A_{n-1})$ by the relation
$$(\Theta_1 - v_0)(\Theta_1 + v_0^{-1}) = 0.$$
Here, t_0 is the image of Θ_1.
(iii) The Hecke algebra $\mathcal{H}(D_n)$ is the subalgebra of the quotient (ii) with $v_0 = 1$, generated by the image $t_{1'}$ of $\Theta_1 t_1 \Theta_1^{-1}$ and images of $t_1, \ldots, t_{n-1}$.

Note that this exhibits $\mathcal{H}(D_n)$ as a subalgebra of $\mathcal{H}_{v,1}(B_n)$. Note also that J_i^A is the image of Θ_i in $\mathcal{H}(A_{n-1})$ for $i = 1, \ldots, n$. We have
$$J_i^A = t_{i-1} \ldots t_1 t_1 \ldots t_{i-1}.$$

Definition 2.5.2. *We define the following families of elements in the Hecke algebras of finite types B, D:*

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

(B) Let j_i^B be the image of Θ_i, $i = 1, \dots, n$, in $\mathcal{H}(B_n)$. We have

$$j_i^B = t_{i-1} \dots t_1 t_0 t_1 \dots t_{i-1} \tag{1}$$

and $J_i^B = (j_i^B)^2$.

(D) Let j_i^D be the image of $\Theta_i \Theta_1^{-1}$, $i = 1, \dots, n$, in $\mathcal{H}(D_n)$. We have

$$j_i^D = t_{i-1} \dots t_2 t_{1'} t_1 t_2 \dots t_{i-1}, i \geq 2, j_1^D = 1$$

and $J_i^D = (j_i^D)^2$.

We have the following

Corollary 2.5.3. *The elements J_i^A (resp. j_i^B, j_i^D) commute with each other. If P is a symmetric polynomial in n variables, $P(J_1^A, \dots, J_n^A)$ (resp. $P(j_1^B, \dots, j_n^B)$) is central in $\mathcal{H}(A_{n-1})$ (resp. in $\mathcal{H}_{v,v_0}(B_n)$). If P is homogeneous of even degree, then $P(j_1^D, \dots, j_n^D)$ is central in $\mathcal{H}(D_n)$.*

3. Markov traces

3.1. Markov traces via Jucys–Murphy elements

Definition 3.1.1. *Let $\mathcal{R}$ be a commutative $\mathcal{A}$-algebra. A family (ϕ_n) of $\mathcal{A}$-linear functionals $\phi_n : \mathcal{H}(X_n) \to \mathcal{R}$, where X is A_{-1}, B or D and $n \in \mathbb{Z}_{\geq 1}$ ($n \in \mathbb{Z}_{\geq 2}$ in type D) is called a Markov trace if there are elements $\mu, \rho \in \mathcal{R}$ such that the following conditions are satisfied:*

(M1) ϕ_n is a trace function, that is, $\phi_n(hh') = \phi_n(h'h)$. This property is a generalisation of the first Markov move from the theory of link invariants.
Recall the embedding $\iota = \iota_{n-1} : \mathcal{H}(X_{n-1}) \to \mathcal{H}(X_n)$ defined in Section 2.4. The functionals ϕ_{n-1} and ϕ_n, $n \geq 2$ must satisfy

(U1) $\phi_n(\iota(h)) = \rho \phi_{n-1}(h)$. This property is a generalisation of a rule for removing an unlinked circle in the theory of link invariants.

(M2) $\phi_n(\iota(h) t_{n-1}) = \mu \phi_{n-1}(h)$. This property is a generalisation of the second Markov move from the theory of link invariants.

Note that if (ϕ_n) is a Markov trace with constants ρ, μ, then, for arbitrary $\lambda_1, \lambda_2 \in \mathcal{R}$, $(\lambda_1 \lambda_2^n \phi_n)$ is again a Markov trace with constants $\lambda_2 \rho, \lambda_2 \mu$.

