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WEIGHTED SOBOLEV TRACE EMBEDDINGS AND THEIR APPLICATIONS

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Abstract This paper addresses three key objectives. First, weighted Sobolev trace embeddings are studied for a class of function spaces defined in the upper half-space. Second, Liouville-type theorems are proved, providing essential insights into the nonexistence of solutions for a class of quasilinear elliptic problems with indefinite boundary conditions under specific constraints. Third, by combining the derived embedding results with the fibering method, the existence of solutions for such problems is established. These findings contribute to a deeper understanding of the analytical and variational properties of these elliptic equations.

Keywords: Hardy–Sobolev inequality; weighted Sobolev trace embeddings; nonlinear boundary conditions; Liouville-type theorems; fibering method; quasilinear elliptic problems

2020 Mathematics Subject Classification 26D10; 46E35; 35J92; 35J62; 35B53

1. Introduction

This study presents new results on weighted Sobolev trace embeddings and investigates their applications to a class of quasilinear elliptic equations with nonlinear boundary conditions. Existence results are established via the fibering method, while Liouville-type results are obtained through a priori estimates. These nonlinear elliptic equations are motivated by challenges that arise in diverse scientific domains, notably in physics and applied mathematics, see [6, 9, 13, 15, 23] and references therein.

1.1. Weighted Sobolev trace embeddings

A fundamental element of the analysis is the Sobolev trace theorem for the upper half-space

$$\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1},\ x_N > 0\}$$

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whose boundary is $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$. This theorem guarantees the existence of a bounded linear operator, known as the *trace operator*, which in the present context asserts that for any smooth function u with compact support in $\mathbb{R}^N$, the following inequality holds:

$$K(N,p)\left(\int_{\mathbb{R}^{N-1}} |u|^{p_*}\,\mathrm{d}x'\right)^{p/p_*} \le \int_{\mathbb{R}^N_+} |\nabla u|^p\,\mathrm{d}x, \tag{1}$$

where $N \ge 3$ and $1 < p < N$. Here, $p_* := \frac{(N-1)p}{N-p}$ denotes the critical exponent associated with the Sobolev trace theorem, which plays a central role in the analysis of trace embeddings and related inequalities. The best constant $K(N,p) > 0$ in inequality (1) is attained by the family of functions

$$u_\varepsilon(x', x_N) = \left(\frac{\varepsilon^{1/p}}{|(x', x_N + \varepsilon)|}\right)^{\frac{N-p}{p-1}}, \tag{2}$$

where $\varepsilon > 0$; see [7, 16, 25] for details. Inequality (1) illustrates that the trace of a Sobolev function on the boundary $\mathbb{R}^{N-1}$ is effectively controlled by the L^p-norm of its gradient in the domain $\mathbb{R}^N_+$, providing a crucial tool for the analysis of elliptic problems with boundary conditions.

Let $C^\infty_\delta(\mathbb{R}^N_+)$ be the space of $C^\infty_0(\mathbb{R}^N)$-functions restricted to $\mathbb{R}^N_+$ and for $0 < p-1 < \gamma$ consider the weighted Sobolev space E defined as the completion of $C^\infty_\delta(\mathbb{R}^N_+)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p\,\mathrm{d}x + \int_{\mathbb{R}^{N-1}} |u|^p\,\mathrm{d}x'\right)^{1/p}. \tag{3}$$

The space E depends on p, N, γ; however, for the sake of simplicity, this dependence is omitted in the notation.

To derive trace embeddings from E into $L^q(\mathbb{R}^{N-1})$ spaces, the following Hardy–Sobolev type inequality is first presented:

Theorem 1 (Hardy type). *If $0 < p-1 < \gamma$, for every $u \in C^\infty_0(\mathbb{R}^N)$, the following inequality holds:*

$$C^p_{p,\gamma}\int_{\mathbb{R}^N_+} \frac{|u|^p}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x + C^{p-1}_{p,\gamma}\int_{\mathbb{R}^{N-1}} |u|^p\,\mathrm{d}x' \le \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p\,\mathrm{d}x, \tag{4}$$

where

$$C_{p,\gamma} := \frac{\gamma - p + 1}{p}. \tag{5}$$

Before proceeding, the following remarks are in order.

Remark 1. The condition $0 < p-1 < \gamma$ is sufficient for the validity of the Hardy-type inequality (4), in agreement with the range assumed in [20]. Determining conditions on

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γ that are also *necessary* for (4) remains open. Moreover, if $u \in C_0^\infty(\mathbb{R}^N_+)$, then (4) holds for every $\gamma \in \mathbb{R}$; see [14]. In this Dirichlet setting, the inequality may be viewed as a Poincaré-type inequality.

Remark 2. Inequality (4) is fundamental in the derivation of the main results. From the definition of E, combined with (4) and a density argument, it follows that

$$C_{p,\gamma}^p \int_{\mathbb{R}^N_+} \frac{|u|^p}{(1+x_N)^{p-\gamma}}\,\mathrm{d}x \le \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p\,\mathrm{d}x, \quad \forall u \in E \tag{6}$$

and

$$C_{p,\gamma}^{p-1} \int_{\mathbb{R}^{N-1}} |u|^p\,\mathrm{d}x' \le \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla u|^p\,\mathrm{d}x, \quad \forall u \in E. \tag{7}$$

It was proved in [1] that if $\gamma > p-1 > 0$ it holds

$$S_{\gamma,p} := \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N_+}(1+x_N)^\gamma |\nabla u|^p\,\mathrm{d}x}{\int_{\mathbb{R}^N_+}(1+x_N)^{\gamma-p}|u|^p\,\mathrm{d}x} = C_{p,\gamma}^p > 0.$$

Thus, $C_{p,\gamma}^p$ is the best constant for inequality (6). It is worth noting that obtaining the best constant for inequality (7) remains an interesting open question. More precisely, define

$$T_{\gamma,p} := \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N_+}(1+x_N)^\gamma |\nabla u|^p\,\mathrm{d}x}{\int_{\mathbb{R}^{N-1}} |u|^p\,\mathrm{d}x'}. \tag{8}$$

It follows from (7) that $C_{p,\gamma}^{p-1} \le T_{\gamma,p}$ whenever $\gamma > p-1$. The question of whether equality holds remains open.

The following result is also a direct consequence of (6):

Corollary 1 (Equivalence of Norms). *In the space E, the functional*

$$\|u\|_\gamma := \left(\int_{\mathbb{R}^N_+} (1+x_N)^\gamma\,|\nabla u|^p\,\mathrm{d}x\right)^{1/p}$$

defines a norm equivalent to the original norm $\|\cdot\|$ given in (3).

Based on the Hardy type inequality presented in Theorem 1, we are able to establish the Sobolev trace embedding in the context of the space E, given by:

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Theorem 2 (Sobolev Trace-type Embedding). *The Sobolev trace embedding*

$$E \hookrightarrow L^q(\mathbb{R}^{N-1}), \quad \forall q \in [p, p_*]$$

is continuous. In other words, there exists a constant $C = C(N, p, q, \gamma) > 0$ such that

$$\left(\int_{\mathbb{R}^{N-1}} |u|^q \, \mathrm{d}x' \right)^{p/q} \leq C \int_{\mathbb{R}^N_+} (1 + x_N)^\gamma |\nabla u|^p \, \mathrm{d}x, \quad \forall u \in E. \tag{9}$$

Moreover, it holds:

(1) If $q = p_*$ and $\gamma > p - 1$ the result is

$$T_{\gamma, p_*} = \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} (1 + x_N)^\gamma |\nabla u|^p \, \mathrm{d}x}{\left(\int_{\mathbb{R}^{N-1}} |u|^{p_*} \, \mathrm{d}x' \right)^{p/p_*}} = K(N, p),$$

where $K(N, p)$ is the best constant for the trace inequality (1).

(2) If $\gamma < p - 1$, then $T_{\gamma, p} = 0$.

1.2. Quasilinear equations involving nonlinear boundary conditions

As an application of Theorem 2, the following class of problems is considered:

$$\begin{cases} -\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \mathbb{R}^N_+, \\ \rho(x', 0)|\nabla u|^{p-2}\nabla u \cdot \nu = h(x')|u|^{q-2}u - m(x')|u|^{s-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases} \tag{$\mathcal{P}$}$$

where ν is the unit outer normal to the boundary $\partial\mathbb{R}^N_+ := \mathbb{R}^{N-1}$. Assume that $h, m \in L^1_{\mathrm{loc}}(\mathbb{R}^{N-1})$ are positive functions and that $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^N_+)$ satisfies

$$\exists \, \rho_0 > 0 \text{ and } \gamma > p - 1 \text{ s.t. } \rho(x) \geq \rho_0 (1 + x_N)^\gamma \quad \text{a.e.} \quad x \in \mathbb{R}^N_+. \tag{H}$$

To establish the variational framework, define the weighted space E_ρ as the closure of $C^\infty_\delta(\mathbb{R}^N_+)$ with respect to the norm

$$\|u\|_\rho := \left(\int_{\mathbb{R}^N_+} \rho(x) |\nabla u|^p \, \mathrm{d}x \right)^{1/p}.$$

We stress that assumption (H) plays a purely technical role and is introduced to ensure the continuous embedding $E_\rho \hookrightarrow E$. Although the analysis was initially carried out for the specific weight $\rho(x) = (1 + x_N)^\gamma$, a closer examination of the proof reveals that no essential property of this particular choice is used. In fact, the argument remains entirely unchanged for any admissible weight that satisfies the assumption (H).

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A weak solution of problem $(\mathcal{P})$ is a function $u \in E_\rho$ such that, for every smooth function φ with compact support in $\mathbb{R}^N$, the following identity is satisfied:

$$\int_{\mathbb{R}^N_+} \rho(x)|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^{N-1}} (h(x')|u|^{q-2}u - m(x')|u|^{s-2}u)\varphi \, \mathrm{d}x'. \tag{10}$$

In the following sections, we establish existence and nonexistence results for problem $(\mathcal{P})$. As observed in [5], the presence of a sign-changing nonlinearity makes the solvability of the problem highly sensitive to the interaction between the competing terms $h\,|u|^{q-2}u$ and $m\,|u|^{s-2}u$. The balance between these terms is determined by qualitative properties of the ratio of the weight functions h and m. In particular, this interplay is captured by conditions (12), (13), and (17) below, which play a decisive role in distinguishing regimes of existence from nonexistence.

1.3. Liouville-type results

Next, inspired by some ideas from [23], we establish nonexistence results for problem $(\mathcal{P})$. To this end, the following notation is adopted:

$$\eta(r,q,t) := \frac{(t-r)^{t-r}}{(q-r)^{q-r}(t-q)^{t-q}} \quad \text{for} \quad r < q < t. \tag{11}$$

First, the case $1 < s < q < p < N$ is analysed.

Theorem 3 (p-sublinear case). *Suppose that assumption (H) holds, and let $m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$. If $1 < s < q < p < N$ and $h/m \in L^\infty(\mathbb{R}^{N-1})$, with*

$$\left\| \frac{h}{m} \right\|_\infty^{p-s} \left(\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0} \right)^{q-s} < \eta(s,q,p), \tag{12}$$

then problem $(\mathcal{P})$ has only the trivial weak solution.

A straightforward computation shows that for sufficiently small $\lambda > 0$, the functions

$$h(x') = \frac{\lambda}{(1+|x'|)^{\theta_1}} \quad \text{and} \quad m(x') = \frac{1}{(1+|x'|)^{\theta_2}}$$

satisfy the assumptions of Theorem 3 provided that $0 < N-1 < \theta_2 \le \theta_1$.

In the second nonexistence result, the case $p < q < s$ is considered, and the implications of this ordering of exponents on the solvability of the problem are explored.

Theorem 4 (p-superlinear case). *Assume that (H) holds, and let $m \in L^\infty(\mathbb{R}^{N-1})$. If $1 < p < q < s \le p_*$ and $h/m \in L^\infty(\mathbb{R}^{N-1})$, with*

$$\left\| \frac{h}{m} \right\|_\infty^{s-p} \left(\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0} \right)^{s-q} < \eta(p,q,s), \tag{13}$$

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then (𝒫) admits only the trivial weak solution.

A direct computation reveals that for sufficiently small $\lambda > 0$, the functions

$$h(x') = \frac{\lambda}{(1+|x'|)^{\theta_1}} \quad \text{and} \quad m(x') = \frac{1}{(1+|x'|)^{\theta_2}}$$

satisfy the hypotheses of Theorem 4 whenever $0 \le \theta_2 \le \theta_1$.

1.4. Existence results

The existence results are established using the fibering method. Theorem 2, combined with the additional conditions on the potentials, more precisely, $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, ensure that the embeddings

$$E_\rho \hookrightarrow L^q(\mathbb{R}^{N-1}, h(x')) \quad \text{and} \quad E_\rho \hookrightarrow L^s(\mathbb{R}^{N-1}, m(x'))$$

are both valid and compact. Consequently, this enables the application of the fibering method, as illustrated in [19, 23].

In the first existence result, two specific combinations of the exponents are examined. More precisely, the cases $1 < s < p < q < p_*$ and $1 < p < s < q < p_*$ are analysed. Particular attention is given to ensuring that the interplay among the exponents aligns with the structural requirements of the problem.

Theorem 5. *If (H) holds,* $\max\{s,p\} < q < p_*$*, and* $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$*, then (𝒫) has a nontrivial nonnegative weak solution. Moreover, u is positive in* $\mathbb{R}^N_+$ *provided that* $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N_+)$.

For the second existence result, a different set of exponent combinations is considered: $1 < q < s < p$ and $1 < q < p < s < p_*$. These cases reflect distinct configurations of the exponents, each leading to unique mathematical challenges and results.

