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Analyzing kinematics and solving active/constrained forces of a 4-dof 3SPS+SP parallel manipulator

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SUMMARY

A novel 3SPS+SP parallel manipulator (PM) with 4-dof is proposed. Its kinematics and statics are analyzed systematically. The analytic formulae for solving the displacement, velocity, acceleration, workspace, active forces and constrained force are derived. The analytic results are verified by using a simulation mechanism of the 3SPS+SP PM.

KEYWORDS: Parallel manipulator, Kinematics, Workspace, Active force, Constrained force.

Nomenclatures

B, m – the base and the moving platform
r_i – the leg and its length
l_i, L_i – the side of m and the side of B
P, S – the prismatic joint and the spherical joint
R_1, R_2, R_3 – the revolute joints
O, o – the center point of B and the center point of m
$\{m\}$ – coordinate o-xyz fixed on m at o
$\{B\}$ – coordinate O-XYZ fixed on B at O
b_i, B_i – the vertices of m and the vertices of B
v_{ri} – the input velocity of leg
e, E – the distances from b_i to o and from B_i to O
$\boldsymbol{\delta}_i$, $\boldsymbol{c}_j$ – the unit vectors of r_i and F_{cj}
$\boldsymbol{v}_r$, $\boldsymbol{a}_r$ – the general inverse velocity and acceleration of active legs
$\boldsymbol{V}$, $\boldsymbol{A}$ – the general forward velocity and acceleration of platform
$\boldsymbol{F}$, $\boldsymbol{T}$ – the concentrated force and concentrated torque applied on m at o
$(F_x\ F_y\ F_z\ T_x\ T_y\ T_z)$ – the components of $(\boldsymbol{F}, \boldsymbol{T})$
$\boldsymbol{F}_{ai}$ – the active forces exerted on r_i
F_{cj}, T_c – the constrained forces $(j = 1, 2)$, constrained torque
$\mathbf{J}$, $\mathbf{H}$ – the general Jacobian matrix and general Hessian matrix
x_l, x_m, x_n – direction cosine between x and X, x and Y, x and Z
y_l, y_m, y_n – direction cosine between y and X, y and Y, y and Z
z_l, z_m, z_n – direction cosine between z and X, z and Y, z and Z
α, β, λ – the rotational angles of m about (Z, X_1, y)
X_o, Y_o, Z_o – the position components of o in $\{B\}$
W – the reachable workspace
$\parallel$, $\perp$ – parallel and perpendicular constraint

1. Introduction

Currently, various 4-dof parallel manipulators (PMs) have attracted a lot of attention because of their merits, such as larger workspace, simple in structure, less solutions for inverse and forward kinematics, and easy controllability.[1–11] In the aspect of type synthesis and kinematics, Fang and Tsai synthesized some $4 \sim 5$-dof PMs by screw theory and reciprocal screws[3]. Dai and Huang *et al.*[4,5] studied mobility of a overconstrained 4-dof PM, and revealed structural characteristics of some 4-dof PMs by the constraint-synthesis. Kong and Gosselin[6] synthesized some 4-dof SP-equivalent PMs by the virtual chain approach. Carricato,[7] Kong and Gosselin,[8] Cornpany,[9] Alizade[10] and Chen[11] *et al.* synthesized, designed some 4-dof PMs with Schonflies motion and studied their dynamic control simulation. Lu and Hu[12] analyzed kinematics and statics of a 4-dof 3SPU + UPR PM. Joshi and Tsai[13] studied Jacobian matrix for mechanisms with less than 6-dof by screw theory. Zhang and Gosselin[14] built kinetostatic modeling of a 4-dof PM with 4 SPU-type active legs and one RRU-passive leg. Merlet[15] studied Jacobian, manipulability, condition number, and accuracy of PMs. Lu[16,17] solved kinematics of some PMs with $3 \sim 6$ legs by CAD variation geometry. In the aspect of statics and dynamics, a Newton–Euler formulation approach,[18] the principle of virtual work[19] and the screw theory[20] are used for analyzing statics and dynamics of some PMs. Dai *et al.*[21] developed a six-component contact force measuring device with a central strut. Lu[22] solved active forces of some PMs by principle of virtual work and CAD functionalities. By using the vector analytic approach or Lagrange equations, others[23–27] studied the kinematics and dynamics of PMs.

In order to remove the redundant self-motion and enlarge the workspace of the platform, this paper focuses on a novel

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Table 1. Synthesized 40 types of possible serial structures leg with 4-dof.

SP	SR	UPR	URR	UPP	URP	UU	PRPR	PPPR	PRRR
PS	PUP	PPU	PUR	PRU	PRRP	PPRR	PRPP	PPPP	RRRP
RS	RPU	RUP	RRU	RUR	RRPR	RPRR	RPRP	RPPP	RRRR
RPC	RCP	RRC	RCR	PCP	PPC	PCR	PRC	CPR	CRR

S-spherical joint, C-cylindrical joint, U-universal joint, P-prismatic joint, R-revolute joint.

3SPS+SP PM with 3 SPS-type legs and one SP-type leg. When the SP-type leg is replaced by a X-type leg which may be various possible 4-dof serial mechanisms (see Table I), other 39 types of 3SPS+X PMs could be created by using a simple combination theory.[5,9] Thus, the analysis results of kinematics and statics for the 3SPS+SP PM can be applied for the 3SPS+X PMs. Comparing them, the SP-type leg is considered as most promising one because of its simplicity in structure, less joints, larger workspace, and easy avoidance of interference.