Remark 3.1.2. *We exclude small n in our definition of the Markov trace because some special treatment is needed for the embeddings $B_0 \to B_1, D_0 \to D_1 \to D_2$. For the geometric traces, these properties are extended to small n, see Section 3.3.*

Definition 3.1.3. *Assume that the involution $v \mapsto v^{-1}$ is extended to $\mathcal{R}$. We say that an element $z \in \mathcal{H}(W, S) \otimes_{\mathcal{A}} \mathcal{R}$ represents a functional $\phi : \mathcal{H}(W, S) \to \mathcal{R}$ if $\phi(h) = \langle z, h \rangle$ for any $h \in \mathcal{H}(W, S)$. It is easy to see that ϕ satisfies the property (M1) if and only if z is in the centre of $\mathcal{H}(W, S)$. In this case, we write $\phi = \mathrm{tr}_z$.*

Let $\mathcal{R} = \mathcal{A}(a)$ and let $\overline{a} = a^{-1}$. Define

$$\zeta_n = \prod_{i=1}^{n} (1 + a^{-1} J_i^X), \zeta_0 = 1,$$

an element in $\mathcal{H}(X_n)$ when $X = A_{-1}, B$ or D. By Corollary 2.5.3, elements ζ_n are central in the corresponding Iwahori–Hecke algebras. We prove

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

Theorem 3.1.4. *The family of functionals* (tr_{ζ_n}) *is a Markov trace with constants* $\rho = 1 + a$, $\mu = v - v^{-1}$.

Proof. We first check the property (U1). We have

$$\langle \zeta_n, \iota(h)\rangle = \langle \zeta_{n-1}(1 + a^{-1}J_n^X), \iota(h)\rangle = \langle \zeta_{n-1}, \iota(h)\rangle + a\langle \zeta_{n-1}J_n^X, \iota(h)\rangle.$$

Applying Proposition 2.4.2 twice – recall that $J_n^X = \mathbb{S}(X_n)^{-1}\mathbb{S}(X_{n-1})$ – we get the equality $\langle \zeta_{n-1}J_n^X, \iota(h)\rangle = \langle \zeta_{n-1}, \iota(h)\rangle$ and (U1) follows.

We now check the property (M2). We have

$$\begin{aligned}\langle \zeta_n, \iota(h)t_{n-1}\rangle &= \langle \zeta_{n-1}(1 + a^{-1}J_n^X), \iota(h)t_{n-1}\rangle = \langle \zeta_{n-1}, \iota(h)t_{n-1}\rangle + a\langle \zeta_{n-1}J_n^X, \iota(h)t_{n-1}\rangle \\ &= \langle \zeta_{n-1}, \iota(h)t_{n-1}^{-1}\rangle + (v - v^{-1})\langle \zeta_{n-1}, \iota(h)\rangle + a\langle \zeta_{n-1}J_n^X, \iota(h)t_{n-1}\rangle.\end{aligned} \tag{2}$$

From Corollary 2.3.2, we get that $\langle \zeta_{n-1}, \iota(h)t_{n-1}^{-1}\rangle = 0$.

For the third summand in the last expression in (2), applying Proposition 2.4.2, we have

$$\langle \zeta_{n-1}J_n^X, \iota(h)t_{n-1}\rangle = \overline{\langle \iota(h)t_{n-1}, \zeta_{n-1}\mathbb{S}(X_{n-1})\rangle} = 0,$$

with the last equality again following from Corollary 2.3.2. We obtain (M2). □

Remark 3.1.5. *Note that in type A the properties (M1), (U1) and (M2) define the Markov trace uniquely, and thus the preceding theorem recovers, up to a change of variable, the classical result: a-coefficients of the Markov trace in type A are represented by the elementary symmetric polynomials in Jucys–Murphy elements.*

Remark 3.1.6. *We will show below that, in types A and B (but not in type D), the traces* tr_{ζ_n} *coincide with the traces coming from geometry. See Section 3.3.*

3.2. General Markov traces in types B, D

In type A, the classical result of Jones and Ocneanu [10] says that a Markov trace with given constants ρ, μ is defined uniquely up to an overall scaling. Thus, Theorem 3.1.4 gives a general description of a Markov trace in type A.

In types B, D, the general description of Markov traces was obtained by Geck and Lambropoulou [3, 5]. We recall their results.

For $n \geq 1$, define the elements

$$T_n = t_{n-1} \dots t_1 t_0 t_1^{-1} \dots t_{n-1}^{-1} \in \mathcal{H}(B_n),\ n \geq 2,\ T_1 = t_0,$$

and

$$U_n = t_{n-1} \dots t_{1'} t_1^{-1} \dots t_{n-1}^{-1} \in \mathcal{H}(D_n),\ n \geq 2,\ U_1 = 1.$$