Theorem 6. *If (H) holds,* $q < \min\{s,p\}$ *with* $s < p_*$*, and* $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$*, then (𝒫) has a nontrivial nonnegative weak solution. Moreover, u is positive in* $\mathbb{R}^N_+$ *provided that* $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N_+)$.

We note that Figures 1 and 2 illustrate that the results established in Theorems 3 and 4, together with Theorems 5 and 6, exhibit a complementary character, as they apply to mutually complementary regions.

To establish the third existence result, consider the functionals defined on E_ρ by

$$H(u) = \int_{\mathbb{R}^{N-1}} h|u|^q \,\mathrm{d}x' \quad \text{and} \quad M(u) = \int_{\mathbb{R}^{N-1}} m|u|^s \,\mathrm{d}x'. \tag{14}$$

Note that under the assumptions of Theorem 4, for $u \in E_\rho$, the following key inequality holds (see Lemma 11):

$$H(u)^{s-p} < \eta(p,q,s) M(u)^{q-p} \|u\|_\rho^{p(s-q)} < \eta(p,q,s) \left(\frac{q}{p}\right)^{s-q} M(u)^{q-p} \|u\|_\rho^{p(s-q)}. \tag{15}$$

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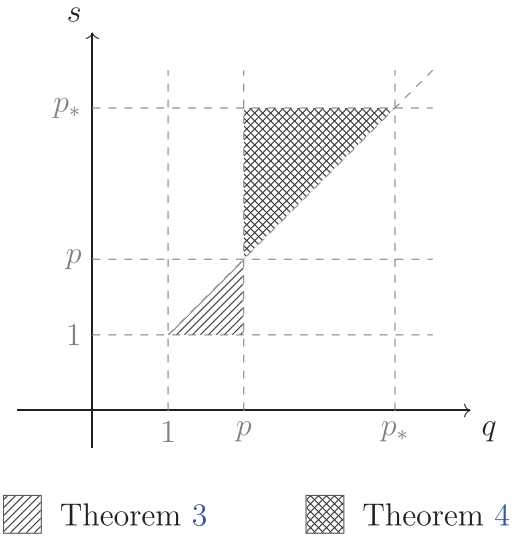


Figure 1. Nonexistence of solutions. The diagonal-striped region corresponds to Theorem 3, and the crosshatched region corresponds to Theorem 4.

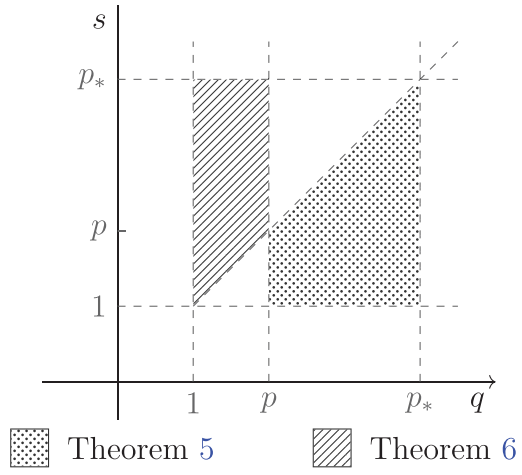


Figure 2. Existence of nontrivial solutions. The diagonal-striped region corresponds to Theorem 6, and the dotted region corresponds to Theorem 5.

Therefore, the existence of solutions for $(\mathcal{P})$ can be considered in a subset of the complementary situation to the inequality (15), precisely:

$$\mathcal{D} := \left\{ u \in E_\rho : H(u)^{s-p} > \left(\frac{q}{p}\right)^{s-q} \eta(p,q,s) M(u)^{q-p} \|u\|_\rho^{p(s-q)} \right\}. \tag{16}$$

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Theorem 7. *Assume* (*H*), $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $p < q < s < p_*$, $\mathcal{D} \neq \emptyset$, *and*

$$\frac{h^{1/q}}{m^{1/s}} \in L^{\frac{sq}{s-q}}(\mathbb{R}^{N-1}). \tag{17}$$

Then, ($\mathcal{P}$) *admits a nontrivial and nonnegative weak solution. Moreover, u is positive in $\mathbb{R}^N_+$ provided that $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N_+)$. We note that Figure 3 illustrates the result established in Theorem 7.*

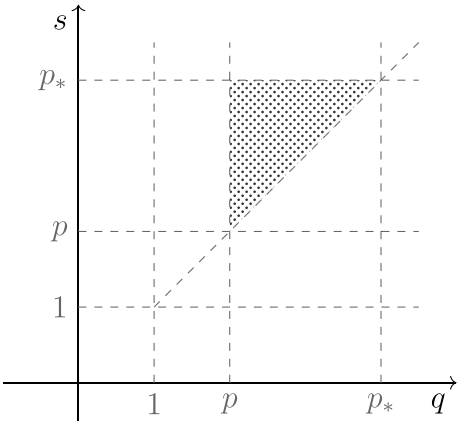


Figure 3. The diagonal dotted region corresponds to Theorem 7.

Remark 3. The functions $h = m = \lambda w$, with w given by

$$w(x') = \frac{1}{(1+|x'|)^\theta},$$

satisfy the assumptions of Theorem 7 for $\theta > N-1$ and λ sufficiently large. First, observe that

$$\left[\frac{h(x')^{1/q}}{m(x')^{1/s}}\right]^{\frac{sq}{s-q}} = h(x') = \lambda(1+|x'|)^{-\theta} \in L^1(\mathbb{R}^{N-1}),$$

whenever $\theta > N-1$. For $u \in E_\rho \setminus \{0\}$ fixed, one has

$$\frac{H(u)^{s-p}}{M(u)^{q-p}} = \lambda^{s-q}\|u\|^{q(s-p)}_{L^q(\mathbb{R}^{N-1},w)}\|u\|^{s(p-q)}_{L^s(\mathbb{R}^{N-1},w)}.$$

Since $s > q$, for λ sufficiently large it holds

$$H(u)^{s-p} > \left(\frac{q}{p}\right)^{s-q}\eta(p,q,s)M(u)^{q-p}\|u\|^{p(s-q)}_\rho,$$

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hence $\mathcal{D} \neq \emptyset$. Conversely, it is easy to verify that for λ small potentials, they satisfy the condition for nonexistence (13).

As stated in [23], the condition $\mathcal{D} \neq \emptyset$ can be interpreted as an indication that h is sufficiently 'large' with respect to m.

The next existence result is derived from the consideration of the set

$$\mathcal{C} := \left\{ u \in E_\rho : H(u)^{p-s} > \left(\frac{q}{s}\right)^{p-s} \eta(s,q,p) M(u)^{p-q} \|u\|_\rho^{p(q-s)} \right\}, \tag{18}$$

where, analogously to the earlier case, the hypothesis is that h is sufficiently *large* in comparison to m, as indicated by the condition $\mathcal{C} \neq \emptyset$. The fourth existence result for problem $(\mathcal{P})$ is stated below:

Theorem 8. *Assume* (H), $1 < s < q < p$ *and* $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$. *If* $\mathcal{C} \neq \emptyset$, *then* $(\mathcal{P})$ *has a nontrivial and nonnegative weak solution. Moreover,* u *is positive in* $\mathbb{R}^N_+$, *if* $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N_+)$. *We note that Figure 4 illustrates the result established in Theorem 8.*

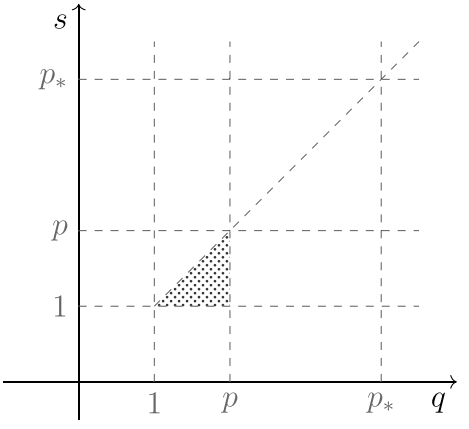


Figure 4. The diagonal dotted region corresponds to Theorem 8.

We observe that the result established in Theorem 7 does not contradict Theorem 4, since the hypothesis $\mathcal{D} \neq \emptyset$ is complementary to the assumption (13). Similarly, Theorem 8 does not contradict Theorem 3, since the hypothesis $\mathcal{C} \neq \emptyset$ is complementary to the assumption (12).

Remark 4. Under the additional hypothesis that $\rho \in C^1(\mathbb{R}^N_+)$, beyond those required for the existence of solutions, it follows from [28, Theorem 1] that all weak solutions of problem $(\mathcal{P})$ belong to $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N_+)$, for some $\alpha \in (0,1)$. Moreover, if ρ is locally Lipschitz in $\overline{\mathbb{R}^N_+}$ and both h and m are locally Lipschitz in $\mathbb{R}^{N-1}$, then [22, Theorem 2] ensures that any such solution satisfies $u \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathbb{R}^N_+})$.

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1.5. Background and related works

A substantial body of research underscores the importance of weighted Sobolev trace embeddings and their applications in studying quasilinear elliptic problems with nonlinear boundary conditions. These embeddings play a crucial role in examining the existence and nonexistence of solutions to these problems, serving as a fundamental tool for their analytical investigation. Without claiming exhaustiveness, the following selection highlights foundational studies closely related to the present research on this class of Sobolev embeddings.

The existence of extremals for the inequality (1) and a complete classification for the case $p = 2$ were established in [7, 16]. In [25], the author extended the analysis of (1) to the general case of $1 < p < N$ using the mass transportation method, thereby resolving the conjecture posed by Escobar in [16] regarding the minimizers of the inequality. It is well known that the Euler–Lagrange equation associated with (1) is the following quasilinear equation with a nonlinear Neumann boundary:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\nabla u \cdot \nu = C|u|^{p_*-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

where ν is the unit outer normal to the boundary $\partial\mathbb{R}^N_+ := \mathbb{R}^{N-1}$.

It is worth noting that the Hardy type inequality in the upper half-space, for the particular case $p = 2$, was addressed by Maz'ya in [24], where the inequality was established for every smooth function u with compact support in $\mathbb{R}^N_+$:

$$\left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{|u|^p}{x_N^p}\,\mathrm{d}x + \alpha(p,\tau)\int_{\mathbb{R}^N_+} \frac{|u|^p}{x_N^{p-\tau}(x_{N-1}^2 + x_N^2)^{\tau/2}}\,\mathrm{d}x \le \int_{\mathbb{R}^N_+} |\nabla u|^p\,\mathrm{d}x,$$

for any smooth function u with compact support on $\mathbb{R}^N_+$ for the specific case $p = 2$. For the general case $1 < p < N$, this inequality was later proved in [27], thereby extending Maz'ya's result and providing an affirmative answer to a question he posed about the validity of such inequalities in the L^p framework.

For $\alpha \in \mathbb{R}$ and $\rho(x) := d\left(x, \partial\mathbb{R}^N_+\right)$ be the distance from the boundary of $\mathbb{R}^N_+$ it was proved in [14] the following inequality

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int_{\mathbb{R}^N_+} \rho^\alpha \frac{|u|^p}{\rho^p} dx \leqslant \int_{\mathbb{R}^N_+} \rho^\alpha |\nabla u|^p dx,\ u \in \mathrm{C}_0^\infty\left(\mathbb{R}^N_+\right).$$

It is worth noting that the authors in [14, 24, 27] considered test functions $u \in C_0^\infty(\mathbb{R}^N_+)$, thereby working under homogeneous Dirichlet boundary conditions. In contrast, the objective here is to investigate this class of inequalities in the upper half-space for arbitrary smooth functions u with compact support in $\mathbb{R}^N$, a framework that plays a fundamental role in the analysis of nonlinear elliptic problems with nonlinear boundary conditions.

Let $\Omega \subset \mathbb{R}^N$ be a smooth domain and denote by $C_\delta^\infty(\Omega)$ the space of $C_0^\infty(\mathbb{R}^N)$-functions restricted to Ω. For $w_\theta(x) := (1+|x|)^\theta$ with $\theta \ge 0$, consider the weighted Lebesgue space

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$L^r(\Omega, w_r)$ and the weighted Sobolev space W is defined as the completion of $C_\delta^\infty(\Omega)$ under the norm $||u||_W := \left(\int_\Omega |\nabla u|^p\,dx + \int_\Omega w_p|u|^p\,dx\right)^{1/p}$. In [18, 26], it was proved the embedding result $W \hookrightarrow L^\sigma(\partial\Omega; w_\theta)$ is continuous if $p \le \sigma \le p_*$ and $(N-1)\left(1-\frac{\sigma}{p_*}\right) \le \theta < N$. In particular, if the upper bound for σ and the lower bound for θ are strict, then the embedding is compact.

The result above consolidates all the embedding properties relevant to the purposes of Lyberopoulos in [23], where the author studied the existence and Liouville-type theorems for indefinite quasilinear elliptic problems with potentials vanishing at infinity of the form:

$$\begin{cases} -\mathrm{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u, & \text{in } \Omega, \\ \rho(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + h(x)|u|^{p-2}u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{PL}$}$$

where Ω is an arbitrary unbounded domain in $\mathbb{R}^N$ with a noncompact smooth boundary $\partial\Omega$, whose outward unit normal is denoted by ν, which includes the upper half-space.

In [23], a quasilinear elliptic problem with Robin boundary conditions was investigated under the assumption that $h(x) \ge 1$ almost everywhere on $\partial\Omega$. Reference is also made to [3], which addresses a related problem in the upper half-space and relies on a fundamental inequality established in [2], analogous to the result derived in [26]. The present contribution extends the analysis in [23] by considering a broader class of weight functions and incorporating Neumann boundary conditions – features not treated in the references listed above

For elliptic equations with Neumann boundary conditions in a half-space, see [8, 10–12, 17, 21]. These works highlight the intricate interplay between boundary conditions and solution behaviour, offering valuable insights into the qualitative properties of solutions in diverse settings. The continued investigation of these connections holds promise for a deeper understanding of the underlying mathematical structures.

