Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 March 2017.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

100.I (Stan Dolan)

Let $x_1, \ldots, x_n$ be a sequence of non-zero real numbers such that

$$x_1 - \frac{1}{x_2} = x_2 - \frac{1}{x_3} = \ldots = x_{n-1} - \frac{1}{x_n} = x_n - \frac{1}{x_1}.$$

(a) Prove that either $x_1 = x_2 = \ldots = x_n$ or n is even and $x_1 x_2 \ldots x_n = 1$.

(b) Find the general solution for $x_1, \ldots, x_n$.

100.J (Michael Fox)

A problem for 2016.

Find a natural number n for which 2^n begins with the digits 2016.

100.K (Muhammed Osman Corbali)

Let the acute-angled triangle ABC have side-lengths a, b, c and circumradius R. Prove that

$$\frac{a^2b^2}{a^2 + b^2 - c^2} + \frac{b^2c^2}{b^2 + c^2 - a^2} + \frac{c^2a^2}{c^2 + a^2 - b^2} \geqslant 9R^2.$$

100.L (Robert M. Young)

A problem for Boxing Day, "Waste not, want not".

Starting with the unit square, an open-top box is formed in the standard way by removing four equal squares from the corners and then folding up the flaps to form the sides. From these deleted squares, four new boxes are then formed in the same way. At the next stage, sixteen new boxes are formed, and so on *ad infinitum*. How should the successive squares be removed so that the sum of the volumes of all the boxes is a maximum?

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Solutions and comments on **100.A**, **100.B**, **100.C**, **100.D** (March 2016).

100.A (Jeremy D. King)

Without using a calculator, determine which of $3^{(3^4)}$ or $4^{(4^3)}$ is bigger.

This popular problem attracted many appreciative comments and a wide variety of approaches.

The solutions involving direct calculation usually revolved around the inequality $2^{19} < 3^{12}$ from which $3^{81} > 2^{19\times 18/12} = 2^{128\frac{1}{4}} > 4^{64}$ follows. To establish $2^{19} < 3^{12}$, the GCHQ Problem Solving Group observed that $3^7 = 2187 > 2176 = 17 \times 2^7$, hence $3^{14} > 289 \times 2^{14} > 288 \times 2^{14}$ which suffices.

Other solutions sought to establish the equivalent inequality $\left(\frac{256}{243}\right)^{16} < 3$. Colin Goldsmith and Edmund Swylan started from $x = 1 + \frac{13}{243} < 1.06$ and used repeated squaring and rounding to show that $x^8 < 1.7$ and $x^{16} < 3$. Thomas Kecker, Zoltan Retkes and Brian Trustrum used the fact that $\left(1 + \frac{1}{n}\right)^n < e$ so that $\left(\frac{256}{243}\right)^{16} = \left(1 + \frac{13}{243}\right)^{16} < \left(1 + \frac{1}{16}\right)^{16} < e < 3$. This last inequality can be established without a calculator – for example, via $e < \left(1 + \frac{1}{n}\right)^{n+1}$ with $n = 5$.

A few solvers estimated logarithms, including the proposer Jeremy King who employed the Maclaurin series for $\ln\left(\frac{1+x}{1-x}\right)$:

$$\ln\left(\frac{9}{8}\right) = \ln\left(\frac{1+\frac{1}{17}}{1-\frac{1}{17}}\right) = 2\left(\frac{1}{17} + \ldots\right) > \frac{2}{17}$$

and $$\ln\left(\frac{4}{3}\right) = \ln\left(\frac{1+\frac{1}{7}}{1-\frac{1}{7}}\right) = 2\left(\frac{1}{7} + \frac{1}{3\times 7^3} + \ldots\right) < 2\left(\frac{1}{7} + \frac{1}{7^3} + \ldots\right) = \frac{7}{24}.$$

Hence

$$\ln\left(\frac{3^{81}}{4^{64}}\right) = 81\ln 3 - 128\ln 2 = 34\ln\left(\frac{9}{8}\right) - 13\ln\left(\frac{4}{3}\right)$$
$$> 34 \times \frac{2}{17} - 13 \times \frac{7}{24} = \frac{5}{24} > 0,$$

as required.

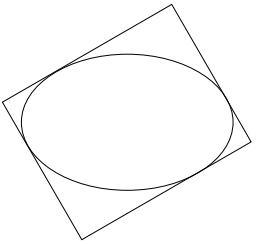
Finally, Gwyn Owen remarked that the same question, but comparing 4^{65} with 3^{82}, is also a good one.

