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ORDERING p-SPACINGS OF GENERALIZED ORDER STATISTICS REVISITED

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In this article we investigate less restrictive conditions on the model parameters and the underlying distribution function upon which the generalized order statistics are based, which enable one to establish the likelihood ratio and the hazard ratio orderings for p-spacings of generalized order statistic. Some previous works in the literature are extended.

1. INTRODUCTION

Kamps [11,12] introduced the concept of generalized order statistics (GOSs) as a unification of several models of random variables arranged in ascending order of magnitude. Specifically, let $n \in \mathbb{N} \equiv \{1, 2, \ldots\}$, $k > 0$, and $\widetilde{m}_n = (m_1, \ldots, m_{n-1}) \in \Re^{n-1}$ be parameters such that

$$\gamma_{r,n} = k + \sum_{j=r}^{n-1} (m_j + 1) > 0, \qquad r = 1, \ldots, n-1$$

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if $n \geq 2$ ($\tilde{m}_n$ arbitrary if $n = 1$). If the random variables $U_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n$, possess a joint density of the form

$$f_{U_{(1,n,\tilde{m}_n,k)},\ldots,U_{(n,n,\tilde{m}_n,k)}}(u_1, \ldots, u_n) = \left(\prod_{j=1}^{n} \gamma_{j,n}\right)\left(\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right)(1-u_n)^{k-1}$$

for all $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n < 1$ with $\gamma_{n,n} = k$, then they are called uniform GOSs. Now, let F be an arbitrary distribution function. The random variables

$$X_{(r,n,\tilde{m}_n,k)} = F^{-1}(U_{(r,n,\tilde{m}_n,k)}), \qquad r = 1, \ldots, n,$$

are called the GOSs based on F, where F^{-1} is the right continuous version of its inverse.

Let F be a distribution function of some nonnegative random variable. For a given positive integer p, $p \leq n$, denote by

$$D_{r,n}^{(p)} = X_{(r+p-1,n,\tilde{m}_n,k)} - X_{(r-1,n,\tilde{m}_n,k)}, \qquad r = 1, \ldots, n-p+1,$$

the p-spacings of the GOSs $\{X_{(i,n,\tilde{m}_n,k)}, i = 1, \ldots, n\}$. Here, $X_{(0,n,\tilde{m}_n,k)} = 0$. For $p = 1$, 1-spacings $D_{r,n}^{(1)}$ are simple spacings in the literature, written as $D_{r,n}$.

Now, we recall the definitions of some stochastic orders. Throughout, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively. Let Z and W be two random variables with respective survival functions $\overline{F}_Z$ and $\overline{F}_W$. We say that Z is smaller than W in the *usual stochastic order*, denoted by $Z \leq_{\mathrm{st}} W$, if $\overline{F}_Z(t) \leq \overline{F}_W(t)$ for all t. Z is smaller than W in the *hazard rate order*, denoted by $Z \leq_{\mathrm{hr}} W$, if $\overline{F}_W(t)/\overline{F}_Z(t)$ is increasing in t (here, $a/0$ is understood to be $+\infty$ whenever $a > 0$). Z is smaller than W in the *likelihood ratio order*, denoted by $Z \leq_{\mathrm{lr}} W$, if Z and W have respective density functions (or mass functions) f_Z and f_W and if $f_W(t)/f_Z(t)$ is increasing in t. We will also write $F_Z \leq_* F_W$ if $Z \leq_* W$, where $\leq_*$ is $\leq_{\mathrm{st}}$, $\leq_{\mathrm{hr}}$, or $\leq_{\mathrm{lr}}$. The likelihood ratio order implies the hazard rate order, which in turn implies the usual stochastic order. See Shaked and Shanthikumar [16] and Müller and Stoyan [15] for more details on their interrelationships and properties.

Let $\{X_{(i,n,\tilde{m}_n,k)},\ i = 1, \ldots, n\}$ and $\{X_{(i,n+1,\tilde{m}_{n+1},k)},\ i = 1, \ldots, n+1\}$ be GOSs based on F, where $\tilde{m}_{n+1} = (\tilde{m}_n, m_n)$. For the special subclass of GOSs with $m_i = m$ for each i, Hu and Zhuang [9,10] proved, among other things, that if $m \geq 0$ and F is DLR (decreasing likelihood ratio), or if $m \in [-1, 0)$, F is DLR, and its hazard rate function $\lambda_F(x)$ is log-convex in $x \in \Re_+$, then

$$D_{r,n}^{(p)} \leq_{\mathrm{lr}} D_{r+1,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p, \tag{1.1}$$

$$D_{r,n+1}^{(p)} \leq_{\mathrm{lr}} D_{r,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p+1, \tag{1.2}$$

$$D_{r,n}^{(p)} \leq_{\mathrm{lr}} D_{r+1,n+1}^{(p)} \quad \text{for } r = 1, \ldots, n-p+1. \tag{1.3}$$

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They also proved that if F is DFR (decreasing failure rate) and $m \geq -1$, then

$$D_{r,n}^{(p)} \leq_{\mathrm{hr}} D_{r+1,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p, \tag{1.4}$$

$$D_{r,n+1}^{(p)} \leq_{\mathrm{hr}} D_{r,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p+1, \tag{1.5}$$

and if F is DFR and there is no further restriction on m, then

$$D_{r,n}^{(p)} \leq_{\mathrm{hr}} D_{r+1,n+1}^{(p)} \quad \text{for } r = 1, \ldots, n-p+1. \tag{1.6}$$

The formal definitions of the aging classes, DLR and DFR, can be found in Sections 2 and 3, respectively. Fang, Hu, Wu, and Zhuang [7] investigated conditions on the underlying distribution function F and the parameters k and $\tilde{m}_n$ to obtain the multivariate likelihood ratio and the multivariate usual stochastic orderings of simple spacing vectors of GOSs with the m_i's unequal. Consider the following two assumptions:

ASSUMPTION A: $k \geq 1$, $m_i \geq 0$ *for each i, and F is DLR.*

ASSUMPTION B: $k > 0$, $m_i \geq -1$ *for each i, F is DLR and* λ_F *is log-convex.*