This paper is organized as follows. Section 2 presents the proofs of Theorems 1 and 2, establishing fundamental inequalities for the subsequent analysis. Section 3 contains the Liouville-type nonexistence results (Theorems 3 and 4). The existence results appear in Section 4, with proofs of Theorems 5, 6, and 7. The final section, Section 5, discusses open problems and potential directions for future research.

1.6. Notations

- $p_* := \frac{(N-1)p}{N-p}$ denotes the critical exponent associated with the Sobolev trace theorem.
- $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ denotes the upper half-space, whose boundary is given by $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$.
- Let $\Omega \subset \mathbb{R}^N$ be a smooth domain. The space $C_0^\infty(\Omega)$ consists of smooth functions with compact support in Ω.
- The space $C_\delta^\infty(\Omega)$ consists of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^N)$.

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2. Proof of Hardy type and trace inequalities

2.1. Proof of Theorem 1

Let $\sigma \in \mathbb{R}$ be a parameter to be specified later. For any $u \in C_0^\infty(\mathbb{R}^N)$, an application of integration by parts yields the following identity:

$$(\sigma+1)\int_{\mathbb{R}^N_+}(1+x_N)^\sigma|u|^p\,\mathrm{d}x+\int_{\mathbb{R}^{N-1}}|u|^p\,\mathrm{d}x'=-\int_{\mathbb{R}^N_+}(1+x_N)^{\sigma+1}(|u|^p)_{x_N}\,\mathrm{d}x$$
$$\leq p\int_{\mathbb{R}^N_+}(1+x_N)^{\sigma+1}|u|^{p-1}|\nabla u|\,\mathrm{d}x.$$

For $a, b \geq 0$ real numbers and $\varepsilon \in (0,1)$. By Young's inequality, the following estimate holds:

$$pab \leq \varepsilon(\sigma+1)a^{\frac{p}{p-1}}+\left(\frac{p-1}{\varepsilon(\sigma+1)}\right)^{p-1}b^p.$$

In particular, consider

$$p(1+x_N)^{\sigma+1}|u|^{p-1}|\nabla u| = p(1+x_N)^{(\sigma+1-\frac{\gamma}{p})}|u|^{p-1}(1+x_N)^{\frac{\gamma}{p}}|\nabla u|,$$

and set

$$a=(1+x_N)^{(\sigma+1-\frac{\gamma}{p})}|u|^{p-1} \quad \text{and} \quad b=(1+x_N)^{\frac{\gamma}{p}}|\nabla u|.$$

Then the integral estimate follows:

$$p\int_{\mathbb{R}^N_+}(1+x_N)^{\sigma+1}|u|^{p-1}|\nabla u|\,\mathrm{d}x \leq$$
$$\varepsilon(\sigma+1)\int_{\mathbb{R}^N_+}(1+x_N)^{(\sigma+1-\frac{\gamma}{p})\frac{p}{p-1}}|u|^p\,\mathrm{d}x+\left(\frac{p-1}{\varepsilon(\sigma+1)}\right)^{p-1}\int_{\mathbb{R}^{N_+}}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x$$

and hence

$$(\sigma+1)\int_{\mathbb{R}^N_+}(1+x_N)^\sigma|u|^p\,\mathrm{d}x+\int_{\mathbb{R}^{N-1}}|u|^p\,\mathrm{d}x' \leq$$
$$\varepsilon(\sigma+1)\int_{\mathbb{R}^N_+}(1+x_N)^{(\sigma+1-\frac{\gamma}{p})\frac{p}{p-1}}|u|^p\,\mathrm{d}x+\left(\frac{p-1}{\varepsilon(\sigma+1)}\right)^{p-1}\int_{\mathbb{R}^{N_+}}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x.$$

Choosing the parameter σ such that

$$\left(\sigma+1-\frac{\gamma}{p}\right)\frac{p}{p-1}=\sigma,$$

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leads to the identity $\sigma = \gamma - p$, and hence $\sigma + 1 = \gamma - p + 1 > 0$. Substituting into the inequality gives

$$(\sigma+1)(1-\varepsilon)\int_{\mathbb{R}^N_+}(1+x_N)^\sigma|u|^p\,\mathrm{d}x + \int_{\mathbb{R}^{N-1}}|u|^p\,\mathrm{d}x' \leq \left(\frac{p-1}{\varepsilon(\sigma+1)}\right)^{p-1}\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x.$$

Multiplying both sides by $(\sigma+1)^{p-1}\varepsilon^{p-1}$, the resulting inequality becomes

$$(\sigma+1)^p(\varepsilon^{p-1}-\varepsilon^p)\int_{\mathbb{R}^N_+}(1+x_N)^\sigma|u|^p\,\mathrm{d}x + (\sigma+1)^{p-1}\varepsilon^{p-1}\int_{\mathbb{R}^{N-1}}|u|^p\,\mathrm{d}x' \leq (p-1)^{p-1}\int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x.$$

Since the function $f(\varepsilon) = \varepsilon^{p-1} - \varepsilon^p$, for $\varepsilon > 0$, attains its maximum at

$$\varepsilon_0 = \frac{p-1}{p}, \quad \text{with} \quad f(\varepsilon_0) = (p-1)^{p-1}\frac{1}{p^p},$$

a direct computation yields

$$\left(\frac{\sigma+1}{p}\right)^p\int_{\mathbb{R}^N_+}(1+x_N)^\sigma|u|^p\,\mathrm{d}x + \left(\frac{\sigma+1}{p}\right)^{p-1}\int_{\mathbb{R}^{N-1}}|u|^p\,\mathrm{d}x' \leq \int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x.$$

The desired inequality follows directly from a suitable choice of σ.

2.2. Proof of Theorem 2

Through a standard density argument, it is sufficient to consider $u \in C_0^\infty(\mathbb{R}^N)$. Applying the Sobolev trace inequality (1) yields

$$K(N,p)\left(\int_{\mathbb{R}^{N-1}}|u|^{p_*}\,\mathrm{d}x'\right)^{p/p_*} \leq \int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{19}$$

On the other hand, by Theorem 1, the following estimate holds

$$C_{p,\gamma}^{p-1}\int_{\mathbb{R}^{N-1}}|u|^p\,\mathrm{d}x' \leq \int_{\mathbb{R}^N_+}(1+x_N)^\gamma|\nabla u|^p\,\mathrm{d}x.$$

Consequently, by interpolation, inequality (9) is obtained for all $p \leq q \leq p_*$.

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From inequality (19), the estimate $K(N,p) \leq T_{\gamma,p_*}$ follows. To establish the reverse inequality, fix $r>0$ and consider a function $\varphi_r \in C^\infty(\mathbb{R}^N_+)$ such that

$$\varphi_r = \begin{cases} 1, & \text{in} \quad B_{r/2}(0) \cap \mathbb{R}^N_+, \\ 0, & \text{in} \quad \mathbb{R}^N_+ \backslash B_r(0). \end{cases} \qquad (\mathcal{R}_0)$$

For any $\varepsilon > 0$ set $\psi_\varepsilon := \varphi_r u_\varepsilon$, where u_ε is given by (2). Since u_ε achieves the best constant in the inequality (1), results in

$$\frac{\int_{\mathbb{R}^N_+} |\nabla \psi_\varepsilon|^p \, \mathrm{d}x}{\left(\int_{\mathbb{R}^{N-1}} |\psi_\varepsilon|^{p_*} \mathrm{d}x\right)^{p/p_*}} = K(N,p) + o(\varepsilon). \tag{20}$$

Therefore,

$$T_{\gamma,p_*} \leq \frac{\int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla \psi_\varepsilon|^p \, \mathrm{d}x}{\left(\int_{\mathbb{R}^{N-1}} |\psi_\varepsilon|^{p_*} \mathrm{d}x\right)^{p/p_*}} \leq \frac{(1+r)^\gamma \int_{\mathbb{R}^N_+} |\nabla \psi_\varepsilon|^p \, \mathrm{d}x}{\left(\int_{\mathbb{R}^{N-1}} |\psi_\varepsilon|^{p_*} \mathrm{d}x\right)^{p/p_*}} = (1+r)^\gamma K(N,p) + o(\varepsilon)$$

and passing to the limit as $r, \varepsilon \to 0^+$ gives $T_{\gamma,p_*} \leq K(N,p)$.

Next, it is proved that $T_{\gamma,p} = 0$ whenever $\gamma < p-1$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ for $|x| \leq 1$ and $\varphi \equiv 0$ for $|x| \geq 2$. Now consider, for $t>0$, $\varphi_t(x) = \varphi(tx)$. Then, observe that

$$\begin{aligned} \int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla \varphi_t(x)|^p \mathrm{d}x =& t^p \int_{B^+_{2/t}} (1+x_N)^\gamma |\nabla \varphi(tx)|^p \mathrm{d}x \\ \leq& \left(1+\frac{2}{t}\right)^\gamma t^{p-N} \int_{B_2^+} |\nabla \varphi|^p \mathrm{d}x \\ \leq& C_1 (t+2)^\gamma t^{p-N-\gamma}. \end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^{N-1}} |\varphi_t|^p \mathrm{d}x' = C_2 t^{1-N}.$$

Thus,

$$0 \leq T_{\gamma,p} \leq \frac{\int_{\mathbb{R}^N_+} (1+x_N)^\gamma |\nabla \varphi_t(x)|^p \mathrm{d}x}{\int_{\mathbb{R}^{N-1}} |\varphi_t|^p \mathrm{d}x'} \leq C(t+2)^\gamma t^{p-1-\gamma} \longrightarrow 0,$$

as $t \longrightarrow 0^+$, and the proof is completed.

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3. Liouville-type results

In this section, the aim is to prove Liouville-type results. To this end, the following estimate that involves the functionals H and M, defined in (14), plays a crucial role in the analysis.

Lemma 9. (Key estimate) *Assume condition* (H) *and* $1 < s < q < p$. *If* $m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ *then* M *is well-defined in* E. *In addition, if* $h/m \in L^\infty(\mathbb{R}^{N-1})$, *then*

$$H(v)^{p-s} \leq \left(\left\| \frac{h}{m} \right\|_\infty^{(p-s)} \left[\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0} \right]^{q-s} \right) M(v)^{p-q} \|v\|_\rho^{p(q-s)}, \quad \forall v \in E_\rho, \tag{21}$$

which implies that H *is well-defined. Furthermore, if* (12) *holds, then*

$$H(v)^{p-s} < \eta(s,q,p) M(v)^{p-q} \|v\|_\rho^{p(q-s)}, \quad \forall v \in E_\rho \backslash \{0\}, \tag{22}$$

where the constant $\eta(s,q,p)$ *was defined in* (11).

Proof. Using Theorem 1 and assumption (H), it follows that $M(v)$ is well-defined in E_ρ. Indeed, let $v \in E_\rho$ then

$$\begin{aligned} M(v) &= \int_{\mathbb{R}^{N-1}} m^{\frac{p-s}{p}} m^{\frac{s}{p}} |v|^s \, \mathrm{d}x' \\ &\leq \|m\|_{L^1(\mathbb{R}^{N-1})}^{(p-s)/p} \left(\int_{\mathbb{R}^{N-1}} m|v|^p \, \mathrm{d}x' \right)^{s/p} \\ &\leq \|m\|_\infty^{s/p} \|m\|_{L^1(\mathbb{R}^{N-1})}^{(p-s)/p} \left(\int_{\mathbb{R}^{N-1}} |v|^p \, \mathrm{d}x' \right)^{s/p} < \infty. \end{aligned}$$

If $s < q < p$, then q can be expressed as a convex combination $q = (1-\alpha)s + \alpha p$, where $\alpha = \frac{q-s}{p-s} \in (0,1)$. Thus, by Hölder's inequality,

$$\begin{aligned} H(v) &= \int_{\mathbb{R}^{N-1}} (h|v|^s)^{1-\alpha} (h|v|^p)^\alpha \, \mathrm{d}x' \\ &\leq \left(\int_{\mathbb{R}^{N-1}} h|v|^s \, \mathrm{d}x' \right)^{1-\alpha} \left(\int_{\mathbb{R}^{N-1}} h|v|^p \, \mathrm{d}x' \right)^{\alpha}. \end{aligned}$$

Using the fact that $1 - \alpha = \dfrac{p-q}{p-s}$, the estimate below is obtained:

$$H(v)^{p-s} \leq \left(\int_{\mathbb{R}^{N-1}} h|v|^s \, \mathrm{d}x' \right)^{p-q} \left(\int_{\mathbb{R}^{N-1}} h|v|^p \, \mathrm{d}x' \right)^{q-s}. \tag{23}$$

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Now, observe that

$$\int_{\mathbb{R}^{N-1}} h|v|^s \,\mathrm{d}x' = \int_{\mathbb{R}^{N-1}} \frac{h}{m}\,(m|v|^s)\,\mathrm{d}x' \leq \left\|\frac{h}{m}\right\|_\infty M(v).$$

Thus, from (7) and (H) it follows

$$\int_{\mathbb{R}^{N-1}} h|v|^p \,\mathrm{d}x' \leq \|m\|_\infty \left\|\frac{h}{m}\right\|_\infty \int_{\mathbb{R}^{N-1}} |v|^p \,\mathrm{d}x' \leq \|m\|_\infty \left\|\frac{h}{m}\right\|_\infty \frac{C_{p,\gamma}^{1-p}}{\rho_0}\|v\|_\rho^p.$$

Therefore, H is well-defined as a consequence of the last two estimates. Moreover, (21) follows by substituting these estimates into (23). □

To proceed, the following auxiliary function is introduced:

$$G(r,v) = H(v)r^{q-p} - M(v)r^{s-p}, \quad r>0 \quad \text{and} \quad v \in E_\rho. \tag{24}$$

Lemma 10. *Assume the hypotheses of Theorem 3. For each fixed $v \in E_\rho \setminus \{0\}$, the function $G(\cdot, v)$ has a unique critical point, which is a maximum, given by*

$$\overline{r}_v = \left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{1/(q-s)}.$$

Moreover,

$$G(\overline{r}_v, v) = \max_{r>0} G(r,v) = \left(\frac{H(v)^{p-s}}{\eta(s,q,p)M(v)^{p-q}}\right)^{1/(q-s)} > 0, \tag{25}$$

where $\eta(s,q,p)$ was defined in (11).