2. The 3SPS+SP PM

A 3SPS+SP parallel manipulator (PM) is composed of a platform m, a base B, and a SP-type leg r_1, 3 SPS-type legs r_i ($i=2, 3, 4$) with linear actuators (see Fig. 1). Where, m is a rectangle with a long sides l_1, a short sides l_2, 4 vertices $b_1b_2b_3b_4$, and a central point o. B is a square with sides $L_i = L$, 4 vertices B_i, and a central point O. The SP-type leg r_1 is perpendicular with m, a spherical joint S is attached to B at B_1, a prismatic joint P is connected to m at b_1. Each of r_i ($i=2, 3, 4$) connects m with B by a spherical joint S on m at b_i, a prismatic joint P along r_i, and a spherical joint S on B at B_i. Let $\{m\}$ be a coordinate frame o-xyz fixed on m at o. Let $\{B\}$ be a coordinate frame O-XYZ fixed on B at O. Since each of the SPS-type legs r_i only bears the axial force along r_i, it obviously has a larger capacity of load bearing and is simple in structure. Since the SP-type leg r_1 bears a axial force along r_1 and a bending moment, its cylinder should be cone-shaped for increasing capability of the load bearing.

In the 3SPS+SP PM, the number of links is $q_0 = 9$ for 1 platform with a cylinder, 3 cylinders, 4 piston-rods, and 1 base; the number of joints is $q = 11$ for 4 prismatic joints P, and 7 spherical joints S; the dofs (degree of freedom) of various joints are $M_1 = 1$ for the prismatic joint, $M_2 = 3$ for the spherical joint; the redundancy dof is $M_0 = 3$ for 3 SPS-type legs rotating about their own axes, and M_0 has no influence on the kinematic characteristics of PM. The dof M of the 3SPS+SP PM can be calculated from a revised Kutzbach–Grubler formula[1,2] as below,

$$M = 6(q_0 - q - 1) + \sum_{i=1}^{g} M_i - M_0 = (6-0) \times (9 - 11 - 1) + (4 \times 1 + 7 \times 3) - 3 = 4 \quad (1)$$

3. Kinematics of the 3SPS+SP PM

3.1. Inverse displacement kinematics

In inverse displacement kinematics of the 3SPS+SP PM, when given four independent pose parameters of platform, the extension of legs should be solved. The position vectors $\boldsymbol{B}_i^B (i=1, 2, 3, 4)$ of points B_i on B in $\{B\}$, the position vectors $\boldsymbol{b}_i^m$ of points b_i on m in $\{m\}$, and the position vectors $\boldsymbol{b}_i^B$ of b_i in $\{B\}$ can be expressed as follows[1,2]:

$$\boldsymbol{B}_i^B = \begin{bmatrix} X_{Bi} \\ Y_{Bi} \\ Z_{Bi} \end{bmatrix}, \quad \boldsymbol{b}_i^m = \begin{bmatrix} x_{bi} \\ y_{bi} \\ z_{bi} \end{bmatrix}, \quad \mathbf{R}_m^B = \begin{bmatrix} x_l & y_l & z_l \\ x_m & y_m & z_m \\ x_n & y_n & z_n \end{bmatrix},$$

$$\boldsymbol{o}^B = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix}, \quad \boldsymbol{b}_i^B = \mathbf{R}_m^B \boldsymbol{b}_i^m + \boldsymbol{o}^B. \quad (2a)$$

where, $\mathbf{R}_m^B$ is a rotation transformation matrix from $\{m\}$ to $\{B\}$; $\boldsymbol{o}^B$ is a vector of point o on m in $\{B\}$, (X_o, Y_o, Z_o) are three components of $\boldsymbol{o}^B$; $(x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n)$ are nine orientation parameters (direction cosine) of m in $\{B\}$, they have following constrained equations:[1,2]

$$x_l^2 + x_m^2 + x_n^2 = 1, \quad y_l^2 + y_m^2 + y_n^2 = 1, \quad z_l^2 + z_m^2 + z_n^2 = 1,$$
$$x_l y_l + x_m y_m + x_n y_n = 0, \quad x_l z_l + x_m z_m + x_n z_n = 0, \quad (2b)$$
$$z_l y_l + z_m y_m + z_n y_n = 0.$$

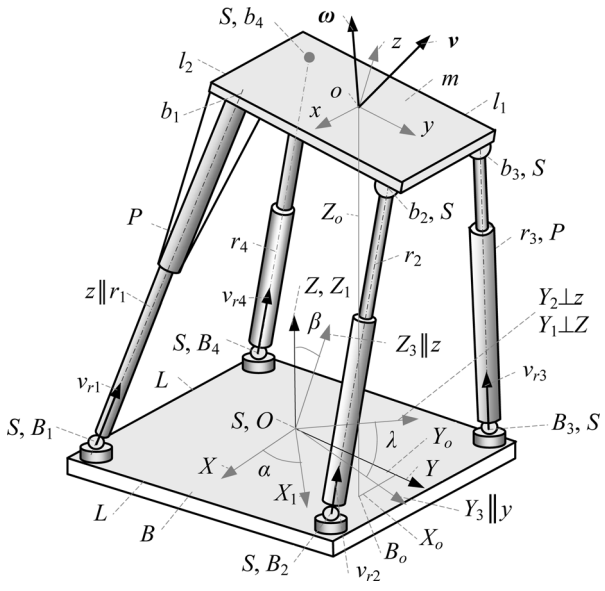


Fig. 1. A model of the 3SPS+SP PM.