In particular, under Assumption A or Assumption B, Fang et al. [7] obtained that

$$\begin{aligned} D_{r,n} &\leq_{\mathrm{lr}} D_{r+1,n} \quad \text{for } r = 1, \ldots, n-p \text{ if } m_1 \geq \cdots \geq m_{n-1}, \\ D_{r,n+1} &\leq_{\mathrm{lr}} D_{r,n} \quad \text{for } r = 1, \ldots, n-p+1, \\ D_{r,n} &\leq_{\mathrm{lr}} D_{r+1,n+1} \quad \text{for } r = 1, \ldots, n-p+1 \text{ if } m_1 \geq \cdots \geq m_n. \end{aligned}$$

From Theorem 2.1 in Fang et al. [7], it is easy to see that if F is DFR and $m_i \geq -1$ for each i, then

$$D_{r,n}^{(p)} \leq_{\mathrm{st}} D_{r+1,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p, \tag{1.7}$$

$$D_{r,n+1}^{(p)} \leq_{\mathrm{st}} D_{r,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p+1, \tag{1.8}$$

$$D_{r,n}^{(p)} \leq_{\mathrm{st}} D_{r+1,n+1}^{(p)} \quad \text{for } r = 1, \ldots, n-p+1. \tag{1.9}$$

The purpose of this article is to investigate conditions on the underlying distribution function F and the model parameters k and $\tilde{m}_n$ under which (1.1)–(1.6) hold for general p-spacings of GOSs with unequal m_i. More precisely, for likelihood ratio orderings, under Assumption A or B, we prove that (1.2) holds, and if, in addition, m_i is decreasing in i, then (1.1) and (1.3) hold. For hazard rate orderings, under the assumption that F is DFR, we prove that if $m_1 \geq \cdots \geq m_{n-1} \geq -1$, then (1.4) holds; if $m_i \geq -1$ for each i, then (1.5) holds; and if $m_1 \geq \cdots \geq m_n$, then (1.6) holds.

The article is organized as follows. Some useful lemmas that will be used in the sequel are presented in Section 2. The main comparison results of p-spacings of GOSs in the likelihood ratio and the hazard rate orders are given in Section 3. We conclude the article with Section 4 on some comments on applications of these main results.

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2. SOME LEMMAS

The first lemma can be used to establish the monotonicity property of a fractional expression in which both numerator and denominator are integrals.

LEMMA 2.1 (Misra and van der Meulen [14]): *Let Θ and $\mathcal{D}$ be subsets of the real line $\Re$ and let X be a random variable, taking values in $\mathcal{D}$, with a distribution function belonging to the family $\mathcal{P} = \{H(\cdot|\theta), \theta \in \Theta\}$, which satisfies*

$$H(\cdot|\theta_1) \leq_{\mathrm{st}} H(\cdot|\theta_2) \quad \textit{whenever } \theta_1, \theta_2 \in \Theta \textit{ and } \theta_1 < \theta_2.$$

Let $\Psi(x,\theta)$ be a real-valued function defined on $\mathcal{D} \times \Theta$. If $\Psi(x,\theta)$ is increasing [decreasing] in (x,θ), then $\mathsf{E}_\theta[\Psi(X,\theta)]$ is increasing [decreasing] in θ.

Denote by $[W|A]$ any random variable whose distribution is the same as the conditional distribution of W given event A. Let $\{X_{(r,n,\tilde{m}_n,k)}, r = 1,\ldots,n\}$ be the GOSs based on distribution function F, and define

$$\overline{F}_u(x) = \frac{\overline{F}(\max\{u, u+x\})}{\overline{F}(u)}, \qquad x \in \Re,$$

the conditional survival function of $[X - u|X > u]$, where X is a random variable with distribution function F.

LEMMA 2.2 (Kamps and Cramer [13]): *For each i, $0 \leq i \leq n-2$, denote*

$$\tilde{\mu}_i = (\mu_{i,1},\ldots,\mu_{i,n-i-1}) = (m_{i+1},\ldots,m_{n-1}). \tag{2.1}$$

Then

$$\left[X_{(r+p-1,n,\tilde{m}_n,k)} - X_{(r-1,n,\tilde{m}_n,k)} | X_{(r-1,n,\tilde{m}_n,k)} = u\right] \overset{\mathrm{st}}{=} X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}$$

for $2 \leq r < r+p \leq n+1$, where $\overset{\mathrm{st}}{=}$ means equality in distribution and $X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}$ is the pth GOS *based on distribution function F_u.*

LEMMA 2.3: *Let F be absolutely continuous.*

(a) If $m_i \geq -1$ for each i, then

$$X_{(p,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} X_{(p,n-1,\tilde{\mu}_1,k)} \quad \textit{for } p = 1,\ldots,n-1, \tag{2.2}$$

where $\tilde{\mu}_1$ is defined in (2.1).

(b) If $\tilde{m}'_n = (m'_1,\ldots,m'_{n-1})$ such that $m_i \geq m'_i$ for each i, then

$$X_{(p,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} X_{(p,n,\tilde{m}'_n,k)} \quad \textit{for } p = 1,\ldots,n.$$

PROOF: We give the proof of part (a); the proof of part (b) is similar. The proof is similar to that of Lemma 2.2 in Xie and Hu [17]. Let $\{B_{j,n}, j = 1,\ldots,p\}$ be a sequence of

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independent random variables such that $B_{j,n}$ is exponential distributed with parameter $\gamma_{j,n}$ and let $\{B_{j,n-1}, j = 1, \ldots, p\}$ be another sequence of independent random variables such that $B_{j,n-1}$ is exponential distributed with parameter $\widetilde{\gamma}_{j,n-1}$ given by

$$\widetilde{\gamma}_{j,n-1} = k + \sum_{l=j}^{n-2}(m_{1+l} + 1) = k + \sum_{l=j+1}^{n-1}(m_l + 1).$$