The proof is omitted as it follows from straightforward calculations.

3.1. Proof of Theorem 3

Assume, for the sake of contradiction, that ($\mathcal{P}$) has a nontrivial weak solution $u_0 \in E_\rho$. From the definition in (10), along with Lemma 9 and a density argument, it follows that

$$\|u_0\|_\rho^p = H(u_0) - M(u_0) = G(1, u_0).$$

On the other hand, (22) gives

$$H(u_0)^{p-s} < \eta(s,q,p)M(u_0)^{p-q}\|u_0\|_\rho^{p(q-s)},$$

which in combination with (25), implies

$$G(\overline{r}_{u_0}, u_0) < \|u_0\|_\rho^p = G(1, u_0).$$

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This leads to a contradiction, since $\overline{r}_{u_0}$ is the maximum of $G(\cdot, u_0)$ and therefore, completing the proof of Theorem 3.

3.2. Proof of Theorem 4

Next, attention is turned to the proof of the second Liouville-type result.

Lemma 11. *Assume condition (H), $p < q < s \leq p_*$. If $m \in L^\infty(\mathbb{R}^{N-1})$, then M is well-defined in E. In addition, if $h/m \in L^\infty(\mathbb{R}^{N-1})$, then*

$$H(v)^{s-p} \leq \left(\left\| \frac{h}{m} \right\|_\infty^{s-p} \left[\frac{\|m\|_\infty C_{p,\gamma}^{1-p}}{\rho_0} \right]^{s-q} \right) M(v)^{q-p} \|v\|_\rho^{p(s-q)}, \quad \forall v \in E_\rho. \tag{26}$$

In particular, H is well-defined. Furthermore, if (13) holds then

$$H(v)^{s-p} < \eta(p,q,s) M(v)^{q-p} \|v\|_\rho^{p(s-q)}, \quad \forall v \in E_\rho \backslash \{0\}. \tag{27}$$

Proof. By Theorem 2 and (H), the result below holds,

$$M(v) = \int_{\mathbb{R}^{N-1}} m|v|^s \, \mathrm{d}x' \leq \|m\|_\infty \int_{\mathbb{R}^{N-1}} |v|^s \, \mathrm{d}x' \leq C\|v\|_\rho^s.$$

Inequality (26) follows by an argument analogous to that used in the proof of Lemma 9. □

Similarly to Lemma 10, an expression can be derived for the maximum value of $G(\cdot, v)$. This expression, combined with hypothesis (13), plays a key role in the proof of Theorem 4. Since the proof involves standard calculations, it is omitted here.

Lemma 12. *Assume condition (H), $p < q < s \leq p_*$. If $h, m \in L^\infty(\mathbb{R}^{N-1})$, then for each $v \in E_\rho \setminus \{0\}$ the function $G(., v)$ defined by (24) has a unique critical point at*

$$\overline{r}_v = \left(\frac{H(v)(q-p)}{M(v)(s-p)} \right)^{1/(s-q)}. \tag{28}$$

Moreover,

$$G(\overline{r}_v, v) = \max_{r>0} G(r, v) = \left(\frac{H(v)^{s-p}}{\eta(p,q,s) M(v)^{q-p}} \right)^{1/(s-q)} > 0. \tag{29}$$

From the definition (10), Lemma 11, and a density argument, it follows that

$$\|u_0\|_\rho^p = H(u_0) - M(u_0) = G(1, u_0).$$

Estimate (27) gives

$$H(u_0)^{s-p} < \eta(p,q,s) M(u_0)^{q-p} \|u_0\|_\rho^{p(s-q)}.$$

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This, combined with (29) implies that

$$G(\overline{r}_{u_0}, u_0) < \|u_0\|_\rho^p = G(1, u_0),$$

contradicting the fact that $\overline{r}_{u_0}$ is the maximum of $G(., u_0)$ and this concludes the proof of Theorem 4.

4. Existence results

In this section, the proof of the existence result is presented, with the Fibering method serving as the main tool. To proceed, a compactness result is first established.

Lemma 13. *Assume condition* (H). *If* $w \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, *then the embedding*

$$E_\rho \hookrightarrow L^q\left(\mathbb{R}^{N-1}, w(x')\right)$$

is continuous for $q \in (1, p_*]$ *and compact for* $q \in (1, p_*)$.

Proof. First, let $1 < q < p$. By Hölder's inequality,

$$\int_{\mathbb{R}^{N-1}} w|u|^q \,\mathrm{d}x' = \int_{\mathbb{R}^{N-1}} w^{\frac{p-q}{p}} w^{q/p}|u|^q \,\mathrm{d}x' \le \|w\|_1^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^{N-1}} w|u|^p \,\mathrm{d}x'\right)^{q/p}.$$

By assumption (H) and Theorem 1, the following estimate holds

$$\int_{\mathbb{R}^{N-1}} w|u|^q \,\mathrm{d}x' \le \|w\|_\infty^{\frac{q}{p}} \|w\|_1^{\frac{p-q}{p}} \left(\int_{\mathbb{R}^{N-1}} |u|^p \,\mathrm{d}x'\right)^{q/p} \le \left(\frac{C_{p,\gamma}^{1-p}}{\rho_0}\right)^{q/p} \|w\|_\infty^{\frac{q}{p}} \|w\|_1^{\frac{p-q}{p}} \|u\|_\rho^q.$$

If $q \in [p, p_*]$, Theorem 2 and (H) imply

$$\int_{\mathbb{R}^{N-1}} w|u|^q \,\mathrm{d}x' \le \|w\|_\infty \int_{\mathbb{R}^{N-1}} |u|^q \,\mathrm{d}x' \le C\|u\|_\rho^q.$$

Now, if $(u_n) \subset E_\rho$ is a bounded sequence, up to a subsequence, we can assume that $u_n \rightharpoonup 0$ in E. By a reflection argument and [4, Theorem 6.3] it follows that $u_n \to 0$ in $L^q(B_R \cap \mathbb{R}^{N-1})$. Given $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that $\|w\|_{L^1(B_R^c \cap \mathbb{R}^{N-1})} \le \varepsilon$. Hence, by Hölder's inequality, Theorem 2 and (H) the conclusion is

$$\begin{aligned}\int_{B_R^c \cap \mathbb{R}^{N-1}} w|u_n|^q \,\mathrm{d}x' &\le \left(\int_{(B_R)^c \cap \mathbb{R}^{N-1}} |w| \,\mathrm{d}x'\right)^{\frac{p_*-q}{p_*}} \left(\int_{(B_R)^c \cap \mathbb{R}^{N-1}} w|u_n|^{p_*} \,\mathrm{d}x'\right)^{q/p_*} \\ &\le C\varepsilon^{\frac{p_*-q}{p_*}} \|u_n\|_\rho^q \\ &\le C_1 \varepsilon^{\frac{p_*-q}{p_*}}\end{aligned}$$

which concludes the proof. □

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The following result on regularity is also essential.

Lemma 14. *Assume (H), $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N_+)$ and suppose that $h, m \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$. If u is a nonnegative solution for (P), then $u \in L^\infty(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^N_+)$ and $u > 0$ in $\mathbb{R}^N_+$.*

Proof. For $m \in \mathbb{N}$, define $u_m = \min\{u, m\}$. Taking $\varphi = u_m^{kp+1}$ as a test function in (10), with fixed $k > 0$, yields

$$\begin{aligned}\int_{\mathbb{R}^N_+} \rho(x)|\nabla u|^{p-2}\nabla u \nabla(u_m^{kp+1})\mathrm{d}x &= \int_{\mathbb{R}^{N-1}} [hu^{q-1} - mu^{s-1}]u_m^{kp+1}\mathrm{d}x' \\ &\le \int_{\mathbb{R}^{N-1}} hu^{kp+q}\mathrm{d}x'.\end{aligned}$$

On the other hand, Theorem 2 gives

$$\left(\int_{\mathbb{R}^{N-1}} |u|^{p_*}\mathrm{d}x'\right)^{p/p_*} \le C\int_{\mathbb{R}^N_+} \rho(x)|\nabla u|^p \mathrm{d}x,$$

which implies

$$\begin{aligned}\int_{\mathbb{R}^N_+} \rho(x)|\nabla u|^{p-2}\nabla u \nabla(u_m^{kp+1})\mathrm{d}x &= \frac{kp+1}{(k+1)^p}\int_{\mathbb{R}^N_+} \rho(x)|\nabla(u_m^{k+1})|^p \mathrm{d}x \\ &\ge C\frac{kp+1}{(k+1)^p}\left(\int_{\mathbb{R}^{N-1}} u_m^{(k+1)p_*}\mathrm{d}x'\right)^{p/p_*}.\end{aligned}$$

Thus,

$$C\frac{kp+1}{(k+1)^p}\left(\int_{\mathbb{R}^{N-1}} u_m^{(k+1)p_*}\mathrm{d}x'\right)^{p/p_*} \le \int_{\mathbb{R}^{N-1}} hu^{kp+q}\mathrm{d}x'. \tag{30}$$

Now, suppose that $q > p$ and let $\zeta \in (p, p_*)$ and $\xi > 1$ such that

$$\frac{1}{\xi} + \frac{p}{\zeta} + \frac{q-p}{p^*} = 1.$$

Since

$$\int_{\mathbb{R}^{N-1}} hu^{kp+q}\mathrm{d}x' = \int_{\mathbb{R}^{N-1}} hu^{(k+1)p}u^{q-p}\mathrm{d}x',$$

Hölder's inequality gives

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} hu^{kp+q}\mathrm{d}x' &\le \left(\int_{\mathbb{R}^{N-1}} h\mathrm{d}x'\right)^{1/\xi}\left(\int_{\mathbb{R}^{N-1}} hu^{(k+1)\zeta}\mathrm{d}x'\right)^{p/\zeta}\left(\int_{\mathbb{R}^{N-1}} hu^{p_*}\mathrm{d}x'\right)^{(q-p)/p_*} \\ &\le C\left(\int_{\mathbb{R}^{N-1}} hu^{(k+1)\zeta}\right)^{p/\zeta},\end{aligned}$$

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which implies,

$$\left(\int_{\mathbb{R}^{N-1}} u_m^{(k+1)p_*}\mathrm{d}x'\right)^{1/p_*} \leq C_1 \frac{k+1}{(kp+1)^{1/p}} \left(\int_{\mathbb{R}^{N-1}} u^{(k+1)\zeta}\mathrm{d}x'\right)^{1/\zeta}.$$

Taking the limit as m goes to infinity, the following result is obtained

$$\|u\|_{(k+1)p_*} \leq C_1^{1/(k+1)} \left[\frac{k+1}{(kp+1)^{1/p}}\right]^{1/(k+1)} \|u\|_{(k+1)\zeta}, \quad \forall k > 0. \tag{31}$$

Next, choose $k = k_1 > 0$ such that $(k_1+1)\zeta = p_*$ and define

$$(k_n+1)\zeta = (k_{n-1}+1)p_*, \quad n \geq 2.$$

Notice that

$$k_n = \left(\frac{p_*}{\zeta}\right)^n - 1, \tag{32}$$

and $\lim_{n\to\infty} k_n = \infty$, since $\zeta \in (p, p_*)$. By induction, the following holds

$$\|u\|_{(k_n+1)p_*} \leq C_1^{1/(k_n+1)} \left[\frac{k_n+1}{(k_np+1)^{1/p}}\right]^{1/(k_n+1)} \|u\|_{(k_{n-1}+1)p_*}. \tag{33}$$

Thus, using (33) and (31) recursively, it follows that

$$\|u\|_{(k_n+1)p_*} \leq C_1^{\sum_{i=1}^n 1/(k_i+1)} \prod_{i=1}^n \left[\frac{k_i+1}{(k_ip+1)^{1/p}}\right]^{1/(k_i+1)} \|u\|_{p_*}.$$

Write

$$\left[\frac{k_i+1}{(k_ip+1)^{1/p}}\right]^{1/(k_i+1)} = \left[\left(\frac{k_i+1}{(k_ip+1)^{1/p}}\right)^{1/\sqrt{k_i+1}}\right]^{1/\sqrt{k_i+1}}$$

and observe that $\left(\frac{k+1}{(kp+1)^{1/p}}\right)^{1/\sqrt{k+1}} > 1$ for $k > 0$ and $\lim\limits_{k\to\infty} \left(\frac{k+1}{(kp+1)^{1/p}}\right)^{1/\sqrt{k+1}} = 1$. Then, there exists $C_2 > 1$ such that

$$\|u\|_{(k_n+1)p_*} \leq C_1^{\sum_{i=1}^n \frac{1}{k_i+1}} C_2^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} \|u\|_{p_*}.$$

Now, given that $\zeta \in (p, p_*)$, by (32) it follows that

$$\sum_{n=1}^{\infty} \frac{1}{k_n+1} = \sum_{n=1}^{\infty} \left(\frac{\zeta}{p_*}\right)^n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{k_n+1}} = \sum_{n=1}^{\infty} \left(\sqrt{\frac{\zeta}{p_*}}\right)^n < \infty.$$

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Consequently, letting n go to infinity produces the following result

$$\|u\|_{\infty,\mathbb{R}^{N-1}} \leq C\|u\|_{p_*}$$

and hence $u \in L^{\infty}(\mathbb{R}^{N-1})$.