Correct solutions were received from: M. Bataille, M. V. Channakeshava, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, C. Goldsmith, G. Howlett, T. Kecker, J. A. Mundie, G. Owen, Z. Retkes, B. N. Roth, V. Schindler, G. Strickland, E. Swylan, G. B. Trustrum and the proposer Jeremy D. King.

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100.B (Stan Dolan)

The diagram illustrates an ellipse and one of its circumscribing rectangles. For a given ellipse, determine the locus of the vertices of the rectangles.



Answer: If the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the locus is the *director circle* (or *orthoptic circle*), $X^2 + Y^2 = a^2 + b^2$.

Almost all solutions featured coordinate geometry arguments, often along the following lines. The line $y = mx + c$ is a tangent to the ellipse if the quadratic $\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$ has discriminant zero. This gives the condition $a^2m^2 + b^2 = c^2$. If the tangent passes through (X, Y), this can be rewritten as

$$a^2m^2 + b^2 = (Y - mX)^2$$

$$\text{or} \qquad \left(a^2 - X^2\right)m^2 + 2XYm + \left(b^2 - Y^2\right) = 0. \qquad (*)$$

For perpendicular tangents through (X, Y), the product of the roots for m is -1 so that $\frac{b^2 - Y^2}{a^2 - X^2} = -1$ and $X^2 + Y^2 = a^2 + b^2$. This shows that every vertex of a circumscribing rectangle lies on the director circle. It is intuitively clear that every such point on the circle arises in this way. Formally, if $A\,(X, Y)$ is on the director circle, so is $C\,(-X, -Y)$ and, by $(*)$, the perpendicular tangents through A and C are pairwise parallel and form a circumscribing rectangle with diagonal AC.

Lienhard Wimmer commented that in general the locus of pairs of tangents to an ellipse that meet at a fixed angle is a quartic curve that may be realised as the intersection of a torus with a plane. Volkhard Schindler noted that this more general locus occurred as a problem in *Mathematics Magazine* with his solution to it published in 1986, pp. 113-114.

Correct solutions were received from: M. Bataille, M. V. Channakeshava, M. G. Elliott, M. D. Fox, GCHQ Problem Solving Group, G. T. Q. Hoare, G. Howlett, J. D. King, J. A. Mundie, Z. Retkes, V. Schindler, C. Starr, E. Swylan, G. Strickland, A. Tee, G. B. Trustrum, L. Wimmer and the proposer Stan Dolan.

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100.C (Muhammed Osman Corbali)

Triangle ABC has side-lengths $a = BC$, $b = AC$, $c = AB$ and ex-radii r_a, r_b, r_c where r_a refers to the escribed circle opposite to vertex A.

Prove that $\dfrac{b^2c^2}{r_a^2} + \dfrac{a^2c^2}{r_b^2} + \dfrac{a^2b^2}{r_c^2} \geqslant \dfrac{4}{3}(a^2 + b^2 + c^2)$.

Several solvers noted that a slightly stronger result holds, namely

$$\frac{b^2c^2}{r_a^2} + \frac{c^2a^2}{r_b^2} + \frac{a^2b^2}{r_c^2} \geqslant 12R^2. \qquad (*)$$

This implies the inequality in **100.C** because of the fact that

$$OH^2 = 9R^2 - (a^2 + b^2 + c^2) \geqslant 0$$

where O is the circumcentre and H is the orthocentre.

Standard formulae give $r_a = \dfrac{\Delta}{s - a}$ and $\Delta = \dfrac{abc}{4R}$ where Δ is the area of triangle ABC. Thus

$$\frac{bc}{r_a} = \frac{4R(s - a)}{a} = 2R\left(\frac{b}{c} + \frac{c}{a} - 1\right)$$

and $$\sum \frac{bc}{r_a} = 2R\left(\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{a} - 3\right) \geqslant 6R$$

by the AM-GM inequality.

Then, by the Cauchy-Schwarz inequality,

$$3\sum \frac{b^2c^2}{r_a^2} \geqslant \left(\sum \frac{bc}{r_a}\right)^2 \geqslant 36R^2$$

which establishes $(*)$. There is equality in both $(*)$ and **100.C** if, and only if, triangle ABC is equilateral.