By Theorem 3.1 in Cramer and Kamps [6],

$$X_{(p,n,\tilde{m}_n,k)} \overset{\text{st}}{=} H_F\left(\sum_{j=1}^{p} B_{j,n}\right) \tag{2.3}$$

and

$$X_{(p,n-1,\tilde{\mu}_1,k)} \overset{\text{st}}{=} H_F\left(\sum_{j=1}^{p} B_{j,n-1}\right), \tag{2.4}$$

where $H_F(x) = F^{-1}(1 - e^{-x})$ for $x \in \Re_+$. Since $\widetilde{\gamma}_{j,n-1} \leq \gamma_{j,n}$ for each j, $B_{j,n} \leq_{\text{lr}} B_{j,n-1}$. Thus, by Theorem 1.C.9 in Shaked and Shanthikumar [16], we have

$$\sum_{j=1}^{p} B_{j,n} \leq_{\text{lr}} \sum_{j=1}^{p} B_{j,n-1}. \tag{2.5}$$

Since F is absolutely continuous, $H_F(x)$ is increasing and differentiable. Since the order $\leq_{\text{lr}}$ is closed under increasing and differentiable transformations (see Theorem 1.C.8 in Shaked and Shanthikumar, [16]), it follows from (2.3)–(2.5) that (2.2) holds. ■

To state the next lemma, We recall the definition of the multivariate likelihood ratio order (see Shaked and Shanthikumar, [16, Sect. 6.E]). Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with density functions f and g, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the *multivariate likelihood ratio order*, denoted by $\mathbf{X} \leq_{\text{lr}} \mathbf{Y}$, if

$$f(\mathbf{x})\, g(\mathbf{y}) \leq f(\mathbf{x} \wedge \mathbf{y})\, g(\mathbf{x} \vee \mathbf{y}), \quad \forall\, \mathbf{x}, \mathbf{y} \in \Re^n,$$

where $\mathbf{x} \wedge \mathbf{y} = \big(\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}\big)$ and $\mathbf{x} \vee \mathbf{y} = \big(\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\}\big)$. It is easy to see that

$$\mathbf{X} \leq_{\text{lr}} \mathbf{Y} \Longrightarrow \frac{g(\mathbf{x})}{f(\mathbf{x})} \text{ is increasing in } \mathbf{x}.$$

Lemma 2.4 follows from Lemmas 2.3–2.5 in Xie and Hu [17] by using the fact that the multivariate order $\leq_{\text{lr}}$ is closed under marginalization.

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LEMMA 2.4 (Xie and Hu [17]): *Let* $\{X_{(r,n,\tilde{m}_n,k)}, r = 1,\ldots,n\}$ *and* $\{X_{(r,n+1,\tilde{m}_{n+1},k)}, r = 1,\ldots,n+1\}$ *be GOSs based on an absolutely continuous distribution* F*, respectively, where* $\tilde{m}_{n+1} = (\tilde{m}_n, m_n)$*. Let* p *be a given positive integer such that* $p < n$*.*

(a) If $m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq -1$*, then*

$$\left(X_{(r-1,n,\tilde{m}_n,k)}, X_{(r+p-1,n,\tilde{m}_n,k)}\right) \leq_{\mathrm{lr}} \left(X_{(r,n,\tilde{m}_n,k)}, X_{(r+p,n,\tilde{m}_n,k)}\right)$$

for $r = 2,\ldots,n-p$*.*

(b) If $m_n \geq -1$*, then*

$$\left(X_{(r-1,n+1,\tilde{m}_{n+1},k)}, X_{(r+p-1,n+1,\tilde{m}_{n+1},k)}\right) \leq_{\mathrm{lr}} \left(X_{(r-1,n,\tilde{m}_n,k)}, X_{(r+p-1,n,\tilde{m}_n,k)}\right)$$

for $r = 2,\ldots,n-p+1$*.*

(c) If $m_1 \geq m_2 \geq \cdots \geq m_n$*, then*

$$\left(X_{(r-1,n,\tilde{m}_n,k)}, X_{(r+p-1,n,\tilde{m}_n,k)}\right) \leq_{\mathrm{lr}} \left(X_{(r,n+1,\tilde{m}_{n+1},k)}, X_{(r+p,n+1,\tilde{m}_{n+1},k)}\right)$$

for $r = 2,\ldots,n-p+1$*.*

Again, by the fact that the multivariate order $\leq_{\mathrm{lr}}$ is closed under marginalization, Lemma 2.4 follows from Theorem 3.6 in Belzunce, Mercader, and Ruiz [4]. Recall that a distribution function F is said to be DLR if F has a log-convex density function.

LEMMA 2.5: *Let* F *be DLR with hazard rate function* λ_F*. If* $k \geq 1$ *and* $m_i \geq 0$ *for each* i*, or if* $k > 0$*,* $m_i \geq -1$ *for each* i *and* λ_F *is log-convex, then*

$$X^u_{(p,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} X^v_{(p,n,\tilde{m}_n,k)}, \quad p = 1,\ldots,n,$$

whenever $u \leq v$*.*

3. MAIN RESULTS

Throughout this section, let F be an absolutely continuous distribution function with $F(0) = 0$ and with the associated hazard rate function λ_F. Let $\{X_{(i,n,\tilde{m}_n,k)},\ i = 1,\ldots,n\}$ and $\{X_{(i,n+1,\tilde{m}_{n+1},k)},\ i = 1,\ldots,n+1\}$ be GOSs based on F, where $\tilde{m}_{n+1} = (\tilde{m}_n, m_n)$. For any random vector (variable) $\mathbf{W}$, denote by $f_{\mathbf{W}}$ and $\overline{F}_{\mathbf{W}}$ the density and survival functions of $\mathbf{W}$, respectively.