Now, suppose that $q < p$ and take $\lambda \in (p, p_*)$. By Hölder's inequality it follows

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} hu^{kp+q}\mathrm{d}x' &= \int_{\mathbb{R}^{N-1}} h^{\frac{\lambda-p}{\lambda}+\frac{p}{\lambda}} u^{(k+q/p)p}\mathrm{d}x' \\ &\leq \|h\|_1^{(\lambda-p)/\lambda}\left(\int_{\mathbb{R}^{N-1}} hu^{\lambda(k+q/p)}\mathrm{d}x'\right)^{p/\lambda}.\end{aligned}$$

Given that $h \in L^{\infty}(\mathbb{R}^{N-1}) \cap L^1(\mathbb{R}^{N-1})$, the application of (30) yields

$$C\frac{kp+1}{(k+1)^p}\left(\int_{\mathbb{R}^{N-1}} u_m^{(k+1)p_*}\mathrm{d}x'\right)^{p/p_*} \leq \left(\int_{\mathbb{R}^{N-1}} u^{\lambda(k+q/p)}\mathrm{d}x'\right)^{p/\lambda}.$$

Taking the limit as $m \to \infty$, the conclusion below holds

$$\left(\int_{\mathbb{R}^{N-1}} u^{(k+1)p_*}\mathrm{d}x'\right)^{1/p_*} \leq c_1\frac{k+1}{(kp+1)^{1/p}}\left(\int_{\mathbb{R}^{N-1}} u^{\lambda(k+q/p)}\mathrm{d}x'\right)^{1/\lambda}, \quad \forall k > 0. \tag{34}$$

Take $k = k_1 > 0$ such that $\lambda(k_1 + q/p) = p_*$ and define

$$(k_n + q/p)\lambda = (k_{n-1} + 1)p_*.$$

Thus, $k_n > (p_*/\lambda)^n - q/p \longrightarrow \infty$. By recursive application of (34) and using similar arguments as in the case $q > p$, it follows that $u \in L^{\infty}(\mathbb{R}^{N-1})$. Finally, consider, for $n \in \mathbb{N}$, the set $\Omega_n := \{x \in \overline{\mathbb{R}^N_+} : u(x) > n\}$. Note that $|\Omega_n| < \infty$, since $u \in L^{p^*}(\mathbb{R}^N_+) \cap L^p(\mathbb{R}^{N-1})$. By choosing $n > 0$ such that $\|u\|_{\infty,\mathbb{R}^{N-1}} < n$ and considering the function φ_n defined by

$$\varphi_n = \begin{cases} u - n, & \text{in } \Omega_n, \\ 0, & \text{in } \overline{\mathbb{R}^N_+} \setminus \Omega_n, \end{cases}$$

to be used as a test function, the inequality

$$0 \leq \int_{\Omega_n} \rho(x)|\nabla u|^p \,\mathrm{d}x = \int_{\mathbb{R}^{N-1}} hu^{q-1}\varphi_n \,\mathrm{d}x' - \int_{\mathbb{R}^{N-1}} mu^{s-1}\varphi_n \,\mathrm{d}x' \leq 0$$

is obtained. This relation implies that u is constant or satisfies $u \leq n$ almost everywhere in $\mathbb{R}^N_+$. Therefore, $u \in L^{\infty}(\mathbb{R}^N_+)$. Furthermore, an application of the Harnack inequality from [29] ensures the strict positivity $u > 0$ throughout $\mathbb{R}^N_+$. □

To proceed with the fibering method, the following result is required.

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Lemma 15. *Let $1 < p < N$ and $q < \min\{s,p\}$ or $q > \max\{s,p\}$. Then, for each $v \in E_\rho \setminus \{0\}$, there exists a unique real number $r_v > 0$ such that the pair (r_v, v) satisfies the equation.*

$$\|v\|^p = r_v^{q-p} H(v) - r_v^{s-p} M(v) = G(r_v, v). \tag{35}$$

Furthermore, the map $r : E_\rho \setminus \{0\} \to \mathbb{R}$, such that $r(v) = r_v$, belongs to $C^1(E_\rho \setminus \{0\}, \mathbb{R})$ and $\mu r_{\mu v} = r_v$ for all $\mu > 0$ and $v \in E_\rho \setminus \{0\}$.

Proof. For each $v \in E_\rho \setminus \{0\}$, it is necessary to establish the existence, uniqueness, and smoothness of $r_v > 0$.

Existence: Consider the continuous function $f : (0,\infty) \times E_\rho \to \mathbb{R}$ defined by

$$f(r, v) = \|v\|^p r^{p-q} + M(v) r^{s-q} - H(v),$$

and note that $f(r, v) = 0$ if and only if (35) holds.

Now observe that if $v \in E_\rho \setminus \{0\}$ and $q > \max\{s,p\}$, it follows

$$\lim_{r \to 0^+} f(r, v) = \infty \quad \text{and} \quad \lim_{r \to +\infty} f(r, v) = -H(v) < 0.$$

On the other hand, if $q < \min\{s,p\}$, it holds that

$$\lim_{r \to 0^+} f(r, v) = -H(v) < 0 \quad \text{and} \quad \lim_{r \to \infty} f(r, v) = \infty.$$

Thus, in either case, by the Intermediate Value Theorem, there exists $r_v \in (0,\infty)$ such that $f(r_v, v) = 0$.

Uniqueness: Fix $v \in E_\rho \setminus \{0\}$ and suppose there exist $r_1, r_2 > 0$ satisfying (35). Then,

$$\|v\|^p r_1^{p-q} + M(v) r_1^{s-q} = H(v) = \|v\|^p r_2^{p-q} + M(v) r_2^{s-q},$$

which is equivalent to

$$\|v\|^p \big(r_1^{p-q} - r_2^{p-q}\big) + M(v)\big(r_1^{s-q} - r_2^{s-q}\big) = 0.$$

Since $r_1, r_2 > 0$, it follows that $r_1 = r_2$. Therefore, the map $r : E_\rho \setminus \{0\} \to \mathbb{R}$ satisfying (35) is well-defined.

Regularity: To prove that r is of class C^1, observe that

$$\frac{\partial f}{\partial r}(r, v) = (p-q) r^{p-q-1} \|v\|^p + (s-q) M(v) r^{s-q-1} \neq 0, \quad \text{in } (0,\infty) \times E_\rho \setminus \{0\}.$$

By the Implicit Function Theorem, there exist open sets $J \subset \mathbb{R}$ and $V \subset E_\rho \setminus \{0\}$ containing r_v and v, respectively, and a C^1-function $\beta : V \to J$ such that

$$\beta(v) = r_v \quad \text{and} \quad f(\beta(w), w) = 0, \quad \forall w \in V.$$

By the uniqueness of the solution, it follows that $r \equiv \beta$ in V, and hence r is a C^1-function in V.

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Finally, for $\mu > 0$ and $v \in E_\rho \setminus \{0\}$, the identity $f(r_{\mu v}, \mu v) = 0$ yields

$$H(v) = \mu^{p-q} r_{\mu v}^{p-q} \|v\|_\rho^p + \mu^{s-q} r_{\mu v}^{s-q} M(v).$$

Conversely, $f(r_v, v) = 0$ and it follows that

$$H(v) = r_v^{p-q} \|v\|_\rho^p + r_v^{s-q} M(v).$$

By combining the equations above, the following result is obtained:

$$0 = \left(\mu^{p-q} r_{\mu v}^{p-q} - r_v^{p-q}\right) \|v\|_\rho^p + \left(\mu^{s-q} r_{\mu v}^{s-q} - r_v^{s-q}\right) M(v).$$

Thus, $(\mu r_{\mu v})^{p-q} = r_v^{p-q}$, completing the proof. □

Remark 5. According to (10), the weak solutions of $(\mathcal{P})$ correspond to the critical points of the functional $J : E_\rho \to \mathbb{R}$, defined by

$$J(v) = \frac{1}{p}\|v\|_\rho^p - \frac{1}{q} H(v) + \frac{1}{s} M(v).$$

Suppose there exists an open set $\mathcal{U} \subset E_\rho \setminus \{0\}$ and a function $r \in C^1(\mathcal{U}, \mathbb{R})$ such that for each $v \in \mathcal{U}$ with $r_v \neq 0$, the pair (r_v, v) satisfies the condition (35), that is,

$$\|v\|_\rho^p = r_v^{q-p} H(v) - r_v^{s-p} M(v). \tag{36}$$

Then, the functional $J(r_v v)$ can express as

$$\begin{aligned} J(r_v v) =& \frac{r_v^p}{p}\|v\|_\rho^p - \frac{r_v^q}{q} H(v) + \frac{r_v^s}{s} M(v) \\ =& \left(\frac{1}{p} - \frac{1}{q}\right) H(v) r_v^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v) r_v^s. \end{aligned}$$

In particular, if $r > 0$ and rv is a critical point of J, it follows that

$$\langle J'(rv), v\rangle = 0,$$

which is equivalent to (36).

The above remark motivates the definition of the *reduced functional* $\mathcal{J}$ given by

$$\mathcal{J}(v) := J(r_v v) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v) r_v^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v) r_v^s. \tag{37}$$

Lemma 16. *Let $\Phi \in C^1(E_\rho \setminus \{0\}, \mathbb{R})$ such that $\langle \Phi'(v), v\rangle \neq 0$ if $\Phi(v) = 1$. If $v_c \in \mathcal{U}$ is a critical point of $\mathcal{J}$ under the constraint $\Phi(v) = 1$, then $u = r_{v_c} v_c$ is a critical point of J.*

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Proof. Let $r \in C^1(\mathcal{U}, \mathbb{R})$ as in Remark 5. Then $\mathcal{J} : \mathcal{U} \to \mathbb{R}$ can define as in (37) and

$$\langle J'(r_v v), v\rangle = 0, \forall v \in \mathcal{U}. \tag{38}$$

In fact, by (36),

$$\langle J'(r_v v), v\rangle = r_v^{p-1}[\|v\|_\rho^p - r_v^{q-p}H(v) + r_v^{s-p}M(v)] = 0$$

If v_c is a critical point of $\mathcal{J}$ under the constraint $\Phi(v) = 1$, by the Lagrange Multiplier Theorem, there exists $\lambda \in \mathbb{R}$ such that

$$\mathcal{J}'(v_c) = \lambda\Phi'(v_c). \tag{39}$$

On the other hand, by the definition of $\mathcal{J}$ and (38) it follows that

$$\langle \mathcal{J}'(v), w\rangle = r_v\langle J'(r_v v), w\rangle + \langle r_v', w\rangle\langle J'(r_v v), v\rangle = r_v\langle J'(r_v v), w\rangle \tag{40}$$

for all $w \in E_\rho$. Then by (38), (40) and (39)

$$0 = r_{v_c}\langle J'(r_{v_c} v_c), v_c\rangle = \langle \mathcal{J}'(v_c), v_c\rangle = \lambda\langle \Phi'(v_c), v_c\rangle.$$

Since $\langle \Phi'(v_c), v_c\rangle \neq 0$, the value λ equals zero and, consequently, by (39) and (40),

$$0 = \mathcal{J}'(v_c) = r_{v_c} J'(r_{v_c} v_c).$$

Hence, $r_{v_c} v_c$ is a critical point of J. □

4.1. Proof of Theorem 5

For each fixed $v \in E_\rho \backslash \{0\}$, by Lemma 15, there exists $r_v > 0$ such that the pair (r_v, v) satisfies the equation

$$\|v\|_\rho^p r_v^{p-q} + M(v) r_v^{s-q} = H(v). \tag{41}$$

The reduced functional $\mathcal{J}$ in (37) is thus well-defined, and the inequality above leads to the expression:

$$\mathcal{J}(v) = \left(\frac{1}{s} - \frac{1}{q}\right) M(v) r_v^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v\|^p r_v^p > 0.$$

Define

$$c_{S^1} := \inf_{v \in S^1} \mathcal{J}(v).$$

where S^1 denotes the unity sphere in E_ρ.

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Consider (v_n) a minimizing sequence. Extracting a subsequence if necessary, assume that $v_n \rightharpoonup v_0$ in E with $\|v_0\|_\rho \leq 1$. By Lemma 13

$$H(v_n) \to H(v_0) \quad \text{and} \quad M(v_n) \to M(v_0).$$

The claim $v_0 \neq 0$ is made. Suppose by contradiction that $v_0 = 0$. By Lemma 15, the following holds:

$$\|v_n\|_\rho^p = r_{v_n}^{q-p} H(v_n) - r_{v_n}^{s-p} M(v_n).$$

Since $\|v_n\|_\rho = 1$, it follows that

$$1 = r_{v_n}^{q-p} H(v_n) - M(v_n) r_{v_n}^{s-p} \leq r_{v_n}^{q-p} H(v_n).$$

Given that $q > p$ and $H(v_n) \to 0$, it follows that $r_{v_n} \to \infty$. On the other hand,

$$\mathcal{J}(v_n) = \left(\frac{1}{s} - \frac{1}{q}\right) M(v_n) r_{v_n}^s + \left(\frac{1}{p} - \frac{1}{q}\right) r_{v_n}^p \geq \left(\frac{1}{p} - \frac{1}{q}\right) r_{v_n}^p.$$

Taking limits in the above inequality leads to a contradiction, which implies that $v_0 \neq 0$. From the previous inequality, up to a subsequence, it can be assumed that $r_{v_n} \to r_0 \geq 0$. Taking limits in (41) yields

$$r_0^{p-q} + M(v_0) r_0^{s-q} = H(v_0) > 0 \tag{42}$$

which implies that $r_0 > 0$.