Theorem 3.2.1 ([3, 5]). *Let* $\mathcal{R} = \mathcal{A}(a, y_1, \dots, y_n, \dots)$, *with infinite number of variables* y_k. *There is a unique Markov trace* $(\mathrm{tr}^X_{n,(y_\bullet)})$, *where X is B or D, with values in* $\mathcal{R}$, *with constants* $\rho = 1 + a$, $\mu = v - v^{-1}$, *normalisation* $\mathrm{tr}^X_{n,(y_\bullet)}(1) = (1 + a)^n$ *and additional properties:*

(B) $\mathrm{tr}^B_{n,(y_\bullet)}(T_1 T_2 \dots T_n) = y_n$, $n \geq 1$, *in type B.*
(D) $\mathrm{tr}^D_{2n,(y_\bullet)}(U_1 U_2 \dots U_{2n}) = y_{2n}$, $n \geq 1$, *in type D.*

Any Markov trace with the chosen ρ, μ *and normalisation is of this form. Moreover, every such Markov trace in type D is obtained by restriction from the Markov trace on* $\mathcal{H}_{v,1}(B_n)$ *(see Section 2.5).*

We will be interested in a particular case of these traces when y_n is of the form $y_n = y^n$. Let from now on $\mathcal{R} = \mathcal{A}(y)$ and denote by $\mathrm{tr}^X_{n,y}$ the Markov trace from Theorem 3.2.1 with $y_n = y^n$ for all $n \geq 1$. Extend the bar involution to $\mathcal{R}$, formally writing $\overline{y}$ — we assume that such an extension exists under

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

any specialisation of y we will use. To simplify the formulas, we adopt the notation $\alpha = v - v^{-1}, \alpha_0 = v_0 - v_0^{-1}$. Define the elements

$$\beta_n(y) = \prod_{i=1}^{n} (1 + (\overline{y} + \alpha_0) j_i^B + a^{-1} J_i^B) \in \mathcal{H}_{v,v_0}(B_n) \otimes_{\mathcal{A}} \mathcal{R}.$$

By Corollary 2.5.3, the element $\beta_n(y)$ is central in $\mathcal{H}_{v,v_0}(B_n) \otimes_{\mathcal{A}} \mathcal{R}$. We have the following

Theorem 3.2.2 *The equality* $\mathrm{tr}^B_{n,y} = \mathrm{tr}_{\beta_n(y)}$ *holds.*

We will use the following lemma, verified by a direct computation.

Lemma 3.2.3.

(a) We have $T_n^{-1} = T_n - \alpha_0$.

(b) The elements T_n, T_n^{-1} *satisfy*

$$T_n \in \mathrm{span}_{\mathcal{A}} \langle t_w, w \in W(B_n) \backslash W(B_{n-1}) \rangle$$

and

$$T_n^{-1} \in \mathrm{span}_{\mathcal{A}} \langle t_w^{-1}, w \in W(B_n) \backslash W(B_{n-1}) \rangle.$$

(c) The element $(j_n^B)^{-1} T_n$ *satisfies*

$$(j_n^B)^{-1} T_n - 1 \in \mathrm{span}_{\mathcal{A}} \langle t_w^{-1}, w \in W(B_n) \backslash W(B_{n-1}) \rangle.$$

The property in Theorem 3.2.1 (B) follows from the more general property (compare [5, Proposition 4.5]).

Proposition 3.2.4. *For any* $h \in \mathcal{H}(B_{n-1})$, *we have* $\mathrm{tr}_{\beta_n(y)}(\iota(h) T_n) = y \mathrm{tr}_{\beta_{n-1}(y)}(h)$.

Proof. To simplify the formulas, introduce the notation $y' = y - \alpha_0, \overline{y'} = \overline{y} + \alpha_0$. We have

$$\langle \beta_n(y), \iota(h) T_n \rangle = \langle \beta_{n-1}(y), \iota(h) T_n \rangle + y' \langle \beta_{n-1}(y) j_n^B, \iota(h) T_n \rangle + a \langle \beta_{n-1}(y) J_n^B, \iota(h) T_n \rangle.$$

Let us examine the summands on the right-hand side of this equation one by one. We will routinely use Corollary 2.3.2. For the first summand, we have, by Lemma 3.2.3 (a), (b),

$$\langle \beta_{n-1}(y), \iota(h) T_n \rangle = \langle \beta_{n-1}(y), \iota(h) T_n^{-1} \rangle + \alpha_0 \langle \beta_{n-1}(y), \iota(h) \rangle = \alpha_0 \langle \beta_{n-1}(y), \iota(h) \rangle.$$