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Let $f_{r,n}^{(p)}(x)$ and $\overline{F}_{r,n}^{(p)}(x)$ denote the density and survival functions of $D_{r,n}^{(p)}$, respectively. Then

$$f_{1,n}^{(p)}(x) = f_{X_{(p,n,\tilde{m}_n,k)}}(x), \qquad x \in \Re_+, \tag{3.1}$$

and, for $r \geq 2$,

$$\begin{aligned} f_{r,n}^{(p)}(x) &= \int_{\Re_+} f_{X_{(r-1,n,\tilde{m}_n,k)},X_{(r+p-1,n,\tilde{m}_n,k)}}(u, x+u)\, du \\ &= \int_{\Re_+} f_{X_{(r-1,n,\tilde{m}_n,k)}}(u) f_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(x)\, du, \qquad x \in \Re_+, \end{aligned} \tag{3.2}$$

where the second equality follows from Lemma 2.2. Similarly,

$$\overline{F}_{1,n}^{(p)}(x) = \overline{F}_{X_{(p,n,\tilde{m}_n,k)}}(x), \qquad x \in \Re_+, \tag{3.3}$$

and, for $r \geq 2$,

$$\overline{F}_{r,n}^{(p)}(x) = \int_{\Re_+} f_{X_{(r-1,n,\tilde{m}_n,k)}}(u)\, \overline{F}_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(x)\, du, \qquad x \in \Re_+. \tag{3.4}$$

3.1. Likelihood Ratio Ordering

In this subsection, we consider the following two assumptions:

ASSUMPTION A: $k \geq 1$, $m_i \geq 0$ *for each i, and F is DLR.*

ASSUMPTION B: $k > 0$, $m_i \geq -1$ *for each i, F is DLR and* λ_F *is log-convex.*

THEOREM 3.1: *Under Assumption A or B, if, in addition,* $m_1 \geq m_2 \geq \cdots \geq m_{n-1}$, *then*

$$D_{r,n}^{(p)} \leq_{\mathrm{lr}} D_{r+1,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p. \tag{3.5}$$

PROOF: To prove (3.5), it suffices to verify that for each fixed $r \in \{1, \ldots, n-p\}$,

$$\Delta_1(\theta) \equiv \frac{f_{r+1,n}^{(p)}(\theta)}{f_{r,n}^{(p)}(\theta)} \text{ is increasing in } \theta \in \Re_+.$$

We consider two cases.

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Case 1: For $r \in \{2, \ldots, n-p\}$, by (3.2), $\Delta_1(\theta)$ can be written as

$$\Delta_1(\theta) = \frac{\displaystyle\int_{\Re_+} f_{X_{(r,n,\tilde{m}_n,k)},X_{(r+p,n,\tilde{m}_n,k)}}(u, \theta+u)\, du}{\displaystyle\int_{\Re_+} f_{X_{(r-1,n,\tilde{m}_n,k)},X_{(r+p-1,n,\tilde{m}_n,k)}}(u, \theta+u)\, du} = \mathsf{E}_\theta\left[\Psi_1(U_1, \theta)\right], \tag{3.6}$$

where

$$\Psi_1(u,\theta) = \frac{f_{X_{(r,n,\tilde{m}_n,k)},X_{(r+p,n,\tilde{m}_n,k)}}(u, \theta+u)}{f_{X_{(r-1,n,\tilde{m}_n,k)},X_{(r+p-1,n,\tilde{m}_n,k)}}(u, \theta+u)}, \qquad (u,\theta) \in \Re_+^2,$$

and the nonnegative random variable U_1 has a distribution function belonging to the family $\mathcal{P}_1 = \{H_1(\cdot|\theta), \theta \in \Re_+\}$ with corresponding densities given by

$$h_1(u|\theta) = d_1(\theta) f_{X_{(r-1,n,\tilde{m}_n,k)}}(u) f_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta), \qquad u \in \Re_+.$$

Here, $d_1(\theta)$ is the normalizing constant and $\tilde{\mu}_{r-1}$ is defined in (2.1). Note that

$$\frac{h_1(u|\theta_2)}{h_1(u|\theta_1)} = \frac{d_1(\theta_2)}{d_1(\theta_1)} \frac{f_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta_2)}{f_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta_1)}$$

is increasing in $u \in \Re_+$ whenever $0 < \theta_1 < \theta_2$ if and only if

$$\frac{f_{X^{u_2}_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta)}{f_{X^{u_1}_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta)} \text{ is increasing in } \theta \text{ whenever } 0 \le u_1 < u_2,$$

which follows from Lemma 2.5. This means that $H_1(\cdot|\theta_1) \le_{\mathrm{lr}} H_1(\cdot|\theta_2)$ and, hence,

$$H_1(\cdot|\theta_1) \le_{\mathrm{st}} H_1(\cdot|\theta_2) \text{ whenever } 0 \le \theta_1 < \theta_2. \tag{3.7}$$

On the other hand, since $m_1 \ge m_2 \ge \cdots \ge m_{n-1} \ge -1$, $\Psi_1(u,\theta)$ is increasing in $(u,\theta) \in \Re_+^2$ by Lemma 2.4(a). Thus, applying Lemma 2.1 in (3.6) yields that $\Delta_1(\theta)$ is increasing in $\theta \in \Re_+$ for $r \in \{2, \ldots, n-p\}$.

Case 2: For $r = 1$, it follows from (3.1) and (3.2) that

$$\Delta_1(\theta) = \int_{\Re_+} f_{X_{(1,n,\tilde{m}_n,k)}}(u) \frac{f_{X^u_{(p,n-1,\tilde{\mu}_1,k)}}(\theta)}{f_{X_{(p,n,\tilde{m}_n,k)}}(\theta)}\, du. \tag{3.8}$$

By Lemmas 2.3(a) and 2.5, we have $X_{(p,n,\tilde{m}_n,k)} \le_{\mathrm{lr}} X_{(p,n-1,\tilde{\mu}_1,k)} \le_{\mathrm{lr}} X^u_{(p,n-1,\tilde{\mu}_1,k)}$ for $u \in \Re_+$, which implies that the integrand in (3.8) is increasing in θ and, hence, $\Delta_1(\theta)$ is increasing in $\theta \in \Re_+$. This completes the proof of the theorem. ■

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THEOREM 3.2: *Under Assumption A or B, we have*

$$D^{(p)}_{r,n+1} \le_{\mathrm{lr}} D^{(p)}_{r,n} \quad \textit{for } r = 1, \ldots, n-p+1. \tag{3.9}$$