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Thus, $\boldsymbol{B}_i^B$ and $\boldsymbol{b}_i^m$ $(i = 1, 2, 3, 4)$ are derived as follows:

$$\begin{aligned}
&\boldsymbol{B}_1^B = \frac{L}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \boldsymbol{B}_2^B = \frac{L}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \boldsymbol{B}_3^B = \frac{L}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix},\\
&\boldsymbol{B}_4^B = \frac{L}{2}\begin{bmatrix}-1\\-1\\0\end{bmatrix} \quad \boldsymbol{b}_1^m = \frac{1}{2}\begin{bmatrix}l_1\\-l_2\\0\end{bmatrix}, \quad \boldsymbol{b}_2^m = \frac{1}{2}\begin{bmatrix}l_1\\l_2\\0\end{bmatrix},\\
&\boldsymbol{b}_3^m = \frac{1}{2}\begin{bmatrix}-l_1\\l_2\\0\end{bmatrix}, \quad \boldsymbol{b}_4^m = \frac{1}{2}\begin{bmatrix}-l_1\\-l_2\\0\end{bmatrix}
\end{aligned} \tag{3a}$$

From Eqs. (2a) and (3a), $\boldsymbol{b}_i^B (i = 1, 2, 3, 4)$ are derived as follows:

$$\begin{aligned}
\boldsymbol{b}_1^B &= \frac{1}{2}\begin{bmatrix}x_l l_1 - y_l l_2 + 2X_o\\ x_m l_1 - y_m l_2 + 2Y_o\\ x_n l_1 - y_n l_2 + 2Z_o\end{bmatrix},\\
\boldsymbol{b}_2^B &= \frac{1}{2}\begin{bmatrix}x_l l_1 + y_l l_2 + 2X_o\\ x_m l_1 + y_m l_2 + 2Y_o\\ x_n l_1 + y_n l_2 + 2Z_o\end{bmatrix},\\
\boldsymbol{b}_3^B &= \frac{1}{2}\begin{bmatrix}-x_l l_1 + y_l l_2 + 2X_o\\ -x_m l_1 + y_m l_2 + 2Y_o\\ -x_n l_1 + y_n l_2 + 2Z_o\end{bmatrix},\\
\boldsymbol{b}_4^B &= \frac{1}{2}\begin{bmatrix}-x_l l_1 - y_l l_2 + 2X_o\\ -x_m l_1 - y_m l_2 + 2Y_o\\ -x_n l_1 - y_n l_2 + 2Z_o\end{bmatrix}.
\end{aligned} \tag{3b}$$

The length $\boldsymbol{r}_i$ $(i = 1, 2, 3, 4)$ and the unit vectors $\boldsymbol{\delta}_i$ of active legs, and the vectors $\boldsymbol{e}_i$ of lines e_i from o to b_i can be expressed as follows:

$$r_i = \left|\boldsymbol{b}_i^B - \boldsymbol{B}_i^B\right|, \quad \delta_i = \left(\boldsymbol{b}_i^B - \boldsymbol{B}_i^B\right)/r_i, \quad \boldsymbol{e}_i = \boldsymbol{b}_i^B - \boldsymbol{o}^B. \tag{4}$$

Based on the geometric constraints ($\boldsymbol{r}_1 \perp m$, i.e., $\boldsymbol{r}_1 \perp x$ and $\boldsymbol{r}_1 \perp y$), from Eqs. (2b), (3) and (4), two constrained equations are derived as follows:

$$\begin{cases} 2(X_o x_l + Y_o x_m + Z_o x_n) + l_1 - L x_l + L x_m = 0\\ 2(X_o y_l + Y_o y_m + Z_o y_n) - l_2 - L y_l + L y_m = 0 \end{cases} \tag{5a}$$

From Eqs. (2a) to (5a), the formulae for solving r_i are derived as follows,

$$\begin{aligned}
r_1^2 &= X_o^2 + Y_o^2 + Z_o^2 + L^2/2 - \left(l_1^2 + l_2^2\right)/4 + L(Y_o - X_o),\\
r_2^2 &= X_o^2 + Y_o^2 + Z_o^2 + L^2/2 - \left(l_1^2 - 3l_2^2\right)/4 - Ll_1 x_m\\
&\quad - Ll_2 y_m - L(Y_o + X_o),\\
r_3^2 &= X_o^2 + Y_o^2 + Z_o^2 + L^2/2 + 3\left(l_1^2 + l_2^2\right)/4 + Ll_2\\
&\quad \times (y_l - y_m) - Ll_1(x_l - x_m) + L(X_o - Y_o),\\
r_4^2 &= X_o^2 + Y_o^2 + Z_o^2 + L^2/2 + \left(3l_1^2 - l_2^2\right)/4 - Ll_1 x_l\\
&\quad - Ll_2 y_l + L(Y_o + X_o).
\end{aligned} \tag{5b}$$

Let $\mathbf{R}_m^B$ be defined by 3 Euler rotations of (Z, Y_1, Z_2), namely, a rotation of α about Z-axis, followed bya rotation of β about Y_1-axis, and a rotation of λ about Z_2-axis. Where, Y_1 is formed by Y rotating about Z by α, and Z_2 is formed by Z_1 rotating about Y_1 by λ. Thus, $\mathbf{R}_m^B$ is derived as,

$$\begin{aligned}
\mathbf{R}_m^B &= \begin{bmatrix} c_\alpha c_\beta c_\lambda - s_\alpha s_\lambda & -c_\alpha c_\beta s_\lambda - s_\alpha c_\lambda & c_\alpha s_\beta\\ s_\alpha c_\beta c_\lambda + c_\alpha s_\lambda & -s_\alpha c_\beta s_\lambda + c_\alpha c_\lambda & s_\alpha s_\beta\\ -s_\beta c_\lambda & s_\beta s_\lambda & c_\beta \end{bmatrix}\\
&= \begin{bmatrix} x_l & y_l & z_l\\ x_m & y_m & z_m\\ x_n & y_n & z_n \end{bmatrix}
\end{aligned} \tag{6a}$$

where, $c_\alpha = \cos\alpha$, $c_\beta = \cos\beta$, $c_\lambda = \cos\lambda$, $s_\alpha = \sin\alpha$, $s_\beta = \sin\beta$, $s_\lambda = \sin\lambda$.