For the second summand, first note that j_n^B is in the centraliser of $\mathcal{H}(B_{n-1})$ in $\mathcal{H}(B_n)$. It follows that

$$y' \langle \beta_{n-1}(y) j_n^B, \iota(h) T_n \rangle = y' \langle \beta_{n-1}(y), \iota(h) (j_n^B)^{-1} T_n \rangle.$$

Thus, we have, by Lemma 3.2.3 (c),

$$y' \langle \beta_{n-1}(y), \iota(h) (j_n^B)^{-1} T_n \rangle = y' \langle \beta_{n-1}(y), \iota(h) \rangle.$$

For the third summand, by Proposition 2.4.2 and Lemma 3.2.3 (b), we have

$$a \langle \beta_{n-1}(y) J_n^B, \iota(h) T_n \rangle = \overline{\langle \iota(h) T_n, \beta_{n-1}(y) \mathbb{S}(B_{n-1}) \rangle} = 0.$$

Collecting all three summands together, we get

$$\langle \beta_n(y), \iota(h) T_n \rangle = (\alpha_0 + y') \langle \beta_{n-1}(y), \iota(h) \rangle = y \langle \beta_{n-1}(y), \iota(h) \rangle,$$

which completes the proof. □

Proof of Theorem 3.2.2. It remains to check the properties (U1) and (M2). Note that the defining expression (1) for j_n^B is reduced as a word in generators t_i and ends in t_{n-1}, so $j_n^B = t_c, j_n^B t_{n-1}^{-1} = t_{c'}$ for some

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

$c, c' \in W(B_n)\backslash W(B_{n-1})$. It follows from Corollary 2.3.2 that, for any $h \in \mathcal{H}(B_{n-1})$ we have

$$\langle(\bar{y}+\alpha_0)\beta_{n-1}j_n^B, \iota(h)\rangle = 0 = \langle(\bar{y}+\alpha_0)\beta_{n-1}j_n^B, \iota(h)t_{n-1}\rangle,$$

which reduces the proof to the proof of Theorem 3.1.4. □

From Theorems 3.2.1 and 3.2.2, we get the following description of a Markov trace $\mathrm{tr}_{n,y}^D$.

Corollary 3.2.5. *Let*

$$P_n(x_1, \ldots, x_n; y) = \prod_{i=1}^{n}(1 + \bar{y}x_i + a^{-1}x_i^2),$$

a symmetric polynomial in the variables (x_i) over $\mathcal{R}$. Write $P_n = P_n^+ + P_n^-$, a sum of even and odd monomial degree parts. Let $\delta_n(y) = P_n^+(j_1^D, \ldots, j_n^D; y)$. Then $\mathrm{tr}_{n,y}^D = \mathrm{tr}_{\delta_n(y)}$.

Proof. Recall that the trace $\mathrm{tr}_{n,(y_\bullet)}^D$ is obtained by restriction of $\mathrm{tr}_{n,(y_\bullet)}^B$ along the embedding $\iota_{D\to B} : \mathcal{H}(D_n) \to \mathcal{H}_{v,1}(B_n)$, so that we have

$$\mathrm{tr}_{n,y}^D(h) = \langle\beta_n(y), \iota_{D\to B}(h)\rangle = \langle P_n(j_1^B, \ldots, j_n^B; y), \iota_{D\to B}(h)\rangle.$$

It remains to observe that for all i we have $j_i^D = t_0 j_i^B$, and so every even degree monomial in j_i^B is equal to the same monomial in j_i^D, since $t_0^2 = 1$ in $\mathcal{H}_{v,1}(B_n)$ and t_0 commutes with j_i^B for all i; and that every odd degree monomial in j_i^B is orthogonal to $\iota_{D\to B}(h)$ with respect to $\langle\,,\rangle$. This completes the proof of the Corollary. □

We have the following analogue of Proposition 3.2.4 in type D.

Proposition 3.2.6. *For $n \geq 1$ and any $h \in \mathcal{H}(D_{2n-2})$, we have*

$$\mathrm{tr}_{2n,y}^D(\iota\circ\iota(h)U_{2n-1}U_{2n}) = y^2\mathrm{tr}_{2n-2,y}^D(h).$$

Proof. Note that we have $\iota_{D\to B}(U_k) = t_0T_k$, and so

$$\mathrm{tr}_{2n,y}^D(\iota\circ\iota(h)U_{2n-1}U_{2n}) = \mathrm{tr}_{2n,y}^B(\iota\circ\iota(h)t_0T_{2n-1}t_0T_{2n}).$$