Next, it will be shown that $v_0 \in S^1$. Otherwise, there exists $\mu > 1$ such that $\mu v \in S^1$. From Lemma 15, there are $r_{v_0} > 0$ such that

$$\|v_0\|_\rho^p r_{v_0}^{p-q} + M(v_0) r_{v_0}^{s-q} = H(v_0).$$

This, combined with (42) and the fact that $\mu > 1$ implies

$$r_0^{p-q} + M(v_0) r_0^{s-q} < r_{v_0}^{p-q} + M(v_0) r_{v_0}^{s-q}.$$

Since $s, p < q$, it follows that $r_0 > r_{v_0}$.

Define the function

$$\psi(t) = \left(\frac{1}{s} - \frac{1}{q}\right) M(v_0) t^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v_0\|_\rho^p t^p, \quad t > 0$$

which is strictly increasing. Thus,

$$c_{S^1} = \liminf_{n\to\infty} \mathcal{J}(v_n) \geq \left(\frac{1}{s} - \frac{1}{q}\right) M(v_0) r_0^s + \left(\frac{1}{p} - \frac{1}{q}\right) r_0^p \|v_0\|_\rho^p = \psi(r_0).$$

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However,

$$\psi(r_0) > \psi(r_{v_0}) = J(r_{v_0}v_0) = J(\mu r_{\mu v_0} v_0) = \mathcal{J}(\mu v_0).$$

This contradicts the definition of c_{S^1}, given that $\mu v_0 \in S^1$. Therefore, $v_0 \in S^1$. From (42) and the uniqueness of the solution r_{v_0}, it follows that $r_0 = r_{v_0}$ and

$$c_{S^1} = \lim_{n\to\infty} \mathcal{J}(v_n) = \mathcal{J}(v_0).$$

Since v_0 is a critical point of $\mathcal{J}$ restricted to S^1, the same holds for $|v_0|$, thus assume $v_0 \geq 0$. Applying Lemma 16 with $\Phi(v) = \|v\|_\rho^p$ implies that $u = r_0 v_0$ is a critical point of J. The proof is completed by invoking Lemma 14.

4.2. Proof of Theorem 6

As in the previous result there exist r_v such that

$$\|v\|_\rho^p r_v^{p-q} + M(v) r_v^{s-q} = H(v), \tag{43}$$

which ensures the reduced functional $\mathcal{J}$ is well-defined and can be expressed as

$$\mathcal{J}(v) = \left(\frac{1}{s} - \frac{1}{q}\right) M(v) r_v^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v\|_\rho^p r_v^p < 0.$$

Now observe that if $v \in S^1$, equation (43) implies

$$0 < r_v^{p-q} \leq r_v^{p-q} + M(v) r_v^{s-q} = H(v).$$

Based on the embedding results, the functionals H and M are bounded in S^1. Therefore, the function r is bounded due to the condition $p > q$. Consequently, the minimization problem

$$-\infty < c_{S^1} = \inf_{v\in S^1} \mathcal{J}(v) < 0$$

can be studied.

Let $(v_n) \subset S^1$ be a minimizing sequence. By extracting a subsequence if necessary, assume that $v_n \rightharpoonup v_0$ weakly in E with $\|v_0\|_\rho \leq 1$. Moreover, Lemma 13 implies

$$H(v_n) \to H(v_0) \quad \text{and} \quad M(v_n) \to M(v_0).$$

Up to a subsequence, $r_{v_n} \to r_0 \geq 0$. Note that $r_0 v_0 \neq 0$, since $r_{v_n} v_n \rightharpoonup r_0 v_0$ and

$$J(r_0 v_0) \leq \liminf J(r_{v_n} v_n) = \liminf \mathcal{J}(v_n) = c_{S^1} < 0.$$

From (43), the equality

$$\|v_n\|_\rho^p r_{v_n}^{p-q} + M(v_n) r_{v_n}^{s-q} = H(v_n)$$

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holds. Passing to the limit and using $\|v_0\|_\rho \leq 1$,

$$\|v_0\|^p r_0^{p-q} + M(v_0) r_0^{s-q} \leq H(v_0).$$

On the other hand, Lemma 15 applied to v_0 gives

$$\|v_0\|_\rho^p r_{v_0}^{p-q} + M(v_0) r_{v_0}^{s-q} = H(v_0),$$

which implies $r_0 \leq r_{v_0}$.

Assuming by contradiction that $r_0 < r_{v_0}$, consider the function

$$\psi(t) := J(tv_0) = \frac{t^p}{p}\|v_0\|_\rho^p - \frac{t^q}{q}H(v_0) + \frac{t^s}{s}M(v_0), \quad t \in [0, r_{v_0}].$$

Since ψ is strictly decreasing on $[0, r_{v_0}]$,

$$c_{S^1} = \liminf J(r_{v_n} v_n) \geq J(r_0 v_0) > J(r_{v_0} v_0) = \mathcal{J}(v_0),$$

contradicting the definition of c_{S^1}.

Note also that, for all $\mu > 0$ and $v \in E_\rho \setminus \{0\}$,

$$\mathcal{J}(\mu v) = \mathcal{J}(v)$$

since $\mu r_{\mu v} = r_v$ by Lemma 15. Taking $\mu = \|v_0\|_\rho^{-1}$, yields $\mu v_0 \in S^1$ and

$$\mathcal{J}(\mu v_0) = \mathcal{J}(v_0) < c_{S^1},$$

which is impossible. Therefore, the equality $r_{v_0} = r_0$ holds. Consequently,

$$1 = \lim_{n\to\infty} \|v_n\|_\rho^p = r_0^{q-p} H(v_0) - r_0^{s-p} M(v_0) = \|v_0\|_\rho^p$$

and

$$c_{S^1} = \lim_{n\to\infty} \mathcal{J}(v_n) = \lim_{n\to\infty} \left(\frac{1}{s} - \frac{1}{q}\right) M(v_n) r_{v_n}^s + \left(\frac{1}{p} - \frac{1}{q}\right) \|v_n\|^p r_{v_n}^p = \mathcal{J}(v_0).$$

Since v_0 is a critical point of $\mathcal{J}$ on S^1, the function $|v_0|$ is also a critical point. Without loss of generality, assume $v_0 \geq 0$. Applying Lemma 16, it follows that $u = r_0 v_0$ is a critical point of J. The proof of Theorem 6 is completed by invoking Lemma 14.

4.3. Proof of Theorem 7

The third existence result is established by considering the following result.

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Lemma 17. *Assume the assumptions of Theorem 8. Then for each $v \in E_\rho \setminus \{0\}$ the function $G(.,v)$ defined by (24) has a unique critical point at*

$$\overline{r}_v = \left(\frac{H(v)(q-p)}{M(v)(s-p)}\right)^{1/(s-q)}. \tag{44}$$

Moreover,

$$G(\overline{r}_v, v) = \max_{r>0} G(r,v) = \left(\frac{H(v)^{s-p}}{\eta(p,q,s)M(v)^{q-p}}\right)^{1/(s-q)} > 0. \tag{45}$$

Under the assumptions of Theorem 7, define the set

$$\mathcal{U} := \{v \in E_\rho \setminus \{0\} : \|v\|_\rho^p < G(\overline{r}_v, v)\},$$

where $\overline{r}_v$ is given by (44).

Remark 6. Let $\mathcal{D}$ be the set defined in (16). Note that $\emptyset \neq \mathcal{D} \subset \mathcal{U}$. In fact, by (44) it follows

$$M(v) = \left(\frac{q-p}{s-p}\right)\overline{r}_v^{q-s}H(v), \tag{46}$$

and therefore

$$G(\overline{r}_v, v) = \overline{r}_v^{q-p}H(v) - \left(\frac{q-p}{s-p}\right)\overline{r}_v^{q-p}H(v) = \left(\frac{s-q}{s-p}\right)H(v)\overline{r}_v^{q-p}. \tag{47}$$

From the definition, for each $v \in \mathcal{D}$, it holds

$$\|v\|^p < \left(\frac{p}{q}\right)\eta(s,q,p)^{\frac{1}{q-s}}H(v)^{\frac{s-p}{s-q}}M(v)^{\frac{p-q}{s-q}}.$$

Thus, by combining (46), (47), and the condition $p < q$, the following estimate is obtained

$$\begin{aligned}\|v\|_\rho^p &< \left(\frac{p}{q}\right)\frac{(s-q)(q-p)^{\frac{q-p}{s-q}}}{(s-p)^{\frac{s-p}{s-q}}}H(v)^{\frac{s-p}{s-q}}\left[\left(\frac{q-p}{s-p}\right)\overline{r}_v^{q-s}H(v)\right]^{\frac{p-q}{s-q}} \\ &= \frac{p}{q}\left(\frac{s-q}{s-p}\right)\overline{r}_v^{q-p}H(v) \\ &< G(\overline{r}_v, v),\end{aligned}$$

which leads to the inclusion $\mathcal{D} \subset \mathcal{U}$.

Several technical properties of $\mathcal{U}$ are derived below, serving as essential components in the proof of Theorem 7.

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Lemma 18. *If $p < q < s < p_*$, for each $v \in \mathcal{U}$ there exists a unique real number $r_v > \overline{r}_v$ such that the pair (r_v, v) satisfies*

$$\|v\|_\rho^p = r_v^{q-p} H(v) - r_v^{s-p} M(v) = G(r_v, v),$$

and $r \in C^1(\mathcal{U}, \mathbb{R})$, where $r(v) = r_v$. Furthermore, for any $v \in \mathcal{U}$ and $\mu > 0$, it holds $\mu v \in \mathcal{U}$, and as a consequence, $\mathcal{U} \cap S^1 \neq \emptyset$.

Proof. If $v \in \mathcal{U}$, the inequality $\|v\|_\rho^p < G(\overline{r}_v, v)$ holds. Given that $G(r, v) = r^{q-p}\left(H(v) - M(v) r^{s-q}\right)$ with $p < q < s$, it follows that

$$\lim_{r \to \infty} G(r, v) = -\infty.$$

By the Intermediate Value Theorem, there exists a real number $r_v > \overline{r}_v$ such that the pair (r_v, v) satisfies $\|v\|_\rho^p = G(r_v, v)$.

To prove the uniqueness of r_v, observe that $(q-p)\overline{r}_v^{q-p} H(v) = (s-p)\overline{r}_v^{s-p} M(v)$, which allows $G(r, v)$ to be rewritten as

$$G(r, v) = H(v)\left(r^{q-p} - \frac{q-p}{s-p}\overline{r}_v^{q-s} r^{s-p}\right).$$

Consequently,

$$\frac{\partial G}{\partial r}(r, v) = (q-p) r^{s-p-1} H(v)\left(r^{q-s} - \overline{r}_v^{q-s}\right) < 0,$$

for all $r > \overline{r}_v$, thereby ensuring the uniqueness of r_v.

To establish the C^1-regularity of the map r, set $r = r_v$ and note that

$$\frac{\partial G}{\partial r}(r_v, v) = (q-p) r_v^{s-p-1} H(v)\left(r_v^{q-s} - \overline{r}_v^{q-s}\right) < 0. \tag{48}$$

Consider the function $f : (0, \infty) \times \mathcal{U} \to \mathbb{R}$ defined by

$$f(r, v) = G(r, v) - \|v\|_\rho^p.$$

Relation (48) implies that $\frac{\partial f}{\partial r}(r_v, v) < 0$. Therefore, an application of the Implicit Function Theorem, together with the argument used in the proof of Lemma 15, yields $r \in C^1(\mathcal{U}, \mathbb{R})$.

Now, let $v \in \mathcal{U}$. By definition,

$$\|v\|_\rho^p < G(\overline{r}_v, v) = \frac{s-q}{s-p}\overline{r}_v^{q-p} H(v),$$

where (47) has been used. Furthermore, by (28), the relation $\mu^{-1}\overline{r}_v = \overline{r}_{\mu v}$ holds for all $\mu > 0$. As a consequence,

$$\|\mu v\|_\rho^p < \frac{s-q}{s-p}\overline{r}_{\mu v}^{q-p} H(\mu v) = G(\overline{r}_{\mu v}, \mu v),$$

shows that $\mu v \in \mathcal{U}$. This concludes the proof. □

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Lemma 19. *Under the assumptions of Theorem 7 it holds*

$$\inf_{v\in\mathcal{U}\cap S^1} M(v) > 0.$$

Proof. Lemma 18 ensures that $\mathcal{U}\cap S^1 \neq \emptyset$. For any $v \in \mathcal{U}\cap S^1$, by combining (47), (28), and the inequality $1 = \|v\|_\rho^p < G(\overline{r}_v, v)$, it follows that

$$\eta(p,q,s)M(v)^{q-p} < H(v)^{s-p}.$$

Applying Hölder's inequality yields

$$\begin{aligned} H(v) = \int_{\mathbb{R}^{N-1}} h|v|^q \,\mathrm{d}x' &= \int_{\mathbb{R}^{N-1}} \frac{h}{m^{q/s}} m^{q/s}|v|^q \,\mathrm{d}x' \\ &\le \left(\int_{\mathbb{R}^{N-1}} \left[\frac{h}{m^{q/s}}\right]^{\frac{s}{s-q}} \mathrm{d}x'\right)^{(s-q)/s} \left(\int_{\mathbb{R}^{N-1}} m|v|^s \,\mathrm{d}x'\right)^{q/s} \\ &\le C_{h,m} M(v)^{q/s}, \end{aligned}$$

where

$$0 < C_{h,m} = \left(\int_{\mathbb{R}^{N-1}} \left[\frac{h^{1/q}}{m^{1/s}}\right]^{\frac{sq}{s-q}} \mathrm{d}x'\right)^{(s-q)/s},$$

which is finite due to (17).