$\boldsymbol{r}_1$ and z are derived from Eqs. (3a), (3b) and (6a) as follows:

$$\begin{aligned}
\boldsymbol{r}_1 &= \boldsymbol{b}_1^B - \boldsymbol{B}_1^B = \frac{1}{2}\begin{bmatrix}x_l l_1 - y_l l_2 + 2X_o - L\\ x_m l_1 - y_m l_2 + 2Y_o + L\\ x_n l_1 - y_n l_2 + 2Z_o\end{bmatrix},\\
z &= \begin{bmatrix} z_l\\ z_m\\ z_n \end{bmatrix}
\end{aligned} \tag{6b}$$

Based on a structure constraint $r_1 \| z$ of the 3SPS+SP PM, from Eq. (6b), leads to

$$\begin{aligned}
\frac{x_l l_1 - y_l l_2 + 2X_o - L}{z_l} &= \frac{x_m l_1 - y_m l_2 + 2Y_o + L}{z_m}\\
&= \frac{x_n l_1 - y_n l_2 + 2Z_o}{z_n}
\end{aligned} \tag{6c}$$

X_o and Y_o are derived from Eq. (6c) as follows:

$$\begin{aligned}
X_o &= \frac{2Z_o z_l + L z_n - l_1 y_m - l_2 x_m}{2z_n},\\
Y_o &= \frac{2Z_o z_m - L z_n + l_1 y_l + l_2 x_l}{2z_n}.
\end{aligned} \tag{7a}$$

By substituting relevant items in Eq. (6a) into Eq. (7a), X_o and Y_o can be expressed by $(Z_o, \alpha, \beta, \lambda)$ as follows:

$$\begin{aligned}
X_o &= \frac{2Z_o c_\alpha s_\beta + L c_\beta - l_1(-s_\alpha c_\beta s_\lambda + c_\alpha c_\lambda) - l_2(s_\alpha c_\beta c_\lambda + c_\alpha s_\lambda)}{2c_\beta},\\
Y_o &= \frac{2Z_o s_\alpha s_\beta - L c_\beta + l_1(s_\alpha c_\beta c_\lambda + c_\alpha s_\lambda) + l_2(c_\alpha c_\beta c_\lambda - s_\alpha s_\lambda)}{2c_\beta}.
\end{aligned} \tag{7b}$$

$r_i (i = 1, 2, 3, 4)$ can be expressed by $(Z_o, \alpha, \beta, \lambda)$ from Eqs. (5b), (6a) and (7b) as follows:

$$\begin{aligned}
r_1^2 &= X_o^2 + Y_o^2 + Z_o^2 + L^2/2 - \left(l_1^2 + l_2^2\right)/4 + L(Y_o - X_o),\\
r_2^2 &= L^2 + l_2^2 + r_1^2 - 2Lr_1 s_\alpha s_\beta - 2Ll_2(-s_\alpha c_\beta s_\lambda + c_\alpha c_\lambda),\\
r_3^2 &= 2L^2 + l_1^2 + l_2^2 + r_1^2 + 2L[(s_\alpha - c_\alpha)(l_1 c_\lambda + l_2 s_\lambda)c_\beta\\
&\quad + (l_1 s_\lambda - l_2 c_\lambda)(s_\alpha + c_\alpha) + r_1(c_\alpha - s_\alpha)s_\beta],\\
r_4^2 &= L^2 + l_1^2 + r_1^2 + 2Lr_1 c_\alpha s_\beta - 2Ll_1(c_\alpha c_\beta c_\lambda - s_\alpha s_\lambda).
\end{aligned} \tag{8}$$

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3.2. Inverse velocity and acceleration

Suppose there are two vectors $\boldsymbol{\eta}$ and ς. A skew-symmetric matrix of $\boldsymbol{\eta}$ must satisfy some mathematic relationships as follows:[1,2]

$$\boldsymbol{\eta}=\begin{bmatrix}\eta_x\\ \eta_y\\ \eta_z\end{bmatrix},\quad \varsigma=\begin{bmatrix}\varsigma_x\\ \varsigma_y\\ \varsigma_z\end{bmatrix},\quad \hat{\boldsymbol{\eta}}=\begin{bmatrix}0 & -\eta_z & \eta_y\\ \eta_z & 0 & -\eta_x\\ -\eta_y & \eta_x & 0\end{bmatrix},$$
$$\boldsymbol{\eta}\times\varsigma=\hat{\boldsymbol{\eta}}\varsigma,\quad \hat{\boldsymbol{\eta}}^T=-\hat{\boldsymbol{\eta}},\quad \boldsymbol{E}_{3\times3}=-\hat{\boldsymbol{\eta}}^2+\boldsymbol{\eta}\boldsymbol{\eta}^T. \tag{9}$$

where $\boldsymbol{\eta}$ may be one of vectors $\boldsymbol{\delta}_i$ $(i=1,2,3)$, $\boldsymbol{e}_i$, $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ and $\boldsymbol{d}$; $\boldsymbol{E}$ is an unit matrix.