PROOF: To prove (3.9), it suffices to verify that for each fixed $r \in \{1, \ldots, n-p+1\}$,

$$\Delta_2(\theta) \equiv \frac{f^{(p)}_{r,n+1}(\theta)}{f^{(p)}_{r,n}(\theta)} \text{ is decreasing in } \theta \in \Re_+.$$

For $r = 1$,

$$\Delta_2(\theta) = \frac{f_{X_{(p,n+1,\tilde{m}_{n+1},k)}}(\theta)}{f_{X_{(p,n,\tilde{m}_n,k)}}(\theta)}$$

is decreasing in $\theta \in \Re_+$ since $X_{(p,n+1,\tilde{m}_{n+1},k)} \le_{\mathrm{lr}} X_{(p,n,\tilde{m}_n,k)}$ by Theorem 3.1(b) in Hu and Zhuang [8]. For $r \in \{2, \ldots, n-p+1\}$, by (3.2), $\Delta_2(\theta)$ can be rewritten as

$$\Delta_2(\theta) = \mathsf{E}_\theta\,[\Psi_2(U_1, \theta)], \tag{3.10}$$

where U_1 is the same as in the proof of Theorem 3.1 with property (3.7), and

$$\Psi_2(u,\theta) = \frac{f_{X_{(r-1,n+1,\tilde{m}_{n+1},k)},X_{(r+p-1,n+1,\tilde{m}_{n+1},k)}}(u, \theta+u)}{f_{X_{(r-1,n,\tilde{m}_n,k)},X_{(r+p-1,n,\tilde{m}_n,k)}}(u, \theta+u)}, \qquad (u,\theta) \in \Re^2_+.$$

Since $m_n \ge -1$, $\Psi_2(u,\theta)$ is decreasing in $(u,\theta) \in \Re^2_+$ by Lemma 2.4(b). Thus, applying Lemma 2.1 in (3.10) yields that $\Delta_2(\theta)$ is decreasing in $\theta \in \Re_+$. This completes the proof of the theorem. ■

THEOREM 3.3: *Under Assumption A or B, if, in addition, $m_1 \ge m_2 \ge \cdots \ge m_n$, then*

$$D^{(p)}_{r,n} \le_{\mathrm{lr}} D^{(p)}_{r+1,n+1} \quad \textit{for } r = 1, \ldots, n-p+1. \tag{3.11}$$

PROOF: It suffices to verify that for each fixed $r \in \{1, \ldots, n-p+1\}$,

$$\Delta_3(\theta) \equiv \frac{f^{(p)}_{r+1,n+1}(\theta)}{f^{(p)}_{r,n}(\theta)} \quad \text{is increasing in } \theta \in \Re_+.$$

For $r = 1$, it follows from (3.1) and (3.2) that

$$\Delta_3(\theta) = \int_{\Re_+} f_{X_{(1,n+1,\tilde{m}_{n+1},k)}}(u) \frac{f_{X^u_{(p,n,\tilde{m}^*_n,k)}}(\theta)}{f_{X_{(p,n,\tilde{m}_n,k)}}(\theta)}\, du, \tag{3.12}$$

where $\tilde{m}^*_n = (m_2, \ldots, m_n)$. By Lemmas 2.3(b) and 2.5, we have

$$X_{(p,n,\tilde{m}_n,k)} \le_{\mathrm{lr}} X_{(p,n,\tilde{m}^*_n,k)} \le_{\mathrm{lr}} X^u_{(p,n,\tilde{m}^*_n,k)} \quad \text{for } u \in \Re_+,$$

which implies that the integrand in (3.12) is increasing in θ and, hence, $\Delta_3(\theta)$ is increasing in $\theta \in \Re_+$ for $r = 1$.

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For $r \in \{2, \ldots, n-p+1\}$, by (3.2), $\Delta_3(\theta)$ can be rewritten as

$$\Delta_3(\theta) = \mathsf{E}_\theta\left[\Psi_3(U_1, \theta)\right], \tag{3.13}$$

where U_1 is the same as in the proof of Theorem 3.1 with property (3.7), and

$$\Psi_3(u,\theta) = \frac{f_{X_{(r,n+1,\tilde{m}_{n+1},k)},X_{(r+p,n+1,\tilde{m}_{n+1},k)}}(u, \theta+u)}{f_{X_{(r-1,n,\tilde{m}_n,k)},X_{(r+p-1,n,\tilde{m}_n,k)}}(u,\theta+u)}, \qquad (u,\theta) \in \Re_+^2.$$

Since $m_1 \geq \cdots \geq m_n$, $\Psi_3(u,\theta)$ is increasing in $(u,\theta) \in \Re_+^2$ by Lemma 2.4(c). Thus, applying Lemma 2.1 in (3.13) yields that $\Delta_3(\theta)$ is increasing in $\theta \in \Re_+$. This completes the proof of the theorem. ■

From Theorems 3.1–3.3, we have the following corollary.

COROLLARY 3.4: *Under Assumption A or B, if, in addition, m_i is decreasing in i, then*

$$D_{r,n}^{(p)} \leq_{\mathrm{lr}} D_{s,l}^{(p)} \text{ whenever } p \geq 1 \text{ and } s - r \geq \max\{0, l-n\}.$$

3.2. Hazard Rate Ordering

In Theorems 3.1–3.3, if, instead, F is assumed to be DFR, then the results can be weakened from the likelihood ratio order to the hazard rate order (see Theorems 3.5–3.7). Recall that F is said to be DFR if $\overline{F}(x)$ is log-convex in x. If the hazard rate function λ_F is well defined, then F being DFR means that $\lambda_F(x)$ is decreasing. It is well known that DLR implies DFR (see Barlow and Proschan [3, p. 77]).

