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Sectional-hyperbolic systems

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Abstract. We introduce a class of vector fields on n-manifolds containing the hyperbolic systems, the singular-hyperbolic systems on 3-manifolds, the multidimensional Lorenz attractors and the robust transitive singular sets in Li *et al* [Robust transitive singular sets via approach of an extended linear Poincaré flow. *Discrete Contin. Dyn. Syst.* **13**(2) (2005), 239–269]. We prove that the closed orbits of a system in such a class are hyperbolic in a persistent way, a property which is false for higher-dimensional singular-hyperbolic systems. We also prove that the singularities in the robust transitive sets in Li *et al* are similar to those in the multidimensional Lorenz attractor. Our results will give a partial negative answer to Problem 9.26 in Bonatti *et al* [*Dynamics Beyond Uniform Hyperbolicity. A Global Geometric and Probabilistic Perspective (Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III)*. Springer, Berlin, 2005].

1. *Introduction*

This work is motivated by the main result in [**LGW**] which proves that certain robust transitive sets with singularities for C^1-vector fields on compact n-manifolds are partially hyperbolic. Such a result was presented as an extension of the main result in [**MPP2**] where it was proved that *every* robust transitive set is partially hyperbolic for $n = 3$. However, [**MPP2**] proved not only partial hyperbolicity but also that every singularity is Lorenz-like and also singular-hyperbolicity, i.e. volume expansion along the central subbundle. It is then natural to ask if these last properties can be proved for the robust transitive sets in [**LGW**] as well.

In this paper we give a positive answer to this question. Indeed, we show that the singularities contained in the robust transitive sets in [**LGW**] are similar to those exhibited by the *multidimensional Lorenz attractor* [**BPV**]. Moreover, we show that the robust

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transitive sets considered in [**LGW**] are *sectional-hyperbolic*, that is, the derivative of their corresponding flows *expand the area of parallelograms along their central subbundle*.

Sectional-hyperbolicity appears as a new kind of hyperbolicity which satisfies some interesting properties. First, we notice that the sectional-hyperbolicity implies singular-hyperbolic and, further, it reduces to singular-hyperbolicity precisely when $n = 3$. In addition, we see that the sectional-hyperbolicity implies the so-called $\mathcal{G}^1$ (or star) property in the sense that the closed orbits in a sectional-hyperbolic system are hyperbolic in a persistent way. Observe that $\mathcal{G}^1$ is false for singular-hyperbolic systems in dimension four by the *wild strange attractor* [**TS**]. Therefore, the class of sectional-hyperbolic systems is properly contained in that of singular-hyperbolic systems. We also prove that the class of sectional-hyperbolic systems contains not only the hyperbolic sets of saddle type and the robust transitive sets [**LGW**] but also the aforementioned multidimensional Lorenz attractor. Our results will give a partial negative answer to [**BDV**, Problem 9.26].

The organization of this paper is as follows. In §2 we present the main definitions including sectional-hyperbolicity and sectional-Axiom A. In §3 we discuss the relationship between sectional-hyperbolicity and robust transitivity as a complement to the main result in [**LGW**]. In §4 we discuss the sectional-hyperbolicity of the multidimensional Lorenz attractor and the wild strange attractor [**TS**]. We also state the $\mathcal{G}^1$ property of sectional-Axiom A vector fields without cycles. Ergodic properties of sectional-hyperbolic systems such as the existence of Sinai–Ruelle–Bowen (SRB) measures and stochastic stability extensions [**APPV**] will appear in a forthcoming work [**BMM**]. This and other related questions and remarks will be discussed in §5.

To finish we have to mention that our results do not use the hypothesis of the existence of periodic orbits. In fact, such a hypothesis is used in [**LGW**] only to avoid the strongly homogeneous hypothesis (see, for instance, [**LGW**, Lemma 1.6, p. 242]). The existence of periodic orbits for robust transitive sets with singularities in dimension three was recently proved in [**BM**] using the theory of singular-hyperbolic systems.

2. *Basic definitions*

In this section we state the definition of sectional-hyperbolic set in a precise way. We first state some basic concepts which are to be used in what follows.