Let $\boldsymbol{V}$ be a general velocity of platform m at o. Let $\boldsymbol{v}$ and $\boldsymbol{\omega}$ be a linear velocity and an angular velocity of m at o. Let $\boldsymbol{v}_i$ $(i=1,\ldots,4)$ be a velocity of m at joint b_i. They can be expressed as,

$$\boldsymbol{V}=\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{\omega}\end{bmatrix}_{6\times1},\quad \boldsymbol{v}=\begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}=\begin{bmatrix}\dot{X}_o\\ \dot{Y}_o\\ \dot{Z}_o\end{bmatrix},\quad \boldsymbol{\omega}=\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix},$$
$$\boldsymbol{v}_i=\boldsymbol{v}+\boldsymbol{\omega}\times\boldsymbol{e}_i \tag{10}$$

A scalar velocity v_{ri} along the ith linear active leg has been derived in ref. [27] as below,

$$v_{ri}=\begin{bmatrix}\boldsymbol{\delta}_i^T & (\boldsymbol{e}_i\times\boldsymbol{\delta}_i)^T\end{bmatrix}_{1\times6}\boldsymbol{V} \tag{11}$$

Let $\boldsymbol{A}$ be a general acceleration of platform m at o. Let $\boldsymbol{a}$ and $\boldsymbol{\varepsilon}$ be the linear acceleration and the angular acceleration of m at o. They can be expressed as follows:

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{\varepsilon}\end{bmatrix}_{6\times1},\quad \boldsymbol{a}=\begin{bmatrix}a_x\\ a_y\\ a_z\end{bmatrix},\quad \boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{bmatrix}. \tag{12}$$

A scalar acceleration a_{ri} along the active leg r_i $(i=1,2,3,4)$ has been derived in ref. [27] as follows:

$$a_{ri}=\begin{bmatrix}\boldsymbol{\delta}_i^T & (\boldsymbol{e}_i\times\boldsymbol{\delta}_i)^T\end{bmatrix}\boldsymbol{A}+\boldsymbol{V}^T\mathbf{h}_{ri}\boldsymbol{V},$$
$$\mathbf{h}_ri=\frac{1}{r_i}\begin{bmatrix}-\hat{\boldsymbol{\delta}}_i^2 & \hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\\ -\hat{\boldsymbol{e}}_i\hat{\boldsymbol{\delta}}_i^2 & r_i\hat{\boldsymbol{e}}_i\hat{\boldsymbol{\delta}}_i+\hat{\boldsymbol{e}}_i\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\end{bmatrix}_{6\times6}. \tag{13}$$

The inverse velocity $\boldsymbol{v}_r$ and acceleration $\boldsymbol{a}_r$ are derived from Eqs. (11) and (13) as follows:

$$\boldsymbol{v}_r=\mathbf{J}_r\boldsymbol{V},\ \boldsymbol{a}_r=\mathbf{J}_r\boldsymbol{A}+\boldsymbol{V}^T\mathbf{H}_r\boldsymbol{V},$$
$$\mathbf{J}_r=\begin{bmatrix}\boldsymbol{\delta}_1^T & (\boldsymbol{e}_1\times\boldsymbol{\delta}_1)^T\\ \boldsymbol{\delta}_2^T & (\boldsymbol{e}_2\times\boldsymbol{\delta}_2)^T\\ \boldsymbol{\delta}_3^T & (\boldsymbol{e}_3\times\boldsymbol{\delta}_3)^T\\ \boldsymbol{\delta}_4^T & (\boldsymbol{e}_4\times\boldsymbol{\delta}_4)^T\end{bmatrix}_{4\times6},\quad \mathbf{H}_r=\begin{bmatrix}\mathbf{h}_{r1}\\ \mathbf{h}_{r2}\\ \mathbf{h}_{r3}\\ \mathbf{h}_{r4}\end{bmatrix},$$
$$\boldsymbol{v}_r=\begin{bmatrix}v_{r1}\\ v_{r2}\\ v_{r3}\\ v_{r4}\end{bmatrix},\quad \boldsymbol{a}_r=\begin{bmatrix}a_{r1}\\ a_{r2}\\ a_{r3}\\ a_{r4}\end{bmatrix}. \tag{14}$$

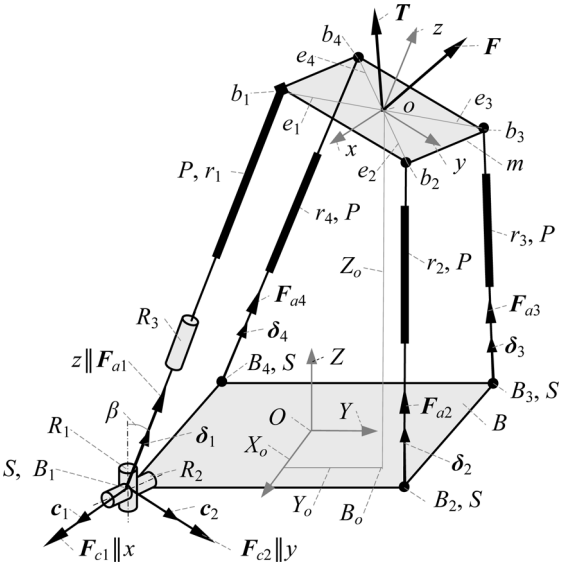


Fig. 2. The statics situation of the 3SPS+SP PM.

Where, $\mathbf{J}_r$ is a 4×6 Jacobian matrix; $\mathbf{H}_r$ is a $4\times6\times6$ Hessian matrix.

3.3. The geometric constraints of constrained forces

The force situation of the 3SPS+SP PM is shown in Fig. 2.