Combining the inequalities obtained gives

$$\eta(p,q,s)M(v)^{q-p} < H(v)^{s-p} \le C_{h,m}^{s-p} M(v)^{(s-p)q/s}.$$

Hence,

$$0 < \eta(p,q,s) C_{h,m}^{p-s} < M(v)^{(s-q)p/s},$$

thereby completing the proof. □

Lemma 20. *If $\mathcal{D}$ is the set defined in (16), then $\mathcal{D}\cap S^1 \neq \varnothing$, where S^1 is the unit sphere in E_ρ. Moreover,*

$$\mathcal{J}(v) < 0, \quad \forall v \in \mathcal{D}. \tag{49}$$

Proof. If $v \in \mathcal{D}$, the computation in Remark 6 shows that

$$\|v\|_\rho^p < \frac{p}{q}\left(\frac{s-q}{s-p}\right)\overline{r}_v^{q-p} H(v). \tag{50}$$

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For every $\mu > 0$, relation (28) implies

$$\overline{r}_{\mu v} = \mu^{-1}\overline{r}_v, \quad \forall v \in E_\rho \setminus \{0\}.$$

Thus,

$$\|\mu v\|_\rho^p < \frac{p}{q}\left(\frac{s-q}{s-p}\right)\mu^{p-q}\overline{r}_v^{q-p}H(\mu v) = \frac{p}{q}\left(\frac{s-q}{s-p}\right)\overline{r}_{\mu v}^{q-p}H(\mu v),$$

which implies that $\mu v \in \mathcal{D}$. In particular, choosing $\mu = \|v\|^{-1}$, it follows that $\mathcal{D}\cap S^1 \neq \varnothing$.

To verify (49), note that the pair (r_v, v) satisfies the equation

$$M(v)r_v^s = H(v)r_v^q - \|v\|_\rho^p r_v^p.$$

The definition of the reduced functional (37), the fact that $\overline{r}_v < r_v$ for each $v \in \mathcal{D} \subset \mathcal{U}$ and inequality (50) imply

$$\begin{aligned}\mathcal{J}(v) &= \left(\frac{1}{s}-\frac{1}{q}\right)H(v)r_v^q + \left(\frac{1}{p}-\frac{1}{s}\right)\|v\|_\rho^p r_v^p \\ &< \left(\frac{1}{s}-\frac{1}{q}\right)H(v)r_v^q + \left(\frac{1}{p}-\frac{1}{s}\right)\frac{p}{q}\left(\frac{s-q}{s-p}\right)H(v)r_v^q.\end{aligned}$$

Note that the last term of the inequality above is zero, and the proof is complete. □

Proof of Theorem 7: Let $r \in C^1(\mathcal{U}, \mathbb{R})$ be the function defined in Lemma 18. For any $v \in S^1$, the following holds

$$1 = r_v^{q-p}H(v) - r_v^{s-p}M(v),$$

which implies

$$r_v < \left(\frac{H(v)}{M(v)}\right)^{1/(s-q)}, \quad v \in S^1. \tag{51}$$

Since H is bounded in S^1, the application of Lemma 19 ensures that r is bounded in $\mathcal{U}\cap S^1$. Consequently, $\mathcal{J}$ is bounded from below in $\mathcal{U}\cap S^1$, and by Lemma 20, it follows that

$$c_{\mathcal{U}\cap S^1} = \inf_{v\in\mathcal{U}\cap S^1}\mathcal{J}(v) < 0. \tag{52}$$

Let $(v_n) \subset \mathcal{U}\cap S^1$ be a minimizing sequence. Up to a subsequence, $v_n \rightharpoonup v_0$ weakly in E with $\|v_0\|_\rho \leq 1$. Lemmas 13 then imply

$$H(v_n) \to H(v_0) \quad \text{and} \quad M(v_n) \to M(v_0).$$

By Lemma 18, the sequence (r_{v_n}) satisfies $r_{v_n} > \overline{r}_{v_n}$. Furthermore, from (51), the sequence (r_{v_n}) is bounded. Up to a subsequence, it can be assumed that $r_{v_n} \to r_0 \geq 0$.

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Consequently,

$$0 > c_{\mathcal{U}\cap S^1} = \liminf \mathcal{J}(v_n) \geq \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0) r_0^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0) r_0^s.$$

which implies that $r_0 > 0$. In addition, from (28) and Lemma 13, it follows

$$\lim_{n\to+\infty} \overline{r}_{v_n} = \overline{r}_{v_0}.$$

and hence

$$\lim_{n\to+\infty} G(\overline{r}_{v_n}, v_n) = G(\overline{r}_{v_0}, v_0).$$

Given that $v_n \in \mathcal{U}$, the following estimates holds

$$\|v_0\|_\rho^p \leq \liminf_{n\to\infty} \|v_n\|_\rho^p \leq \liminf_{n\to\infty} G(\overline{r}_{v_n}, v_n) = G(\overline{r}_{v_0}, v_0).$$

The next step is to prove that $v_0 \in \mathcal{U}$. Assume by contradiction that $v_0 \notin \mathcal{U}$, that is, $\|v_0\|_\rho^p = G(\overline{r}_{v_0}, v_0)$. Since $\|v_n\|_\rho^p = G(r_{v_n}, v_n)$, taking to the limit it follows

$$G(\overline{r}_{v_0}, v_0) = \|v_0\|_\rho^p \leq \liminf_{n\to\infty} \|v_n\|_\rho^p = \liminf_{n\to\infty} G(r_{v_n}, v_n) = G(r_0, v_0),$$

which implies that $\overline{r}_{v_0} = r_0$ because $\overline{r}_{v_0}$ is the global maximum of $G(., v_0)$. Then, $r_{v_n} \to \overline{r}_{v_0}$ and from the definition of $\mathcal{J}$ and (28) it follows that

$$c_{\mathcal{U}\cap S^1} = \lim_{n\to\infty} \mathcal{J}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0) \overline{r}_{v_0}^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0) \overline{r}_{v_0}^s$$

and $(s-p)M(v_0)\overline{r}_{v_0}^s = H(v_0)(q-p)\overline{r}_{v_0}^q$. Consequently,

$$\begin{aligned} c_{\mathcal{U}\cap S^1} &= M(v_0)\overline{r}_{v_0}^s \left[\frac{(s-p)}{q-p}\left(\frac{1}{p} - \frac{1}{q}\right) + \left(\frac{1}{s} - \frac{1}{p}\right)\right] \\ &= M(v_0)\overline{r}_{v_0}^s \frac{(s-p)}{p}\left(\frac{1}{q} - \frac{1}{s}\right) > 0 \end{aligned}$$

because $p < q < s$, which contradicts (52) and hence $v_0 \in \mathcal{U}$.
Claim: $r_0 = r_{v_0}$.

Assuming the validity of the claim and taking the limit in

$$1 = \|v_n\|_\rho^p = G(r_{v_n}, v_n),$$

yields

$$1 = G(r_{v_0}, v_0) = \|v_0\|_\rho^p.$$

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Thus, it follows that $v_0 \in \mathcal{U} \cap S^1$, and additionally,

$$c_{\mathcal{U}\cap S^1} = \lim_{n\to\infty} \mathcal{J}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0) r_{v_0}^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0) r_{v_0}^s = \mathcal{J}(v_0).$$

Therefore, by Lemma 16, the element $r_{v_0} v_0$ is a nonnegative and nontrivial critical point of $\mathcal{J}$ in E_ρ. The proof is then completed by applying Lemma 14.

It remains to prove that $r_0 = r_{v_0}$. Since $v_0 \in \mathcal{U}$, Lemma 18 ensures the existence of $\mu_0 > 0$ such that $\mu_0 v_0 \in \mathcal{U} \cap S^1$. Moreover, by Lemma 18, the inequalities $r_{v_0} > \overline{r}_{v_0}$ and $G(r_{v_0}, v_0) = \|v_0\|_\rho^p$ hold. Taking the limit in the identity $\|v_n\|_\rho^p = G(r_{v_n}, v_n)$ yields

$$\|v_0\|_\rho^p \leq G(r_0, v_0).$$

Thus,

$$G(r_{v_0}, v_0) = \|v_0\|_\rho^p \leq G(r_0, v_0).$$

Since $G(r, v_0)$ is strictly decreasing for $r \geq \overline{r}_{v_0}$ and $r_{v_0} > \overline{r}_{v_0}$, it follows that

$$r_0 \leq r_{v_0},$$

which implies

$$\overline{r}_{v_0} \leq r_0 \leq r_{v_0}.$$

Assume, by contradiction, that $r_0 < r_{v_0}$. Given that $G(r, v_0)$ is strictly decreasing for $r \in (r_0, r_{v_0})$, one has

$$\|v_0\|_\rho^p = G(r_{v_0}, v_0) < G(r, v_0), \quad \forall r \in [r_0, r_{v_0}).$$

Now, consider the function

$$z(r) = J(r v_0), \quad r \in (r_0, r_{v_0}).$$

A straightforward computation gives

$$z'(r) = r^{p-1} \left(\|v_0\|_\rho^p - G(r, v_0)\right) < 0,$$

showing that z is strictly decreasing on (r_0, r_{v_0}). As a consequence,

$$c_{\mathcal{U}\cap S^1} = \liminf_{n\to\infty} J(r_{v_n} v_n) \geq J(r_0 v_0) > J(r_{v_0} v_0) = J(r_{\mu_0 v_0} \mu_0 v_0) = \mathcal{J}(\mu_0 v_0),$$

where $\mu_0 v_0 \in \mathcal{U} \cap S^1$. This contradicts the definition of $c_{\mathcal{U}\cap S^1}$. Therefore, the equality $r_0 = r_{v_0}$ holds. □

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4.4. Proof of Theorem 8

The proof of this theorem begins with the establishment of several key lemmas essential to the argument.

Lemma 21. *Assume the assumptions in Theorem 8. For each fixed $v \in E_\rho \setminus \{0\}$, the function $G(.,v)$ has a unique critical point which is a maximum and is given by*

$$\overline{r}_v = \left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{1/(q-s)}. \tag{53}$$

Moreover,

$$G(\overline{r}_v, v) = \max_{r>0} G(r,v) = \left(\frac{H(v)^{p-s}}{\eta(s,q,p)M(v)^{p-q}}\right)^{1/(q-s)} > 0, \tag{54}$$

where $\eta(s,q,p)$ was defined in (11).

Similar to the previous existence result, the set under consideration is

$$\mathcal{U} = \{v \in E_\rho \setminus \{0\} : \|v\|_\rho^p < G(\overline{r}_v, v)\},$$

with $\overline{r}_v$ given by (53).

Remark 7. Consider $s < q < p$ and let $\mathcal{C}$ be the set defined in (18). Note that $\mathcal{C} \subset \mathcal{U}$ and hence $\mathcal{U} \neq \emptyset$. In fact, note that $v \in \mathcal{C}$ if, and only if,

$$\|v\|_\rho^p < \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} \left(\frac{H(v)^{p-s}}{\eta(s,q,p)M(v)^{p-q}}\right)^{1/(q-s)} = \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G(\overline{r}_v, v).$$

Then, $\|v\|_\rho^p < G(\overline{r}_v, v)$ and $v \in \mathcal{U}$, since $s < q < p$.

Lemma 22. *If $s < q < p$, for each $v \in \mathcal{U}$ there exists a unique positive real number $r_v > \overline{r}_v$ such that the pair (r_v, v) satisfies*

$$\|v\|_\rho^p = r_v^{q-p} H(v) - r_v^{s-p} M(v) = G(r_v, v),$$

and $r \in C^1(\mathcal{U}, \mathbb{R})$, where $r(v) = r_v$. Furthermore, for any $v \in \mathcal{U}$ and $\mu > 0$, it holds $\mu v \in \mathcal{U}$, and as a consequence, $\mathcal{U} \cap S^1 \neq \emptyset$.

Proof. If $v \in \mathcal{U}$, the inequality $\|v\|_\rho^p < G(\overline{r}_v, v)$ holds. Given that $s < q < p$,

$$\lim_{r\to\infty} G(r,v) = 0.$$

By the Intermediate Value Theorem, there exists a positive real number $r_v > \overline{r}_v$ such that the pair (r_v, v) verifies $\|v\|_\rho^p = G(r_v, v)$. To prove that r_v is unique, note that G is

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decreasing for all $r > \overline{r}_v$. First note that by (53)

$$(q-p)H(v) = (s-p)\overline{r}_v^{s-q}M(v).$$

Consequently,

$$\begin{aligned}\frac{\partial G}{\partial r}(r,v) &= (q-p)r^{q-p-1}H(v) - (s-p)r^{s-p-1}M(v)\\ &= (s-p)M(v)r^{q-p-1}\overline{r}_v^{s-q} - (s-p)r^{s-p-1}M(v)\\ &= (s-p)r^{q-p-1}M(v)\left(\frac{1}{\overline{r}_v^{q-s}} - \frac{1}{r^{q-s}}\right) < 0,\end{aligned}$$

for all $r > \overline{r}_v$, thereby implying the uniqueness of r_v.