Throughout this paper M will denote a compact connected boundaryless manifold and $\chi^r(M)$ will denote the space of C^r vector fields on M with the C^r topology, $r \geq 1$. We fix $X \in \chi^1(M)$ and denote by X_t the flow generated by X in M. Define the non-wandering set of X as the set $\Omega(X)$ consisting of those points p such that for every neighborhood U of p and every $T > 0$ there is a $t > T$ such that $X_t(U) \cap U \neq \emptyset$. A compact invariant set Λ of X is *singular* if it contains a singularity of X and *non-trivial* if it does not reduce to a single orbit. We say that Λ is *isolated* if it has a compact neighborhood U, called the *isolating block*, such that $\Lambda = \Lambda_X(U)$ where

$$\Lambda_X(U) = \bigcap_{t \in \mathbb{R}} X_t(U)$$

is the maximal invariant set of X_t in U. We say that Λ is *transitive* if it coincides with the accumulation point set of a positive orbit contained on Λ.

In the following definitions we fix $r \geq 1$ and $X \in \chi^r(M)$ unless otherwise stated.

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Definition 2.1. An isolated set Λ of X is C^r *robust transitive* if there is an isolating block U of Λ such that $\Lambda_Y(U)$ is a non-trivial transitive set of Y for every vector field Y sufficiently C^r close to X.

A *subbundle* F_Λ over $\Lambda \subset M$ is a continuous map $x \in \Lambda \mapsto F_x$ where F_x is a linear subspace of $T_x M$. We say that F_Λ has constant dimension $\dim(F_\Lambda)$ if the integer-valued map $x \in \Lambda \mapsto \dim(F_x)$ takes the constant value $\dim(F_\Lambda)$. If Λ is an invariant set of X we say that F_Λ is invariant if $DX_t(x)(F_x) = F_{X_t(x)}$ for every $x \in \Lambda$ and every $t \in \mathbb{R}$. All subbundles over a transitive set have constant dimension. The Jacobian of a linear map T will be denoted by $\mathrm{Det}(T)$.

Definition 2.2. An invariant subbundle F_Λ over Λ is:

(1) *contracting* if there are positive constants K, λ such that

$$\|DX_t/_{F_x}\| \le Ke^{-\lambda t} \quad \text{for all } x \in \Lambda, \text{ for all } t > 0;$$

(2) *expanding* if F_Λ is contracting for the reversed vector field $-X$;

(3) *volume expanding* if K, λ above can be chosen such that

$$|\mathrm{Det}(DX_t/_{F_x})| \le Ke^{-\lambda t} \quad \text{for all } x \in \Lambda, \text{ for all } t > 0.$$

Next, we introduce the notion of a sectionally expanding subbundle. Previously we observed that if F_Λ is an invariant subbundle over a compact set Λ then there is a disjoint union $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$ of compact sets Λ_i such that each subbundle F_{Λ_i} has constant dimension. This decomposition will be called the F_Λ*-constant decomposition of* Λ.

Definition 2.3. An invariant subbundle F_Λ over a compact invariant subset Λ of X is *sectionally expanding* if each subbundle F_{Λ_i} in the F_Λ-constant decomposition of Λ satisfies one of the following two alternatives:

(1) $\dim(F_{\Lambda_i}) = 1$ and F_{Λ_i} is expanding;

(2) $\dim(F_{\Lambda_i}) \ge 2$ and there are positive constants K, λ such that for every $x \in \Lambda_i$ and every two-dimensional subspace $L_x \subset F_x$ one has

$$|\mathrm{Det}(DX_t(x)/_{L_x})| \ge Ke^{\lambda t} \quad \text{for all } t > 0.$$

The sectionally expanding property expresses nothing but the fact that the derivative of X_t along F_Λ *expands the area of parallelograms* instead of *the length of vectors*. Actually the difference between sectionally expanding and expanding subbundles is that the first one allows at most one negative Lyapunov exponent whereas the second one does not. In particular, an expanding subbundle is sectionally expanding but not conversely.

Now we recall the classical definition of a hyperbolic set.

Definition 2.4. A compact invariant set H of X is *hyperbolic* if there is a continuous invariant tangent bundle decomposition $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ such that:

(1) E^s_H is a contracting subbundle;

(2) E^u_H is an expanding subbundle;

(3) E^X_x is the subbundle generated by $X(x)$ in $T_x M$ for all $x \in H$.

We say that a hyperbolic set H is of *saddle-type* if $E^s_x \ne 0 \ne E^u_x$ for all $x \in H$.