When ignored the friction of all joints in the 3SPS+SP PM, the whole workloads can be simplified as a workload wrench $(\boldsymbol{F},\boldsymbol{T})$ applied onto m at central point o, which includes the inertia wrench of the platform, the damping wrench of the platform, the equivalent inertia wrench of legs, the gravity of platform and legs, and the external working wrench (such as machining or operating wrench). $\boldsymbol{F}$ is a concentrated force and $\boldsymbol{T}$ is a concentrated torque. $(\boldsymbol{F},\boldsymbol{T})$ or its components $(F_x,F_y,F_z,T_x,T_y,T_z)$ are balanced by four active forces $\boldsymbol{F}_{ai}$ $(i=1,2,3,4)$ applied on and along r_i and two the constrained forces $\boldsymbol{F}_{cj}$ $(j=1,2)$ exerted on the SP-type leg r_1 at B_1. Since $\boldsymbol{F}_{cj}$ limit to the movement of r_1, the power done by $\boldsymbol{F}_{cj}$ must be zero during the movement of r_1.

In the SP-type leg r_1, a spherical joint S on B at B_1 can be replaced by three intersected revolute joints R_k $(k=1,2,3)$. The unit vectors $\boldsymbol{R}_k$ of $\boldsymbol{R}_k$ are determined as follows:

$$\boldsymbol{R}_1=\boldsymbol{Z}=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad \boldsymbol{R}_2=\begin{bmatrix}c_\alpha & -s_\alpha & 0\\ s_\alpha & c_\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}s_\alpha\\ c_\alpha\\ 0\end{bmatrix},$$
$$\boldsymbol{R}_3=\boldsymbol{\delta}_1=\frac{\boldsymbol{b}_1^B-\boldsymbol{B}_1^B}{r_1} \tag{15}$$

Let $\boldsymbol{v}_{rl}$ be a translation velocity along r_1, $\boldsymbol{F}_{cj}\cdot v_{rl}=0$, i. e. $\boldsymbol{F}_{cj}\perp r_1$ must be satisfied. Let $\rho_j\times\boldsymbol{F}_{cj}$ be the torques of $\boldsymbol{F}_{cj}$ about R_k, $\boldsymbol{R}_k\cdot(\rho_j\times\boldsymbol{F}_{cj})=0$ must be satisfied. In this case, $\boldsymbol{F}_{cj}$ must intersect or be parallel with each of $\boldsymbol{R}_k$. Let $\boldsymbol{c}_j$ be the unit vectors of $\boldsymbol{F}_{cj}$ $(j=1,2)$, so that $\boldsymbol{c}_j$ must be perpendicular to r_1 in any direction. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are perpendicular to r_1, when set $\boldsymbol{c}_1\|\boldsymbol{x}$ and $\boldsymbol{c}_2\|\boldsymbol{y}$, $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ can be determined as follows:

$$\boldsymbol{c}_1=[\,x_l\quad x_m\quad x_n\,]^T,\quad \boldsymbol{c}_2=[\,y_l\quad y_m\quad y_n\,]^T \tag{16}$$

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3.4. The auxiliary velocity and acceleration equations
Since the constrained forces $\boldsymbol{F}_{cj}$ $(j=1,\ 2)$ limit to the movement of platform m, the power done by $\boldsymbol{F}_{cj}$ must be zero during the movement of PM.[12,27] Thus, there must be

$$F_{cj}\boldsymbol{c}_j \cdot \boldsymbol{v} + (\boldsymbol{d} \times F_{cj}\boldsymbol{c}_j)\cdot \boldsymbol{\omega} = 0, \quad (j = 1, 2) \tag{17}$$

where $\boldsymbol{d}$ is a vector of line B_1o. When deleted F_{cj} from Eq. (17), leads to

$$\mathbf{J}_a \boldsymbol{V} = \mathbf{0}, \quad \mathbf{J}_a = \begin{bmatrix} \boldsymbol{c}_1^T & (\boldsymbol{d}\times \boldsymbol{c}_1)^T \\ \boldsymbol{c}_2^T & (\boldsymbol{d}\times \boldsymbol{c}_2)^T \end{bmatrix}_{2\times 6} \tag{18}$$

By differentiating Eq. (18) with respect to time, from Eq. (9), an auxiliary acceleration equation is derived as follows:

$$\begin{gathered} \mathbf{J}_a \boldsymbol{A} + \dot{\mathbf{J}}_a \boldsymbol{V} = \mathbf{0}, \\ \dot{\mathbf{J}}_a = \begin{bmatrix} \dot{\boldsymbol{c}}_1^T & (\dot{\boldsymbol{d}}\times \boldsymbol{c}_1 + \boldsymbol{d}\times \dot{\boldsymbol{c}}_1)^T \\ \dot{\boldsymbol{c}}_2^T & (\dot{\boldsymbol{d}}\times \boldsymbol{c}_2 + \boldsymbol{d}\times \dot{\boldsymbol{c}}_2)^T \end{bmatrix}_{2\times 6} = \boldsymbol{V}^T \begin{bmatrix} \mathbf{h}_{a1} \\ \mathbf{h}_{a2} \end{bmatrix}, \\ \mathbf{h}_{aj} = \begin{bmatrix} \mathbf{0}_{3\times 3} & -\hat{\mathbf{c}}_j \\ \hat{\mathbf{c}}_j & -\hat{\mathbf{c}}_j\hat{\mathbf{d}} \end{bmatrix}_{6\times 6}, (j = 1,_2). \end{gathered} \tag{19}$$