To establish the regularity of r, note that for every $v \in \mathcal{U}$, setting $r = r_v$ yields $\frac{\partial G}{\partial r}(r_v, v) < 0$. Consider the function $f : (0,\infty) \times \mathcal{U} \to \mathbb{R}$ defined by

$$f(r,v) = G(r,v) - \|v\|_\rho^p.$$

Since $\frac{\partial f}{\partial r}(r_v, v) < 0$, an application of the Implicit Function Theorem, combined with the argument used in the proof of Lemma 15, yields that $r \in C^1(\mathcal{U}, \mathbb{R})$. Now, observe that given $\mu > 0$

$$\mu\overline{r}_{\mu v} = \overline{r}_v, \quad \forall v \in E_\rho \setminus \{0\}. \tag{55}$$

Indeed, relation (53) gives

$$\overline{r}_{\mu v} = \left(\frac{M(\mu v)(p-s)}{H(\mu v)(p-q)}\right)^{1/(q-s)} = \frac{\mu^{\frac{s}{q-s}}}{\mu^{\frac{q}{q-s}}}\left(\frac{M(v)(p-s)}{H(v)(p-q)}\right)^{1/(q-s)} = \frac{1}{\mu}\overline{r}_v.$$

Using (53), it is possible to write:

$$H(v) = \left(\frac{p-s}{p-q}\right)M(v)\overline{r}_v^{s-q}$$

and therefore

$$G(\overline{r}_v, v) = \overline{r}_v^{q-p}H(v) - \overline{r}_v^{s-p}M(v) = \frac{q-s}{p-q}\overline{r}_v^{s-p}M(v).$$

Suppose that $v \in \mathcal{U}$, that is

$$\|v\|_\rho^p < G(\overline{r}_v, v) = \frac{q-s}{p-q}\overline{r}_v^{s-p}M(v).$$

Relation (55) yields:

$$\|\mu v\|_\rho^p < \mu^{p-s+s}\frac{q-s}{p-q}\overline{r}_v^{s-p}M(v) = \frac{q-s}{p-q}\overline{r}_{\mu v}^{s-p}M(\mu v) = G(\overline{r}_{\mu v}, \mu v),$$

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Since $\mu v \in \mathcal{U}$ for any $\mu > 0$, the choice $\mu = \|v\|_\rho^{-1}$ ensures that $\mathcal{U} \cap S^1 \neq \emptyset$. □

Lemma 23. *Let $s < q < p$. If $v \in \mathcal{C}$, where $\mathcal{C}$ is the set defined in (18), then*

$$\overline{r}_v < \left(\frac{q}{s}\right)^{\frac{1}{q-s}} \overline{r}_v < r_v, \tag{56}$$

where r is the function given in Lemma 22. Moreover,

$$\mathcal{J}(v) < 0, \quad \forall v \in \mathcal{C} \quad \text{and} \quad \mathcal{C} \cap S^1 \neq \emptyset.$$

Proof. As shown in the proof of Lemma 22, $G(r, v)$ is decreasing for $r > \overline{r}_v$. Since $\overline{r}_v < r_v$, to establish (56), it suffices to demonstrate that if $v \in \mathcal{C}$, then

$$G\left(\left(\frac{q}{s}\right)^{\frac{1}{q-s}} \overline{r}_v, v\right) > G(r_v, v).$$

As indicated in Remark 7, v to belong to $\mathcal{C}$ is equivalent to satisfying the following condition:

$$G(r_v, v) = \|v\|_\rho^p < \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G(\overline{r}_v, v). \tag{57}$$

Thus, since $s < q < p$,

$$\begin{aligned} G\left(\left(\frac{q}{s}\right)^{\frac{1}{q-s}} \overline{r}_v, v\right) &= \left(\frac{q}{s}\right)^{\frac{q-p}{q-s}} \overline{r}_v^{q-p} H(v) - \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \overline{r}_v^{s-p} M(v) \\ &> \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \overline{r}_v^{q-p} H(v) - \left(\frac{q}{s}\right)^{\frac{s-p}{q-s}} \overline{r}_v^{s-p} M(v) \\ &= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G(\overline{r}_v, v) \\ &> G(r_v, v), \end{aligned}$$

which conclude (56). To establish that $\mathcal{J} < 0$ in $\mathcal{C}$, by the identity from (53) it follows that

$$M(v) = \frac{p-q}{p-s} \overline{r}_v^{q-s} H(v).$$

and by (56)

$$\overline{r}_v^{q-s} < \frac{s}{q} r_v^{q-s}.$$

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Therefore,

$$\begin{aligned}\mathcal{J}(v) &= \left(\frac{1}{p}-\frac{1}{q}\right)H(v)r_v^q + \left(\frac{1}{s}-\frac{1}{p}\right)M(v)r_v^s \\ &= \left(\frac{q-p}{pq}\right)H(v)r_v^q + \left(\frac{p-s}{sp}\right)\left(\frac{p-q}{p-s}\right)\overline{r}_v^{q-s}H(v)r_v^s \\ &< \left(\frac{q-p}{pq}\right)H(v)r_v^q + \left(\frac{p-q}{sp}\right)\frac{s}{q}H(v)r_v^q \\ &= \frac{1}{p}H(v)r_v^q\left(\frac{q-p}{q}+\frac{p-q}{q}\right) \\ &= 0.\end{aligned}$$

Let $v \in \mathcal{C}$ and $\mu > 0$, then, by multiplying (57) by μ^p and by (55) it follows

$$\begin{aligned}\|\mu v\|_\rho^p &< \mu^p\left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G(\overline{r}_v, v) \\ &= \mu^p\left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} (\overline{r}_v^{q-p}H(v) - \overline{r}_v^{s-p}M(v)) \\ &= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} (\mu^{p-q}\mu^q\overline{r}_v^{q-p}H(v) - \mu^{p-s}\mu^s\overline{r}_v^{s-p}M(v)) \\ &= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} (\overline{r}_{\mu v}^{q-p}H(\mu v) - \overline{r}_{\mu v}^{s-p}M(\mu v)) \\ &= \left(\frac{s}{q}\right)^{\frac{p-s}{q-s}} G(\overline{r}_{\mu v}, \mu v),\end{aligned}$$

which implies that $\mu v \in \mathcal{C}$. Choosing $\mu = \|v\|_\rho^{-1}$ ensures that $\mathcal{C} \cap S^1 \neq \emptyset$. □

The proof of Theorem 8 is now presented. By Lemma 22, the function r is bounded on S^1. For any $v \in S^1$, the estimate below holds:

$$1 = r_v^{q-p}H(v) - r_v^{s-p}M(v) \leq r_v^{q-p}H(v),$$

then $r_v \leq H(v)^{\frac{1}{p-q}}$. Since $1 < q < p$ and H is bounded in S^1 by Lemma 13, the boundedness of r in S^1 is ensured. Consequently, $\mathcal{J}$ remains bounded in S^1. Furthermore, by Remark 7,

$$-\infty < c_{\mathcal{U}\cap S^1} = \inf_{v\in\mathcal{U}\cap S^1} \mathcal{J}(v) < 0.$$

Let $(v_n) \subset \mathcal{U}\cap S^1$ a minimizing sequence. There exist $v_0 \in E_\rho$ such that, going if necessary to a subsequence, $v_n \rightharpoonup v_0$ in E and by Lemma 13,

$$H(v_n) \to H(v_0) \quad \text{and} \quad M(v_n) \to M(v_0). \tag{58}$$

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Up to a subsequence, $r_{v_n} \to r_0 \geq 0$. In particular, note that $r_0 > 0$, otherwise

$$0 > c_{\mathcal{U}\cap S^1} = \lim_{n\to\infty} \mathcal{J}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0) r_0^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0) r_0^s = 0,$$

which is impossible. Furthermore, note that $v_0 \neq 0$ because both $H(v_0)$ and $M(v_0)$ cannot be 0 simultaneously. The next step is to prove that $v_0 \in \mathcal{U}$. By observing (53) and (58), it is obtained that

$$\overline{r}_{v_n} = \left(\frac{M(v_n)(p-s)}{H(v_n)(p-q)}\right)^{1/(q-s)} \longrightarrow \left(\frac{M(v_0)(p-s)}{H(v_0)(p-q)}\right)^{1/(q-s)} = \overline{r}_{v_0}$$

and hence

$$G(\overline{r}_{v_n}, v_n) \longrightarrow G(\overline{r}_{v_0}, v_0),$$

which implies

$$\|v_0\|_\rho^p \leq \liminf_{n\to\infty} \|v_n\|_\rho^p \leq \liminf_{n\to\infty} G(\overline{r}_{v_n}, v_n) = G(\overline{r}_{v_0}, v_0).$$

Suppose that $v_0 \notin \mathcal{U}$, that is,

$$\|v_0\|_\rho^p = G(\overline{r}_{v_0}, v_0).$$

Conversely,

$$\|v_0\|_\rho^p \leq \liminf_{n\to\infty} \|v_n\|_\rho^p = \liminf_{n\to\infty} G(r_{v_n}, v_n) = G(r_0, v_0) \tag{59}$$

and therefore $G(\overline{r}_{v_0}, v_0) = \|v_0\|_\rho^p \leq G(r_0, v_0)$. Since by Lemma 21 $\overline{r}_{v_0}$ is the global maximum of $G(., v_0)$, it follows that $\overline{r}_{v_0} = r_0$. By (53), it can be written that

$$M(v_0)\overline{r}_{v_0}^s = \left(\frac{p-q}{p-s}\right) H(v_0)\overline{r}_{v_0}^q.$$

Thus,

$$\begin{aligned} c_{\mathcal{U}\cap S^1} &= \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0)\overline{r}_{v_0}^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0)\overline{r}_{v_0}^s \\ &= \left(\frac{q-p}{pq}\right) H(v_0)\overline{r}_{v_0}^q + \left(\frac{p-s}{sp}\right)\left(\frac{p-q}{p-s}\right) H(v_0)\overline{r}_{v_0}^q \\ &= \frac{H(v_0)\overline{r}_{v_0}^q}{p}\left(\frac{q-p}{q} + \frac{p-q}{s}\right) \\ &> 0. \end{aligned}$$

This leads to a contradiction, and therefore $v_0 \in \mathcal{U}$.

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Now, it is asserted that $r_0 = r_{v_0}$. To begin, let $\mu = \|v_0\|^{-1}$. According to Lemma 22, it follows that $\mu v_0 \in \mathcal{U} \cap S^1$. Observe now that

$$\begin{aligned} G(\mu r_{\mu v_0}, v_0) =& \frac{1}{\mu^p}(r_{\mu v_0}^{q-p} H(\mu v_0) - r_{\mu v_0}^{s-p} M(\mu v_0)) \\ =& \frac{1}{\mu^p} G(r_{\mu v_0}, \mu v_0) \\ =& \frac{1}{\mu^p} \|\mu v_0\|_\rho^p \\ =& \|v_0\|_\rho^p \\ =& G(r_{v_0}, v_0). \end{aligned}$$

Then, given that $r_{v_0} > \overline{r}_{v_0}$ and $\mu r_{\mu v_0} > \overline{r}_{\mu v_0} = \overline{r}_{v_0}$, by Lemma 22 it follows that

$$\mu r_{\mu v_0} = r_{v_0}. \tag{60}$$

Furthermore, since $\overline{r}_{v_n} < r_{v_n}$, taking the limit as $n \to \infty$ yields $\overline{r}_{v_0} \leq r_0$. Then, by (59), it follows that

$$\overline{r}_{v_0} \leq r_0 \leq r_{v_0}.$$

Suppose by contradiction that $r_0 < r_{v_0}$ and consider the function

$$\sigma(t) := \frac{\partial}{\partial t} J(tv_0) = t^{p-1}(\|v_0\|_\rho^p - G(t, v_0)),$$

for $t \in (r_0, r_{v_0})$. Note that $\sigma(t) < 0$, hence $J(tv_0)$ is decreasing on t and by (60)

$$c_{\mathcal{U}\cap S^1} = \liminf_{n\to\infty} J(r_{v_n} v_n) \geq J(r_0 v_0) > J(r_{v_0} v_0) = J(r_{\mu v_0} \mu v_0) = \mathcal{J}(\mu v_0),$$

with $\mu v_0 \in \mathcal{U} \cap S^1$, which is impossible. Therefore, $r_0 = r_{v_0}$, consequently

$$1 = \lim_{n\to\infty} \|v_n\|_\rho^p = \lim_{n\to\infty} G(r_{v_n}, v_n) = G(r_{v_0}, v_0) = \|v_0\|_\rho^p,$$

which ensures that $v_0 \in \mathcal{U} \cap S^1$, and

$$c_{\mathcal{U}\cap S^1} = \lim_{n\to\infty} \mathcal{J}(v_n) = \left(\frac{1}{p} - \frac{1}{q}\right) H(v_0) r_{v_0}^q + \left(\frac{1}{s} - \frac{1}{p}\right) M(v_0) r_{v_0}^s = \mathcal{J}(v_0).$$

Finally, by Lemma 16, $r_{v_0} v_0$ is a nonnegative and nontrivial critical point of J in E_ρ. Lemma 14 completes the proof of Theorem 8.

5. Final comments

This section outlines potential research directions that emerge from the results established in the present work.

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- The condition $1 < p < \gamma+1$ is sufficient to establish the Hardy type inequality (4), in accordance with the condition given in [20]. Determining a necessary condition on γ for the validity of (4) remains an open problem.
- An interesting question concerns the validity of inequality (4) when considering a singular weight such as $x_N^{p-\gamma}$ instead of $(1+x_N)^{p-\gamma}$.
- Determining the optimal constant in inequality (7) is a natural objective. From (7), it follows that $C_{p,\gamma}^{p-1} \leq T_{\gamma,p}$ whenever $\gamma > p-1$. The validity of equality in this case remains to be investigated.
- Another open problem concerns the validity of the trace embedding (9) in the borderline case $p = N$ and $q \in [N, \infty)$.
- It is of particular interest to establish whether the conditions (12)–(13) are optimal for the derivation of Liouville-type nonexistence results, as presented in Theorems 3–4.
- The sharpness of the assumptions (12)–(13) required in Theorems 3–4 remains an important topic for further analysis.
- The weighted Sobolev trace embeddings established herein may also be employed to investigate the multiplicity of solutions for the class of problems defined in $(\mathcal{P})$, following the variational framework developed in [23].
- Since the compact embeddings obtained play a crucial role in the existence results, it is natural to question whether existence theorems can still be derived in scenarios where compactness fails.

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Availability of data and materials All relevant data generated or analysed during the study are included in this article.

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