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A closed orbit of X is hyperbolic if it is a hyperbolic set of X. If $U \subset M$ and $r \geq 1$, denote by $\mathcal{G}^r(U)$ the set of $X \in \chi^r(M)$ such that every closed orbit in U of every vector field Y sufficiently C^r-close to X is hyperbolic. The *index* $\mathrm{Ind}(O)$ of a hyperbolic closed orbit O of X is the dimension of its stable subbundle E^s_O. The following definition comes from [**LGW**].

Definition 2.5. A compact invariant set Λ of X is *C^r strongly homogeneous* if there exist a non-negative integer $\mathrm{Ind}(\Lambda)$ and a neighborhood $U \subset M$ of Λ such that $\mathrm{Ind}(O) = \mathrm{Ind}(\Lambda)$, for every hyperbolic periodic orbit $O \subset U$ of every vector field that is C^r close to X.

The following is the concept of dominated splitting which is an extension of the concept of a hyperbolic set.

Definition 2.6. A *dominated splitting* over a compact invariant set Λ of X is a continuous invariant direct sum $T_\Lambda M = E^{ss}_\Lambda \oplus E^c_\Lambda$ with $E^s_x \neq 0$, $E^c_x \neq 0$ for every $x \in \Lambda$ for which there are positive constants K, λ such that

$$\|DX_t/_{E^{ss}_x}\| \cdot \|DX_{-t}/_{E^c_{X_t(x)}}\| \leq Ke^{-\lambda t} \quad \text{for all } x \in \Lambda, \text{ for all } t > 0.$$

A compact invariant set Λ is said to be *partially hyperbolic* if it exhibits a dominated splitting $T_\Lambda M = E^{ss}_\Lambda \oplus E^c_\Lambda$ such that the subbundle E^{ss}_Λ is contracting. In that case we say that E^c_Λ is the *central subbundle* of Λ. A *singular-hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and whose central subbundle is volume expanding.

The definition of singular-hyperbolic set motivates the following.

Definition 2.7. A *sectional-hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and whose central subbundle is sectionally expanding.

Let us present some examples to clarify the definition of sectional-hyperbolic set.

Example 2.8. Every saddle-type hyperbolic set Λ is a sectional-hyperbolic set. Indeed, the hyperbolic splitting $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ of Λ induces a sectional-hyperbolic splitting $T_\Lambda M = E^{ss}_\Lambda \oplus E^c_\Lambda$ for Λ defined by $E^{ss}_\Lambda = E^s_\Lambda$ and $E^c_\Lambda = E^X_\Lambda \oplus E^u_\Lambda$.

The following example says that alternative (2) in Definition 2.3 suffices to characterize non-trivial sectional-hyperbolic transitive sets.

Example 2.9. Let Λ be a transitive set which is not a singularity of a C^1 vector field X. Then, Λ is sectional-hyperbolic if and only if Λ has a partially hyperbolic splitting $T_\Lambda M = E^{ss}_\Lambda \oplus E^c_\Lambda$ with $\dim(E^c_\Lambda) \geq 2$ such that E^c_Λ satisfies the alternative (2) of Definition 2.3.

Example 2.10. Every sectional-hyperbolic set is singular-hyperbolic. The converse is true in dimension three but not in dimensions of at least four (see Theorem 4.3 below).

The following generalizes the concept of a Lorenz-like singularity introduced in [**MPP1**].

Definition 2.11. A *generalized Lorenz-like singularity* of a C^1 vector field X is a singularity σ of X with the following property. σ has a real negative eigenvalue λ_0, at least one eigenvalue different from λ_0 with negative real part and at least one eigenvalue with positive real part such that if we define

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- $\lambda_-(\sigma) = \max\{\mathrm{Re}(\lambda) : \lambda \neq \lambda_0$ is an eigenvalue of σ with negative real part$\}$,
- $\lambda_+(\sigma) = \min\{\mathrm{Re}(\lambda) : \lambda$ is an eigenvalue of σ with positive real part$\}$,

then

$$\lambda_-(\sigma) < \lambda_0 < 0 < -\lambda_0 < \lambda_+(\sigma).$$

Example 2.12. Every Lorenz-like singularity [**MPP1**] of a vector field X in a 3-manifold is generalized Lorenz-like. An example of a generalized Lorenz-like singularity in higher dimension is the singularity of the multidimensional Lorenz attractor. Not every singularity of a sectional-hyperbolic set is generalized Lorenz-like.