THEOREM 3.5: *If F is DFR and $m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq -1$, then*

$$D_{r,n}^{(p)} \leq_{\mathrm{hr}} D_{r+1,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p.$$

PROOF: It suffices to prove that, for $r = 1, \ldots, n-p$,

$$\Delta_4(\theta) \equiv \frac{\overline{F}_{r+1,n}^{(p)}(\theta)}{\overline{F}_{r,n}^{(p)}(\theta)} \quad \text{is increasing in } \theta \in \Re_+.$$

For fixed $r \in \{2, \ldots, n-p\}$, from (3.4) we get that

$$\Delta_4(\theta) = \frac{\displaystyle\int_{\Re_+} f_{X_{(r,n,\tilde{m}_n,k)}}(u)\, \overline{F}_{X^u_{(p,n-r,\tilde{\mu}_r,k)}}(\theta)\, du}{\displaystyle\int_{\Re_+} f_{X_{(r-1,n,\tilde{m}_n,k)}}(u)\, \overline{F}_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta)\, du} = \mathsf{E}_\theta\left[\Psi_4(U_2,\theta)\right], \tag{3.14}$$

where

$$\Psi_4(u,\theta) = \frac{f_{X_{(r,n,\tilde{m}_n,k)}}(u)\, \overline{F}_{X^u_{(p,n-r,\tilde{\mu}_r,k)}}(\theta)}{f_{X_{(r-1,n,\tilde{m}_n,k)}}(u)\, \overline{F}_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta)}, \qquad (u,\theta) \in \Re_+^2,$$

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and the nonnegative random variable U_2 has a distribution function belonging to the family $\mathcal{P}_2 = \{H_2(\cdot|\theta), \theta \in \Re_+\}$ with corresponding densities given by

$$h_2(u|\theta) = d_2(\theta) f_{X_{(r-1,n,\tilde{m}_n,k)}}(u)\, \overline{F}_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta), \qquad u \in \Re_+.$$

Here, $d_2(\theta)$ is the normalizing constant and $\tilde{\mu}_r$ is defined in (2.1). Observe the following:

- $H_2(\cdot|\theta_1) \leq_{\mathrm{st}} H_2(\cdot|\theta_2)$ whenever $0 \leq \theta_1 < \theta_2$. To see it, we prove that $H_2(\cdot|\theta_1) \leq_{\mathrm{lr}} H_2(\cdot|\theta_2)$ whenever $0 \leq \theta_1 < \theta_2$, which is equivalent to

$$X^{u_1}_{(p,n-r+1,\tilde{\mu}_{r-1},k)} \leq_{\mathrm{hr}} X^{u_2}_{(p,n-r+1,\tilde{\mu}_{r-1},k)} \text{ whenever } 0 \leq u_1 < u_2, \tag{3.15}$$

  which follows from Theorem 3.3 in Hu and Zhuang [8], since F being DFR implies that $F_{u_1} \leq_{\mathrm{hr}} F_{u_2}$ whenever $0 \leq u_1 < u_2$.
- $\Psi_4(u, \theta)$ is increasing in $\theta \in \Re_+$ for each fixed u since, by Lemma 2.3 (a),

$$X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)} \leq_{\mathrm{lr}} X^u_{(p,n-r,\tilde{\mu}_r,k)},$$

  which implies $X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)} \leq_{\mathrm{hr}} X^u_{(p,n-r,\tilde{\mu}_r,k)}$.
- $\Psi_4(u, \theta)$ is increasing in $u \in \Re_+$ for each fixed θ. To see it, rewrite $\Psi_4(u, \theta)$ as

$$\Psi_4(u,\theta) = \frac{\displaystyle\int_\theta^\infty f_{X_{(r,n,\tilde{m}_n,k)}}(u) f_{X^u_{(p,n-r,\tilde{\mu}_r,k)}}(t)\, dt}{\displaystyle\int_\theta^\infty f_{X_{(r-1,n,\tilde{m}_n,k)}}(u) f_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(t)\, dt} = \mathsf{E}_u\left[\Psi_5(T,u)\right], \tag{3.16}$$

  where

$$\Psi_5(t,u) = \frac{f_{X_{(r,n,\tilde{m}_n,k)}}(u) f_{X^u_{(p,n-r,\tilde{\mu}_r,k)}}(t)}{f_{X_{(r-1,n,\tilde{m}_n,k)}}(u) f_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(t)} = \frac{f_{X_{(r,n,\tilde{m}_n,k)},X_{(r+p,n,\tilde{m}_n,k)}}(u, u+t)}{f_{X_{(r-1,n,\tilde{m}_n,k)},X_{(r+p-1,n,\tilde{m}_n,k)}}(u, u+t)}, \qquad (t,u) \in \Re_+^2,$$

  and the nonnegative random variable T has a distribution function belonging to the family $\mathcal{P}_3 = \{H_3(\cdot|u), u \in \Re_+\}$ with corresponding densities given by

$$h_3(t|u) = d_3(u) f_{X_{(r-1,n,\tilde{m}_n,k)}}(u) f_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(t) \cdot 1_{\{t \geq \theta\}}, \qquad t \in \Re_+.$$

  Here, $d_3(u)$ is the normalizing constant and 1_A is the indicator function of set A. From (3.15), it follows that $H_3(\cdot|u_1) \leq_{\mathrm{hr}} H_3(\cdot|u_2)$ for $0 \leq u_1 < u_2$ and, hence, $H_3(\cdot|u_1) \leq_{\mathrm{st}} H_3(\cdot|u_2)$ for $0 \leq u_1 < u_2$. Since $m_1 \geq \cdots \geq m_{n-1} \geq -1$, by Lemma 2.4(a) we know that $\Psi_5(t,u)$ is increasing in $(t,u) \in \Re_+^2$. Then applying Lemma 2.1 in (3.16) yields that $\Psi_4(u,\theta)$ is increasing in $u \in \Re_+$ for each fixed θ.

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Combining the above observations and again applying Lemma 2.1 in (3.14), we conclude that $\Delta_4(\theta)$ is increasing in $\theta \in \Re_+$ for $r \in \{2, \ldots, n-p\}$.