Using the property (M1) and Proposition 3.2.4 twice, we get the result. □

3.3. Geometric Markov traces

Markov traces $\mathrm{tr}_n^B : \mathcal{H}_{v,v}(B_n) \to \mathcal{A}(a)$, $\mathrm{tr}_n^D : \mathcal{H}(D_n) \to \mathcal{A}(a)$ studied in [7, 20] are specialisations of traces $\mathrm{tr}_{n,y}^X$ satisfying the extension of the properties (U1) and (M2) for the embeddings $B_0 \to B_1$ and $D_1 \to D_2$, $D_0 \to D_1$. Since these traces are of geometric origin, we call them 'geometric Markov traces'. In particular, these traces satisfy

$$\mathrm{tr}_1^B(t_0) = v - v^{-1}, \mathrm{tr}_1^D(t_1) = v - v^{-1}, \mathrm{tr}_2^D(t_1t_{1'}) = (v - v^{-1})^2.$$

The condition on tr_1^B forces $y = v - v^{-1}$ and $\bar{y} + \alpha_0 = 0$. Thus, we get that $\mathrm{tr}_n^B = \mathrm{tr}_{\zeta_n}$, the trace from Theorem 3.1.4.

The conditions on $\mathrm{tr}_1^D, \mathrm{tr}_2^D$ imply that

$$y^2 = \mathrm{tr}_2^D(U_1U_2) = \mathrm{tr}_2^D(t_{1'}t_1^{-1}) = \mathrm{tr}_2^D(t_1t_{1'}) - \alpha\mathrm{tr}_2^D(t_{1'}) = \alpha^2 - \alpha^2(1+a) = -\alpha^2 a.$$

It is easy to see that geometric Markov traces $\mathrm{tr}_n^B, \mathrm{tr}_n^D$ take values in $\mathcal{A}[a]$. Let $\mathrm{tr}_n^{X,k} : \mathcal{H}(X_n) \to \mathcal{A}$ be the coefficient of tr_n^X at a^k. We summarise the above discussion in the following

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

Theorem 3.3.1. *The functional* $\mathrm{tr}_n^{B,k}$ *is represented by the kth elementary symmetric polynomial in Jucys–Murphy elements:*

$$\mathrm{tr}_n^{B,k}(h) = \langle e_k(J_1^B, \ldots, J_n^B), h\rangle.$$

Let $e'_k(x_1, \ldots, x_n; \alpha)$ *be the symmetric polynomial in* (x_i) *appearing as a degree* $2k$ *homogeneous part in*

$$\prod_{i=1}^{n}(1 + \alpha x_i - x_i^2) = \prod_{i=1}^{n}(v - x_i)(v^{-1} + x_i),$$

where $\alpha = v - v^{-1}$. *Then*

$$\mathrm{tr}_n^{D,k}(h) = (-1)^k\langle e'_k(j_1^D, \ldots, j_n^D; \alpha), h\rangle.$$

The symmetric polynomial e'_k can be written explicitly as

$$e'_k = \sum_{\substack{i+j=2k,\\ i,j\geq 0}} (-1)^i v^{j-i} e_i e_j.$$

Note that the highest a-coefficient is always represented by the 'full twist' element $\mathbb{S}^{-1} = e_n(J_1^X, \ldots, J_n^X)$, $\mathrm{tr}_n^{X,n} = \mathrm{tr}_{\mathbb{S}^{-1}}$. This recovers the symmetry of the HOMFLY-PT polynomial observed by Kálmán [11] that was categorified in [4] in type A and in [2] in general type.

Remark 3.3.2. *The statement analogous to Theorem 3.3.1 was lifted to the level of Khovanov–Rozansky homology in [2]. Namely, there it is shown that the object representing the kth degree of the invariant has a filtration by the objects representing products of Jucys–Murphy elements. For an arbitrary Weyl group, the object representing the kth degree was constructed in [2] in terms of some explicit character sheaves on a reductive group G with the Weyl group W. It was also explained how these character sheaves correspond to exterior powers of the reflection representation of W. We expect that, if the analogous filtration categorifying expressions in Theorem 3.3.1 can be constructed in types B and D, the arguments in this paper can be lifted to the level of Khovanov–Rozansky homology: the pairing* $\langle , \rangle$ *is categorified to the* Hom*-pairing on the Hecke category; the 'Serre property' of Proposition 2.4.2 has a categorical analogue in the description of the Serre functor of the Hecke category [1, 4]; elements* j_n^X *can be expressed using the braid group action on the Hecke category. Such an extension would, for example, give a geometric interpretation to the special properties of Markov traces in types* B_n *and* D_n *involving the elements* T_n, U_n. *We hope to address this in a future work.*

We would also like to remark that, in type A, the fact that Jucys–Murphy elements and their symmetric polynomials come from the extended affine Hecke algebra has an interpretation in terms of geometric representation theory, see [6, 15 and 19]. We do not know if this interpretation can be extended to other types.