Remark 2.13. If σ is a generalized Lorenz-like singularity of a vector field X, then σ is hyperbolic and so it has an associated sectional-hyperbolic splitting $T_\sigma M = E^{ss}_\sigma \oplus E^c_\sigma$ according to Example 2.8. On the other hand, σ is equipped with another sectional-hyperbolic splitting $T_\sigma M = \tilde{E}^{ss}_\sigma \oplus \tilde{E}^c_\sigma$ where $\tilde{E}^{ss}_\sigma$ is the eigenspace associated to the eigenvalues different from λ_0 with negative real part and $\tilde{E}^c_\sigma$ is the eigenspace associated to the eigenvalues in $\{\lambda_0\} \cup \{\lambda \mid \lambda$ has positive real part$\}$. In particular, the sectional-hyperbolic splitting of a sectional-hyperbolic set is not unique.

We finish with the definition of a sectional-Axiom A vector field extending the definition of singular-Axiom A vector fields in [**MPP1**, p. 3396]. First, recall that a *sink* of a vector field X is a hyperbolic closed orbit with zero unstable subbundle E^u. A *source* of X is a sink for the time reversed vector field $-X$.

Definition 2.14. A C^r vector field X on a closed manifold is *sectional-Axiom A* if $\Omega(X)$ is the closure of the closed orbits of X and is a disjoint union of isolated transitive sets

$$\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k,$$

where each Λ_i is either a sink or a sectional-hyperbolic set for X or $-X$. We say that X *has no cycles* if there are no regular orbits linking the Λ_i in a cyclic way.

3. *Sectional-hyperbolicity and robust transitivity*

Here after we fix a C^1 vector field X on a compact manifold M. We also fix a C^1 strongly homogeneous and C^1 robust transitive singular set Λ of X such that every singularity $\sigma \in \Lambda$ of X is hyperbolic of index $\mathrm{Ind}(\sigma) > \mathrm{Ind}(\Lambda)$.

Our first result gives a characterization of the singularities in Λ.

THEOREM 3.1. *Every singularity of X in Λ is generalized Lorenz-like.*

Proof. It follows from [**LGW**, Theorem A] that Λ has a partially hyperbolic splitting

$$T_\Lambda M = E^{ss}_\Lambda \oplus E^c_\Lambda. \tag{1}$$

Let $\sigma \in \Lambda$ be a singularity of X. Then, σ is hyperbolic by hypothesis so there is another (hyperbolic) splitting $T_\sigma M = E^s_\sigma \oplus E^u_\sigma$ on σ. The splitting (1) induces a splitting $T_\sigma M = E^{ss}_\sigma \oplus E^c_\sigma$ on σ with $\dim(E^{ss}_\sigma) = \mathrm{Ind}(\Lambda)$. As E^{ss}_σ is contracting we have $E^{ss}_\sigma \subset E^s_\sigma$. By [**LGW**, Lemma 4.1, p. 253], E^s_σ splits into a dominated splitting $E^s_\sigma = E^{ss}_\sigma \oplus E_\sigma$ for some subspace E_σ.

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We claim that $\dim(E_\sigma) = 1$. Indeed, there is a strong stable manifold W^{ss}_σ tangent to E^{ss}_σ at σ. Note that $W^{ss}_\sigma \cap \Lambda = \{\sigma\}$ by [**LGW**, Lemma 4.3, p. 255]. Consequently, the periodic orbits of Λ accumulate on the direct sum $E_\sigma \oplus E^u_\sigma$.

By making a C^1-perturbation if necessary we can assume that there is a weak contracting eigenvalue $\lambda_E(\sigma)$ of σ whose eigenspace E'_σ is contained in E_σ. Analogously there is a weak expanding eigenvalue $\lambda_u(\sigma)$ with positive real part whose eigenspace E^*_σ is contained in E^u_σ.

By central manifold arguments (e.g. [**HPS**]) we have that the periodic orbits of Λ accumulate on the direct sum $E'_\sigma \oplus E^*_\sigma$. Then, by the Connecting lemma [**H**], there is a homoclinic loop γ associated to σ issuing from σ tangent to E^*_σ and returning to σ tangent to E'_σ.