For $r = 1$, from (3.3) and (3.4) it follows that

$$\Delta_4(\theta) = \int_{\Re_+} f_{X_{(1,n,\tilde{m}_n,k)}}(u) \frac{\overline{F}_{X^u_{(p,n-1,\tilde{\mu}_1,k)}}(\theta)}{\overline{F}_{X_{(p,n,\tilde{m}_n,k)}}(\theta)}\, du. \tag{3.17}$$

By Lemma 2.3(a) and (3.15), we have

$$X_{(p,n,\tilde{m}_n,k)} \leq_{\mathrm{lr}} X_{(p,n-1,\tilde{\mu}_1,k)} \leq_{\mathrm{hr}} X^u_{(p,n-1,\tilde{\mu}_1,k)} \quad \text{for } u \in \Re_+,$$

which implies that the integrand in (3.17) is increasing in θ and, hence, $\Delta_4(\theta)$ is increasing in $\theta \in \Re_+$ for $r = 1$. This completes the proof of the theorem. ■

THEOREM 3.6: *If F is DFR and $m_i \geq -1$ for each i, then*

$$D^{(p)}_{r,n+1} \leq_{\mathrm{hr}} D^{(p)}_{r,n} \quad \textit{for } r = 1, \ldots, n-p+1.$$

PROOF: It suffices to prove that, for $r = 1, \ldots, n-p+1$,

$$\Delta_5(\theta) \equiv \frac{\overline{F}^{(p)}_{r,n+1}(\theta)}{\overline{F}^{(p)}_{r,n}(\theta)} \text{ is decreasing in } \theta \in \Re_+.$$

Since $m_i \geq -1$ for each i, from Theorem 3.1(b) in Hu and Zhuang [8], it follows that

$$D^{(p)}_{1,n+1} = X_{(p,n+1,\tilde{m}_{n+1},k)} \leq_{\mathrm{lr}} X_{(p,n,\tilde{m}_n,k)} = D^{(p)}_{1,n},$$

implying that $\Delta_5(\theta)$ is decreasing in $\theta \in \Re_+$ for $r = 1$.

For fixed $r \in \{2, \ldots, n-p+1\}$, from (3.4) we get that

$$\Delta_5(\theta) = \mathsf{E}_\theta\left[\Psi_6(U_2, \theta)\right], \tag{3.18}$$

where U_2 is the same as in the proof of Theorem 3.5 and

$$\Psi_6(u,\theta) = \frac{f_{X_{(r-1,n+1,\tilde{m}_{n+1},k)}}(u)\, \overline{F}_{X^u_{(p,n-r+2,\tilde{\mu}^*_{r-1},k)}}(\theta)}{f_{X_{(r-1,n,\tilde{m}_n,k)}}(u)\, \overline{F}_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta)}, \qquad (u,\theta) \in \Re^2_+.$$

Here, $\tilde{\mu}^*_{r-1} = (m_r, \ldots, m_n)$ and $\tilde{\mu}_{r-1} = (m_r, \ldots, m_{n-1})$. Observe the following:

- $H_2(\cdot|\theta_1) \leq_{\mathrm{st}} H_2(\cdot|\theta_2)$ whenever $0 \leq \theta_1 < \theta_2$, as shown in the proof of Theorem 3.5.
- $\Psi_6(u,\theta)$ is decreasing in $\theta \in \Re_+$ for each fixed u since, by Lemma 2.3(a),

$$X^u_{(p,n-r+2,\tilde{\mu}^*_{r-1},k)} \leq_{\mathrm{lr}} X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)},$$

which implies $X^u_{(p,n-r+2,\tilde{\mu}^*_{r-1},k)} \leq_{\mathrm{hr}} X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}$.

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- $\Psi_6(u,\theta)$ is decreasing in $u \in \Re_+$ for each fixed θ. To see it, rewrite $\Psi_6(u,\theta)$ as

$$\Psi_6(u,\theta) = \frac{\displaystyle\int_\theta^\infty f_{X_{(r-1,n+1,\tilde{m}_{n+1},k)},X_{(r+p-1,n+1,\tilde{m}_{n+1},k)}}(u,u+t)\,dt}{\displaystyle\int_\theta^\infty f_{X_{(r-1,n,\tilde{m}_n,k)},X_{(r+p-1,n,\tilde{m}_n,k)}}(u,u+t)\,dt} = \mathsf{E}_u\left[\Psi_7(T,u)\right], \tag{3.19}$$

where

$$\Psi_7(t,u) = \frac{f_{X_{(r-1,n+1,\tilde{m}_{n+1},k)},X_{(r+p-1,n+1,\tilde{m}_{n+1},k)}}(u,u+t)}{f_{X_{(r-1,n,\tilde{m}_n,k)},X_{(r+p-1,n,\tilde{m}_n,k)}}(u,u+t)}, \qquad (t,u) \in \Re_+^2,$$

and T is the same as in the proof of Theorem 3.5 with the property that $H_3(\cdot|u_1) \le_{\mathrm{st}} H_3(\cdot|u_2)$ for $0 \le u_1 < u_2$. Since $m_n \ge -1$, by Lemma 2.4(b), $\Psi_7(t,u)$ is decreasing in $(t,u) \in \Re_+^2$. Then applying Lemma 2.1 in (3.19) yields that $\Psi_6(u,\theta)$ is decreasing in $u \in \Re_+$ for each fixed θ.

Combining the above observations and again applying Lemma 2.1 in (3.18), we conclude that $\Delta_5(\theta)$ is decreasing in $\theta \in \Re_+$ for $r \in \{2,\ldots,n-p+1\}$. This completes the proof of the theorem. ■

THEOREM 3.7: *If F is DFR and $m_1 \ge m_2 \ge \cdots \ge m_n$, then*

$$D_{r,n}^{(p)} \le_{\mathrm{hr}} D_{r+1,n+1}^{(p)} \quad \textit{for } r = 1,\ldots,n-p+1.$$

PROOF: It suffices to prove that, for $r = 1,\ldots,n-p+1$,

$$\Delta_6(\theta) \equiv \frac{\overline{F}_{r+1,n+1}^{(p)}(\theta)}{\overline{F}_{r,n}^{(p)}(\theta)} \text{ is increasing in } \theta \in \Re_+.$$

For $r = 1$, from (3.3) and (3.4) it follows that

$$\Delta_6(\theta) = \int_{\Re_+} f_{X_{(1,n+1,\tilde{m}_{n+1},k)}}(u)\frac{\overline{F}_{X^u_{(p,n,\tilde{m}^*_n,k)}}(\theta)}{\overline{F}_{X_{(p,n,\tilde{m}_n,k)}}(\theta)}\,du, \tag{3.20}$$

where $\tilde{m}_n^* = (m_2,\ldots,m_n)$. By Lemma 2.3(b) and (3.15), we have

$$X_{(p,n,\tilde{m}_n,k)} \le_{\mathrm{lr}} X_{(p,n,\tilde{m}^*_n,k)} \le_{\mathrm{hr}} X^u_{(p,n,\tilde{m}^*_n,k)} \quad \text{for } u \in \Re_+,$$

implying that the integrand in (3.20) is increasing in θ and, hence, $\Delta_6(\theta)$ is increasing in $\theta \in \Re_+$ for $r = 1$.