Acknowledgements. The results of this paper were obtained while the authors participated in Yulia's Dream programme. We would like to thank the programme's organisers for this wonderful initiative. We would like to thank Mykhailo Barkulov, another participant of the programme, for helpful discussions.

We have conducted numerical experiments using SageMath software [18] throughout the work on this paper and would like to thank its developers.

K.T. was funded by the Deutsche Forschungsgemeinschaft SFB 1624 grant, Projektnummer 506632645.

References

[1] A. Beilinson, R. Bezrukavnikov and I. Mirković, Tilting exercises, *Mosc. Math. J.* **4**(2004), no. 3, 547–557.

[2] R. Bezrukavnikov and K. Tolmachov, Monodromic model for Khovanov–Rozansky homology, *J. Reine Angew. Math.* **2022** (2022), no. 787, 79–124.

[3] M. Geck, Trace functions on Iwahori-Hecke algebras, *Banach Center Publ.* **42**(1998), no. 1, 87–109.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050

[4] E. Gorsky, M. Hogancamp, A. Mellit and K. Nakagane, Serre duality for Khovanov–Rozansky homology, *Sel. Math.* **25**(2019), no. 5, 79.
[5] M. Geck and S. Lambropoulou, Markov traces and knot invariants related to Iwahori-Hecke algebras of type B, *J. Reine Angew. Math.* **485** (1997), 191–214.
[6] E. Gorsky, A. Neguţ and J. Rasmussen, Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology, arXiv preprint arXiv:1608.07308, 2016.
[7] Y. Gomi, The Markov traces and the fourier transforms, *J. Algebra* **303**(2006), no. 2, 566–591.
[8] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, Vol. **21**., Oxford University Press, 2000.
[9] Q. P. Ho and P. Li, Relative Serre duality for Hecke categories, arXiv preprint arXiv:2504.12798, 2025.
[10] V. F. R. Jones, "Hecke algebra representations of braid groups and link polynomials" in *New Developments in the Theory of Knots*, World Scientific, 1987, 20–73.
[11] T. Kálmán, Meridian twisting of closed braids and the Homfly polynomial, *Math. Proc. Camb. Philos. Soc.* **146** (2009), no. 03, 649.
[12] M. Khovanov, Triply-graded link homology and Hochschild homology of Soergel bimodules, *Int. J. Math.* **18**(2007), no. 08, 869–885.
[13] S. Lambropoulou, "Solid torus links and Hecke algebras of B-type" in *Proceedings of the Conference on Quantum Topology (Manhattan, KS. 1993)*, River Edge, NJ, World Sci. Publ. Co., Inc., 1994, 225–245.
[14] G. Lusztig, "Singularities, character formulas, and a q-analog of weight multiplicities" in *Analyse et topologie sur les espaces singuliers (II-III) - 6 - 10 juillet 1981, Astérisque, no. 101–102*, 1983.
[15] A. Oblomkov and L. Rozansky, Affine braid group, JM elements and knot homology, *Transform Groups* **24**(2019), no. 2, 531–544.
[16] A. Ram, Seminormal representations of Weyl groups and Iwahori-Hecke algebras, *Proc. Lond. Math. Soc.* **75**(1997), no. 1, 99–133.
[17] A. Ram and J. Ramagge, "Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory" In *A Tribute to C. S. Seshadri: Perspectives in Geometry and Representation Theory*, Hindustan Book Agency, Gurgaon, 2003, 428–466.
[18] Sage Developers, *SageMath, the Sage Mathematics Software System* (Version 10.6), 2025, https://www.sagemath.org.
[19] K. Tolmachov, Linear structure on a finite Hecke category in type A, arXiv preprint arXiv:2403.19734, 2024.
[20] B. Webster and G. Williamson, The geometry of Markov traces, *Duke Math. J.* **160**(2011), no. 2, 401–419.
[21] B. Webster and G. Williamson, A geometric construction of colored HOMFLYPT homology, *Geom. Topol.* **21**(2017), no. 5, 2557–2600.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 17 Nov 2026 at 06:14:06, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089526101050