If there were another eigenvalue $\beta \neq \lambda_E(\sigma)$ with eigenspace contained in E_σ, then by bifurcating γ we would obtain a periodic orbit of index $\geq \mathrm{Ind}(\Lambda) + 1$ where the additional contracting direction comes from β. However, this is absurd because Λ is strongly homogeneous of index $\mathrm{Ind}(\Lambda)$. Consequently, $E_\sigma = E'_\sigma$ and so $\dim(E_\sigma) = 1$ or 2 depending on whether $\lambda_E(\sigma)$ is real or complex.

If $\lambda_E(\sigma)$ is real we are done.

If $\lambda_E(\sigma)$ is complex we get a contradiction as follows. We have that $\lambda_u(\sigma)$ may be real or complex. If $\lambda_u(\sigma)$ were real then we can apply the analysis in [**AH**, Ch. 6] to prove that γ bifurcates to a periodic orbit which either has index $\geq \mathrm{Ind}(\Lambda) + 1$ or $\leq \mathrm{Ind}(\Lambda) - 1$. In any case we get a contradiction. So we can assume that $\lambda_u(\sigma) = \rho + i\Omega$ is a complex number with $\rho > 0$. Also denote $\lambda_E(\sigma) = \lambda + i\omega$ with $\lambda < 0$. It turns out that γ is a *bifocal homoclinic orbit* (see [**FS**] for a definition).

By using central manifold arguments [**HPS**] we can assume that γ is associated to a four-dimensional manifold tangent to the direct sum $E'_\sigma \oplus E^*_\sigma$. We can also assume by small perturbation that $-\lambda \neq \rho$ and that the stable and unstable manifolds intersect almost transversely along γ.

Hence by the theorem in [**Sh**, p. 54] we have that γ carries a countable set of periodic orbits of saddle-type (note that the analytic condition required in [**Sh**] is not necessary, see [**FS**] or [**Wig**, p. 261]). Increasing the eigenspace E^{ss}_σ, we obtain that these periodic orbits have index $\geq \mathrm{Ind}(\Lambda) + 1$ which is absurd because Λ is C^1 strongly homogeneous of index $\mathrm{Ind}(\Lambda)$. This contradiction proves the claim.

Now we use the claim to conclude the proof. By the claim we have $\dim(E_\sigma) = 1$, and so E_σ is the eigenspace associated to a real eigenvalue λ_0. It turns out that λ_0 is negative because $E_\sigma \subset E^s_\sigma$. On the one hand, since E^{ss}_σ has non-zero dimension $\mathrm{Ind}(\Lambda)$ and the sum $E^{ss}_\sigma \oplus E_\sigma$ is direct, we conclude that σ has at least one eigenvalue $\neq \lambda_0$ with negative real part. On the other hand, σ must have an eigenvalue with positive real part for otherwise σ would be an attracting singularity. Therefore, $\Lambda = \{\sigma\}$ contradicting the fact that Λ is non-trivial.

Now, recall the definition of $\lambda_-(\sigma)$, $\lambda_+(\sigma)$ in Definition 2.11. Since $E^s_\sigma = E^{ss}_\sigma \oplus E_\sigma$ is dominated we conclude that $\lambda_-(\sigma) < \lambda_0$. In addition, if $\lambda_u(\sigma)$ is the weak expanding eigenvalue of σ then $-\lambda_0 < \mathrm{Re}(\lambda_u(\sigma))$ for otherwise we would produce a periodic orbit of index $\neq \mathrm{Ind}(\Lambda)$ by just bifurcating a homoclinic loop obtained from the Connecting lemma as before. Since $\lambda_u(\sigma)$ is the weak expanding eigenvalue we have $\mathrm{Re}(\lambda_u(\sigma)) = \lambda_+(\sigma)$

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so therefore $-\lambda_0 < \lambda_+(\sigma)$. It follows that $\lambda_-(\sigma) < \lambda_0 < 0 < -\lambda_0 < \lambda_+(\sigma)$ and so σ is generalized Lorenz-like. □

Remark 3.2. The proof of Theorem 3.1 implies that the singularities $\sigma \in \Lambda$ of X have constant index $\mathrm{Ind}(\sigma) = \mathrm{Ind}(\Lambda) + 1$ (observe that we have only required that $\mathrm{Ind}(\sigma) > \mathrm{Ind}(\Lambda)$). This brings a negative answer to [**BDV**, Problem 9.26], at least for the C^1 strongly homogeneous robust transitive sets considered in Theorem 3.1.