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For fixed $r \in \{2, \ldots, n-p+1\}$, from (3.4) we get that

$$\Delta_6(\theta) = \mathsf{E}_\theta \left[\Psi_8(U_2, \theta)\right], \tag{3.21}$$

where U_2 is the same as in the proof of Theorem 3.5 with the property that $H_2(\cdot|\theta_1) \leq_{\mathrm{st}} H_2(\cdot|\theta_2)$ whenever $0 \leq \theta_1 < \theta_2$, and

$$\Psi_8(u, \theta) = \frac{f_{X_{(r,n+1,\tilde{m}_{n+1},k)}}(u)\, \overline{F}_{X^u_{(p,n-r+1,\tilde{\mu}^*_r,k)}}(\theta)}{f_{X_{(r-1,n,\tilde{m}_n,k)}}(u)\, \overline{F}_{X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)}}(\theta)}, \qquad (u, \theta) \in \Re^2_+.$$

Here, $\tilde{\mu}^*_r = (m_{r+1}, \ldots, m_n)$ and $\tilde{\mu}_{r-1} = (m_r, \ldots, m_{n-1})$. Observe the following:

- $\Psi_8(u, \theta)$ is increasing in $\theta \in \Re_+$ for each fixed u since, by Lemma 2.3(b),

$$X^u_{(p,n-r+1,\tilde{\mu}_{r-1},k)} \leq_{\mathrm{hr}} X^u_{(p,n-r+1,\tilde{\mu}^*_r,k)}.$$

- $\Psi_8(u, \theta)$ is increasing in $u \in \Re_+$ for each fixed θ. To see it, rewrite $\Psi_8(u, \theta)$ as

$$\Psi_8(u, \theta) = \frac{\displaystyle\int_\theta^\infty f_{X_{(r,n+1,\tilde{m}_{n+1},k)}, X_{(r+p,n+1,\tilde{m}_{n+1},k)}}(u, u+t)\, dt}{\displaystyle\int_\theta^\infty f_{X_{(r-1,n,\tilde{m}_n,k)}, X_{(r+p-1,n,\tilde{m}_n,k)}}(u, u+t)\, dt} = \mathsf{E}_u \left[\Psi_9(T, u)\right], \tag{3.22}$$

where

$$\Psi_9(t, u) = \frac{f_{X_{(r,n+1,\tilde{m}_{n+1},k)}, X_{(r+p,n+1,\tilde{m}_{n+1},k)}}(u, u+t)}{f_{X_{(r-1,n,\tilde{m}_n,k)}, X_{(r+p-1,n,\tilde{m}_n,k)}}(u, u+t)}, \qquad (t, u) \in \Re^2_+,$$

and T is the same as in the proof of Theorem 3.5 with the property that $H_3(\cdot|u_1) \leq_{\mathrm{st}} H_3(\cdot|u_2)$ for $0 \leq u_1 < u_2$. Since $m_1 \geq \cdots \geq m_n$, by Lemma 2.4(c), $\Psi_7(t, u)$ is increasing in $(t, u) \in \Re^2_+$. Then applying Lemma 2.1 in (3.22) yields that $\Psi_8(u, \theta)$ is increasing in $u \in \Re_+$ for each fixed θ.

Again applying Lemma 2.1 in (3.21), we conclude that $\Delta_6(\theta)$ is increasing in $\theta \in \Re_+$ for $r \in \{2, \ldots, n-p+1\}$. This completes the proof of the theorem. ■

From Theorems 3.5–3.7, we have the following corollary.

COROLLARY 3.8: *If F is DFR and if m_i is decreasing in i with $m_i \geq -1$ for each i, then*

$$D^{(p)}_{r,n} \leq_{\mathrm{hr}} D^{(p)}_{s,l} \ \textit{whenever } p \geq 1 \textit{ and } s - r \geq \max\{0, l - n\}.$$

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4. COMMENTS ON APPLICATIONS

Choosing the parameters k, n, and $\tilde{m}_n = (m_1, \ldots, m_{n-1})$ appropriately, several other models of ordered random variables are seen to be particular cases. For example, we have the following:

- $\mathfrak{F}_1$: ordinary order statistics, record values, order statistics with noninteger sample size, k-record values (see Kamps [11,12]),
- $\mathfrak{F}_2$: a subclass of sequential order statistics (see Cramer and Kamps [5,6]),
- $\mathfrak{F}_3$: progressive type II right censored order statistics (see Balakrishnan [1] and Balakrishnan and Aggarwala [2]),
- $\mathfrak{F}_4$: order statistics under multivariate imperfect repair (see Belzunce et al. [4]).

The models in family $\mathfrak{F}_1$ correspond to GOSs with $m_1 = \cdots = m_{n-1}$, whereas the models in families $\mathfrak{F}_2$–$\mathfrak{F}_4$ correspond to GOSs with the m_i's unequal. Hu and Zhuang [9,10] gave some comparison results on ordering p-spacings of the models in family $\mathfrak{F}_1$. Therefore, the main results in Section 3 can be applied to provide stochastic comparisons of p-spacings of these models in families $\mathfrak{F}_2$–$\mathfrak{F}_4$ with respect to the likelihood ratio and the hazard rate orders.

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