Correct solutions were received from: M. Bataille, M. V. Channakeshava, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, M. Lukarevski (2 solutions), J. A. Mundie, P. Nüesch, Z. Retkes (2 solutions), V. Schindler, E. Swylan and the proposer Muhammed Osman Corbali.

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100.D (Isaac Sofair)

(a) Prove the polynomial identity

$$(1-x)^m \sum_{k=0}^{n-1} \binom{m+k-1}{k} x^k + x^n \sum_{k=0}^{m-1} \binom{n+k-1}{k} (1-x)^k \equiv 1.$$

(b) Show that, for $0 < x < 1$,

$$m(1-x)^m \sum_{k=0}^{n-1} \binom{m+k}{k} x^k + nx^n \sum_{k=0}^{m-1} \binom{n+k}{k} (1-x)^k < m+n-1.$$

First, an apology: as several respondents spotted, the inequality in (b) is not strict when $m = n = 1$. Solvers split evenly between those providing algebraic arguments and those using probabilistic ones.

As representative of the algebraic ones, the similar solutions of Michel Bataille and Stan Dolan (below) were notably succinct.

For $m, n \geqslant 1$, let $F(n, m) = \sum_{k=0}^{n-1} \binom{m+k-1}{k} x^k$. Then. by standard properties of binomial coefficients,

$$(1-x)F(n,m) = \begin{cases} F(n, m-1) - \binom{m+n-2}{n-1} x^n & \text{for } m \geqslant 2 \\ 1 - x^n & \text{for } m = 1 \end{cases}.$$

Repeating this process m times yields

$$(1-x)^m F(n,m) = 1 - x^n \sum_{k=0}^{m-1} \binom{n+k-1}{k} (1-x)^k$$

as required for (a).

For (b), note that

$$m\binom{m+k}{k} = (m+k)\binom{m+k-1}{k} \leqslant (m+n-1)\binom{m+k-1}{k}$$

with equality if $k = n-1$.

Thus

$$m(1-x)^m \sum_{k=0}^{n-1} \binom{m+k}{k} x^k + nx^n \sum_{k=0}^{m-1} \binom{n+k}{k} (1-x)^k$$

$$\leqslant (m+n-1)(1-x)^m \sum_{k=0}^{n-1} \binom{m+k-1}{k} x^k + (m+n-1)x^n \sum_{k=0}^{m-1} \binom{n+k-1}{k} (1-x)^k$$

$$\leqslant m+n-1,$$

by (a) with the inequality being strict unless $m = n = 1$.

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M. G. Elliott's solution and one of the three from the GCHQ Problem Solving Group considered repeatedly tossing a biased coin (where the probability of a head is $1 - x$) and stopping when either m heads or n tails have occurred. Then $\binom{m+k-1}{k}(1-x)^m x^k$ gives the probablity of stopping because m heads and $0 \leqslant k \leqslant n-1$ tails have occurred; similarly $\binom{n+k-1}{k}(1-x)^k x^n$ gives the probability of stopping because n tails and $0 \leqslant k \leqslant m-1$ heads have occurred. The identity in (a) expresses the fact that the sum of these probabilities is 1. This establishes (a) for $0 < x < 1$; since it is a polynomial identity, it holds for all x.

For (b), note that since $m\binom{m+k}{k} = (m+k)\binom{m+k-1}{k}$ and $n\binom{n+k}{k} = (n+k)\binom{n+k-1}{k}$, the left-hand side of (b) is just the expected number of tosses until either m heads or n tails have occurred. And this is less than or equal to $m + n - 1$ since there cannot be more than $m + n - 1$ tosses altogether. The inequality is strict unless $m = n = 1$ when there is only one term in each sum.

The GCHQ Problem Solving Group noted that setting $x = \sin^2\theta$ in (a) gives the family of trigonometric identities

$$\cos^{2m}\theta \sum_{k=0}^{n-1}\binom{m+k-1}{k}\sin^{2k}\theta + \sin^{2n}\theta \sum_{k=0}^{m-1}\binom{n+k-1}{k}\cos^{2k}\theta \equiv 1$$

and the proposer, Isaac Sofair, commented that the identity in (a) arose from the *Problem of points*, an early and historically important problem in probability theory which engaged the attention of Pascal, Fermat, Montmort, Lagrange and Laplace.

Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group (3 solutions), G. Howlett, E. Swylan and the proposer Isaac Sofair.

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