To prove our second result we shall need the following technical lemma. If O is a hyperbolic periodic orbit of a C^1 vector field Z, we denote by $T_O M = E^s_O \oplus E^Z_O \oplus E^u_O$ the hyperbolic splitting over O.

LEMMA 3.3. *There exist a neighborhood $\mathcal{U}$ of X in $\chi^1(M)$ and a neighborhood U of Λ in M such that if $Y \in \mathcal{U}$ then there exist $0 < \lambda < 1$ and a neighborhood $\mathcal{V} \subset \mathcal{U}$ of Y such that if $Z \in \mathcal{V}$ and p is a periodic point of Z with period t_p and orbit contained in U, then*

$$\|DZ_{-t_p}(p)/_{E^u_p}\| \le \lambda^{t_p}.$$

Proof. The proof is similar to the proof of [**MPP2**, Theorem 3.6(a), p. 412]. Indeed, in the present case, one uses the Franks lemma for flows ([**MPP2**] or [**BGV**, Appendix, p. 24]) with the C^1 strongly homogeneous hypothesis playing the role of the property of the non-existence of sinks or sources used there. □

The second result of this section is a complement of [**LGW**, Theorem A].

THEOREM A. *Λ is a sectional-hyperbolic set of X.*

Proof. As observed in the proof of Theorem 3.1, we have that Λ has a partially hyperbolic splitting (1). On the other hand, every singularity of Λ is hyperbolic by hypothesis. Then, to prove that Λ is sectional-hyperbolic, it suffices to show that the central subbundle E^c_Λ of this splitting is sectionally expanding. For this we first prove that

$$\liminf_{t\to\infty} |\mathrm{Det}\, DX_{-t}(x)/_{L_x}| = 0, \tag{2}$$

for every $x \in \Lambda$ and every two-dimensional subspace $L_x \subset E^c_x$. To prove it we let $\mathcal{U}$ and U be the neighborhoods of X and Λ, respectively, in Lemma 3.3. Put $Y = X$ in Lemma 3.3 and fix the number λ and the neighborhood $\mathcal{V} \subset \mathcal{U}$ in that lemma.

If (2) were not true then we obtain a contradiction as follows. Let $\gamma < 0$ be a real number close to 0. Since the singularities are generalized Lorenz-like by Theorem 3.1 we can follow [**MPP2**, p. 405] and apply the ergodic closing lemma for flows [**H**] to find: $Z \in \mathcal{V}$, a periodic point p with period t_p of Z with orbit lying in U, and a two-dimensional subspace $L_p \subset E^c_p$ such that

$$|\mathrm{Det}\, DZ_{-t_p}(p)/L_p| \ge (e^\gamma)^{t_p}.$$

As

$$\|DZ_{-t_p}(p)\| \ge \sqrt{|\mathrm{Det}\, DZ_{-t_p}(p)/_{L_p}|},$$

we would obtain

$$\|DZ_{-t_p}(p)\| \ge (e^{\gamma/2})^{t_p}.$$

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This together with the conclusion of Lemma 3.3 yields

$$\lambda^{t_p} \geq (e^{\gamma/2})^{t_p},$$

which is absurd since λ is fixed and γ is close to 0. Therefore (2) is true.

Once we prove (2) we can prove that E^c_Λ is sectionally expanding by using the arguments in the proof of [**MPP2**, Lemma 3.9, p. 401] together with the compactness of the Grassmann manifold of two-dimensional planes (e.g. [**MS**, p. 55]). The proof follows. □

4. *Multidimensional Lorenz and wild strange attractors*

For completeness we give a brief description of the multidimensional Lorenz attractor [**BPV**]. Consider an expanding map $f : T^k \to T^k$ of the k-dimensional torus $T^k = S^1 \times \overset{k}{\cdots} \times S^1$, $k \geq 2$. Denote by D^2 the closed unit ball in $\mathbb{R}^2$ and denote $N = T^k \times D^2$. Note that N carries a foliation by 2-disks $\mathcal{F}^s = \{* \times D^2 : * \in T^k\}$ with leaf space $N/\mathcal{F}^s = T^k$. Denote by $\mathrm{Int}(N)$ the interior of N. Let $F : N \to \mathrm{Int}(N)$ be a C^∞ diffeomorphism which preserves and contracts $\mathcal{F}^s$ such that the map induced by F in the leaf space $N/\mathcal{F}^s = T^k$ is f. It follows that the maximal invariant set $A_F = \bigcap_{n\geq 0} F^n(N)$ is a hyperbolic attractor with stable foliation $\mathcal{F}^s$ of F. Consider the suspension X^F of F which is a C^∞ vector field defined in the suspended manifold M^F with a global cross section $\Sigma = N \times 0$. To prove transitivity of the attractor [**BPV**] impose the condition that the modulus of the derivative of f is uniformly larger than $\max\{2, 2(\Delta/R)\}$ where R and Δ are respectively the injectivity radius of the exponential map and the diameter of T^k.

Pick a point $\xi \in A_F \subset \Sigma$ and consider the piece of orbit $\overline{\xi F(\xi)}$ of X^F from ξ to its first return point $F(\xi)$ to Σ. Consider a small neighborhood of the form $U \times 0$ of ξ. Let $V = X_{[0,1]}(U \times 0)$; then V is a neighborhood of $\overline{\xi F(\xi)}$ in M^F. By surgery on V we can replace the flow of the vector field restriction $X^F/_V$ by a Cherry-like one. More precisely, we create a singularity σ inside V with stable manifold W^s_σ of dimension $k+1$. Observe that σ has a strong stable manifold W^{ss}_σ of dimension k. In this way we obtain a new vector field Y having $\Sigma = T^k \times D^2$ as a cross section. Moreover, there is a return map Φ induced by Y in Σ such that:

(P1) Φ admits $\mathcal{F}^s$ as a contracting C^∞ foliation;

(P2) the quotient space of $\mathcal{F}^s$ is diffeomorphic to T^k and the map $\phi : T^k \setminus \{\xi\} \to T^k$ induced by Φ in the leaf space of $\mathcal{F}^s$ has derivative larger that $\max\{2, 2(\Delta/R)\}$.

These properties are preserved by small C^1 perturbations Z of Y. The *multidimensional Lorenz attractor* is the attracting set of Y defined by

$$A(Y) = \bigcap_{T>0} \mathrm{closure}\left(\bigcup_{t>T} Z_t(\Sigma)\right).$$

The following corollary of Theorem A shows that the multidimensional Lorenz attractor is sectional-hyperbolic.

COROLLARY 4.1. *$A(Y)$ is sectional-hyperbolic.*

Proof. We can prove this in a direct way by verifying the required properties in the previous construction. Here we proceed in an indirect way using Theorem A. More precisely, we shall prove that $A(Y)$ is a C^1 strongly homogeneous and C^1 robust transitive set with a

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unique singularity σ which is hyperbolic and satisfies $\mathrm{Ind}(\sigma) > \mathrm{Ind}(A(Y))$. That $A(Y)$ has a unique singularity σ which is hyperbolic is obvious from the construction and $A(Y)$ is C^1 robust transitive by part (1) of the main theorem in [**BPV**, p. 885]. Next we observe that $A(Y)$ is C^1 strongly homogeneous as the dimension Ind of the stable direction of every periodic orbit P close to $A(Y)$ of a vector field Z that is C^1 close to Y is precisely 2. Hence $\mathrm{Ind}(A(Y)) = 2$. Observe also that the stable dimension $\mathrm{Ind}(\sigma)$ of the singularity σ is precisely 3. Then,

$$\mathrm{Ind}(\sigma) = 3 > 2 = \mathrm{Ind}(A(Y)),$$

and so Theorem A applies. □

Recall that, given $U \subset M$ and $r \geq 1$, we denote by $\mathcal{G}^r(U)$ the set of C^r vector fields X such that every closed orbit in U of every vector field that is C^r close to X is hyperbolic. Observe that $\mathcal{G}^1(U)$ is the local version of the space $\mathcal{G}^1(M)$ considered in [**H**].

LEMMA 4.2. *Consider $r \geq 1$ and take Λ to be a sectional-hyperbolic set of a C^r vector field X on a compact manifold M. Then, there is an open subset $U \subset M$ containing Λ such that every compact invariant* non-singular *subset in U of every vector field that is C^r close to X is hyperbolic. In particular, $X \in \mathcal{G}^r(U)$ and if Λ is transitive, then Λ is C^r strongly homogeneous as well.*

Proof. By the compactness of the Grassmann space of two-dimensional subspaces we have that the sectionally expansiveness of a subbundle is preserved by C^r perturbations. In particular, there is a neighborhood U of Λ such that every compact invariant non-singular set $H \subset U$ of every Y that is C^r close to X has a partially hyperbolic splitting $T_H M = E_H^{ss,Y} \oplus E_H^{c,Y}$ with sectionally expanding central subbundle $E_H^{c,Y}$.

One can prove that H is hyperbolic for Y by repeating [**MPP3**, Lemmas 1 to 3, p. 3397]. Indeed, let N_H be the normal bundle induced by Y in H (this bundle is well defined since H has no singularities). By projecting DY_t onto N_H we obtain the so-called Poincaré flow P_t over H. To prove that H is hyperbolic it suffices to prove that P_t is hyperbolic. Notice that P_t already has a contracting subbundle, e.g. the one induced by the contracting subbundle $E_H^{ss,Y}$. Moreover, the sectional expansivity of $E_H^{c,Y}$ implies that if $F_H = N_H \cap E_H^{c,Y}$ then the product $m(P_t/_{F_H}) \cdot m(DX_t/_{E_H^Y})$ grows exponentially as $t \to \infty$, where E_H^Y is the one-dimensional subbundle generated by Y in H. But Y has no singularities in H so $m(DY_t/_{E_H^Y})$ is uniformly bounded. It follows that $m(P_t/_{F_H})$ grows exponentially as $t \to \infty$. So, the subbundle F_H is expanding for P_t proving that P_t is hyperbolic.

By taking H as a periodic orbit of Y we obtain that $X \in \mathcal{G}^1(U)$. On the other hand, if Λ is transitive, then it is connected and so the dimension of its strong stable subbundle E_Λ^{ss} is a constant denoted by $\dim(\Lambda)$. In such a case Λ is C^r strongly homogeneous with index $\dim(\Lambda)$. This proves the lemma. □

COROLLARY 4.3. *The wild strange attractor* [**TS**] *is singular but not sectional-hyperbolic.*

Proof. The attractor is singular-hyperbolic by construction [**TS**, p. 138]. It is not sectional-hyperbolic by [**TS**, Lemma 4.2 and Theorem 4]. □

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Let us finish with some global results related to the sectional-hyperbolic systems.

It turns out that the class of sectional-Axiom A vector fields includes the Axiom A vector fields, the singular-Axiom A vector fields in 3-manifolds [**MPP3**] and the multidimensional Lorenz attractors.

The result below extends [**MPP3**, Theorem B, p. 3396] to higher dimensions.

THEOREM B. *A C^r sectional-Axiom A vector field without cycles of a closed n-manifold M is in $\mathcal{G}^r(M)$ for any $r \geq 1$.*

Proof. The proof is similar to the proof of [**MPP1**, Theorem B, p. 3399] with Lemma 4.2 playing the role of Proposition 1 there. Indeed, write $\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k$ to be the decomposition of $\Omega(X)$ given by Definition 2.14. Fix $1 \leq i \leq k$. On the one hand, by Lemma 4.2, one can find a neighborhood $\mathcal{U}_i$ of X in the space of C^r vector fields, and a neighborhood U_i of Λ_i, such that if $Y \in \mathcal{U}_i$ then every closed orbit of Y contained in U_i is hyperbolic. On the other hand, since X has no cycles, by using filtration, one can find a neighborhood $\mathcal{V}$ of X such that if $Y \in \mathcal{V}$ then every closed orbit of Y lies in U_i for some $1 \leq i \leq k$. Define $U = \mathcal{V} \cap \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k$. Hence $\mathcal{U}$ is a neighborhood of X and if $Y \in \mathcal{U}$ then every closed orbit of Y is hyperbolic. This proves $X \in \mathcal{G}^r(M)$ as desired. □

5. *Final remarks and questions*

To complement Theorem B let us mention that there are vector fields in $\mathcal{G}^r(M)$ which are not sectional-Axiom A for all $r \geq 1$. However, Theorem B motivates the question: How large is the class of sectional-Axiom A vector fields? In particular, are the sectional-Axiom A vector fields dense or generic in $\mathcal{G}^r(M)$? Another question is: What amount of singular-hyperbolic systems' theory in dimension three (e.g. [**BDV**, Ch. 9]) can be extended to sectional-hyperbolic systems in dimensions of at least four? For example, ergodic properties of sectional-hyperbolic systems such as the existence of SRB measures and stochastic stability will appear in a forthcoming work [**BMM**].

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