where $\boldsymbol{d} = \boldsymbol{B}_1^B - \boldsymbol{o}^B$, $\dot{\boldsymbol{d}} = -\dot{\boldsymbol{o}}^B = -\boldsymbol{v}$. Equation (9), leads to

$$\begin{aligned} \dot{\boldsymbol{c}}_j &= \boldsymbol{\omega}\times \boldsymbol{c}_j = \boldsymbol{\omega}\hat{\mathbf{c}}_j, \dot{\boldsymbol{d}}\times \boldsymbol{c}_j + \boldsymbol{d}\times \dot{\boldsymbol{c}}_j \\ &= -\boldsymbol{v}\times \boldsymbol{c}_j + \boldsymbol{d}\times(\boldsymbol{\omega}\times \boldsymbol{c}_j) = -\boldsymbol{v}\hat{\boldsymbol{c}}_j - \boldsymbol{\omega}\hat{\boldsymbol{c}}_j\hat{\mathbf{d}}. \end{aligned} \tag{20}$$

3.5. General inverse/forward velocity and acceleration
Based on the unification approach of velocity/acceleration of limited-dof PMs with linear active legs[27] a general inverse velocity $\boldsymbol{v}_c$ and a general acceleration $\boldsymbol{a}_c$ of the 3SPS+SP PM are derived from Eqs. (14), (18) and (19) as follows:

$$\begin{gathered} \boldsymbol{v}_c = \mathbf{J}V,\ \boldsymbol{a}_c = \mathbf{J}\boldsymbol{A} + \boldsymbol{V}^T\mathbf{H}\boldsymbol{V}, \\ \boldsymbol{v}_c = \begin{bmatrix} v_{r1} \\ v_{r2} \\ v_{r3} \\ v_{r4} \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_c = \begin{bmatrix} a_{r1} \\ a_{r2} \\ a_{r3} \\ a_{r4} \\ 0 \\ 0 \end{bmatrix}, \\ \mathbf{J} = \begin{bmatrix} \mathbf{J}_r \\ \mathbf{J}_a \end{bmatrix} = \begin{bmatrix} \delta_1^T & (e_1\times \delta_1)^T \\ \delta_2^T & (e_2\times \delta_2)^T \\ \delta_3^T & (e_3\times \delta_3)^T \\ \delta_4^T & (e_4\times \delta_4)^T \\ c_1^T & (d\times c_1)^T \\ c_2^T & (d\times c_2)^T \end{bmatrix}_{6\times 6}, \ \mathbf{H} = \begin{bmatrix} \mathbf{h}_{r1} \\ \mathbf{h}_{r2} \\ \mathbf{h}_{r3} \\ \mathbf{h}_{r4} \\ \mathbf{h}_{a1} \\ \mathbf{h}_{a2} \end{bmatrix} \end{gathered} \tag{21}$$

where, $\mathbf{J}$ is a general 6×6 Jacobian matrix; $\mathbf{H}$ is a general $6\times 6\times 6$ Hessian matrix.

In fact, 6 components of $\boldsymbol{V}$ for the 3SPS+SP PM cannot be any set of values, only four are independent. Thus, when given four independent components of $\boldsymbol{V}$, other two components can be determined, then v_{ri} $(i = 1, 2, 3, 4)$ can be solved from Eq. (21).

From Eq. (21), the general forward velocity and acceleration equations of the 3SPS+SP PM are derived as follows:

$$\boldsymbol{V} = \mathbf{J}^{-1}\boldsymbol{v}_c, \quad \boldsymbol{A} = \mathbf{J}^{-1}(\boldsymbol{a}_c - \boldsymbol{V}^T\mathbf{H}\boldsymbol{V}) \tag{22}$$

Each of items in $\mathbf{J}$, $\mathbf{J}^{-1}$, $\mathbf{H}$ and its sub-matrix $\mathbf{h}_{ri}$ and $\mathbf{h}_{aj}$ does not include any partial differentiation.

4. The Reachable Workspace

A reachable workspace W of PM is defined as all poses that can be reached by the central point of platform. When given the maximum extension $r_{\max}$ and the minimum extension $r_{\min}$ of legs, W of the 3SPS+SP PM can be constructed by using its simulation mechanism[12,16] or some relative analytic formulae. Since the 3SPS+SP PM has a symmetric plane P (B_1OZ) in structure, W must be plane symmetry. Therefore, only 3 upper boundary surfaces S_{uk} $(k=1,\ 2,\ 3)$ and 3 lower boundary surfaces S_{lk} are required to construct. Where, S_{u1} includes 3 vertices $p_2p_6p_5$, S_{u2} includes $p_1p_7p_5$, S_{u3} includes $p_1p_2p_5$, S_{l1} includes $p_2p_6p_8$, S_{l2} includes $p_1p_7p_8$, and S_{l3} includes $p_1p_2p_8$. Their construction procedures are explained as follows:

Set $L=100$, $l=l_1=l_2=50$, $r_{\max}=200$, $r_{\min}=140$, $\delta r=5$cm, $n_1=(r_{\max}-r_{\min})/\delta r$. Construct two lines ob_3 and OB_2. When set $ob_3\perp OB_2$, a rotation dof of platform m about leg r_1 is limited, and the simulation mechanism has 3-dof. When given the 3 extensions of four legs, the position components $(X_o,\ Y_o,\ Z_o)$ of m can be solved automatically by its simulation mechanism. The construction procedures of S_{ul} are described as follows:

1. Let $r_3=r_{\max}$, $r_1=r_{\min}+j\delta r(j=0,1,\ldots,n_1-1)$; and transform the driving dimensions of $(r_2,\ r_4)$ into driven dimensions.
2. Construct a line L_o from O to B_o. Construct a plane P by passing line OB_1 and perpendicular to B. Give the angle θ between L_o and plane P an initial driving dimension $\theta=0^\circ$.
3. Set $j=0$ and $\delta\theta=1^\circ$, and increase θ by $\delta\theta$ at each increment step from 0 to θ_1 which is corresponding to one of the driven dimensions of (r_2, r_4) are equal or close to $r_{\max}$ or $r_{\min}$.
4. Construct S_{u1} by a CAD variation geometry.[12,16] The construction procedures of S_{u2} are similar to that of S_{u1}, except that set $r_4=r_{\max}$, $r_2=r_{\min}+j\delta r(j=0,1,\ldots,n_1-1)$; and transform the driving dimensions of (r_1, r_3) into driven dimensions.

The construction procedure of S_{u3} is similar to that of S_{u1}, except that set $r_1=r_{\max}$, $r_3=r_{\min}+j\delta r(j=0,1,\ldots,n_1-1)$; and transform the driving dimensions of (r_2, r_4) into driven dimensions.

The construction procedures of S_{lk} $(k=1,2,3)$ are similar to that of S_{uk}, except that set $r_3=r_{\min}$, $r_4=r_{\min}$, $r_1=r_{\min}$, respectively.

Thus, from S_{uk} and S_{lk}, the whole workspace can be constructed by the reflection command about plane P (see Fig. 3).

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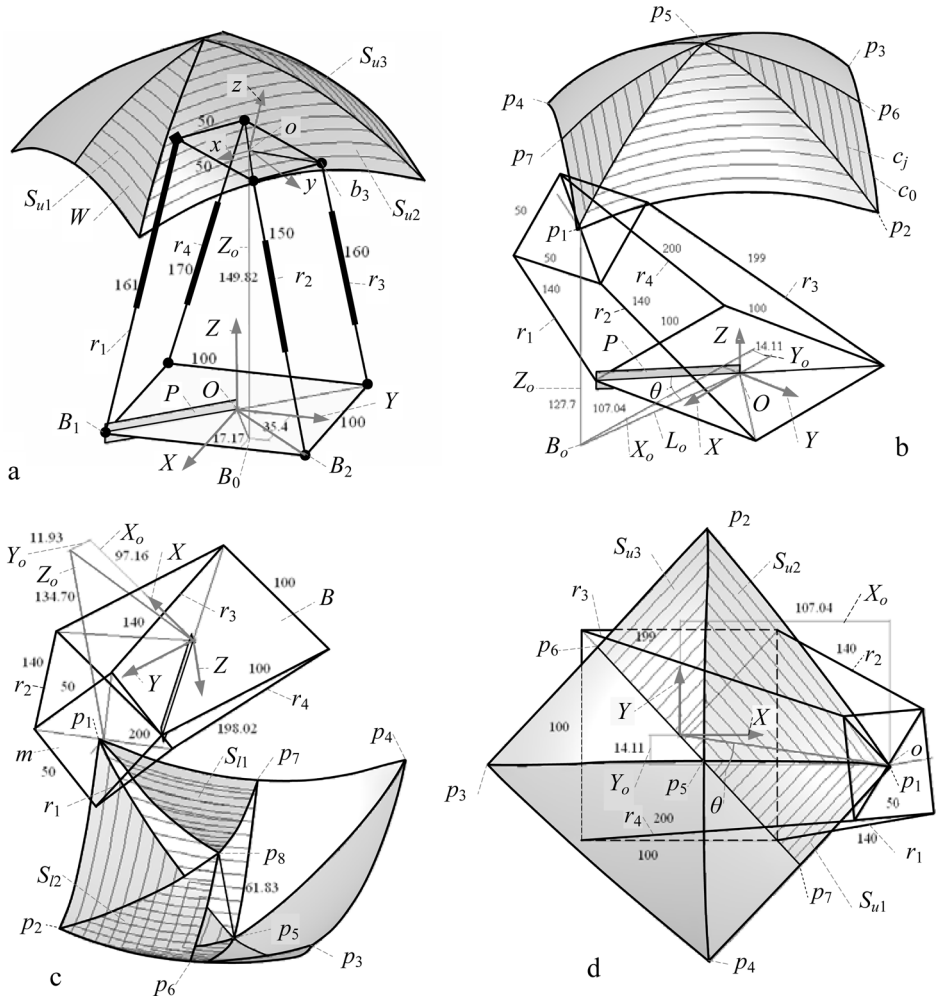


Fig. 3. A reachable workspace W of the 3SPS+SP PM. (a) A simulation mechanism (b) Isometric view (c) The up view (d) The top view.

When deleting a geometric constraint $ob_3 \perp OB_2$, and varying angle between line ob_3 and plane P, m can be rotated in $\{B\}$. In this case, when varying r_i $(i = 1, 2, 3, 4)$ in range $r_{\min} \sim r_{\max}$, all positions of o of simulation mechanism are verified within W.

5. Active and Constrained Forces

Based on the Newton–Euler formulation approach,[18] there is

$$[F_{a1} \quad F_{a2} \quad F_{a3} \quad F_{a4} \quad F_{c1} \quad F_{c2}]\boldsymbol{v}_c + [F_x \quad F_y \quad F_z \quad T_x \quad T_y \quad T_z]\boldsymbol{V} = 0 \qquad (23a)$$

By substituting $\boldsymbol{v}_c = \mathbf{J}\boldsymbol{V}$ in Eq. (21) into Eq. (23(a)), the active/constrained forces of the 3SPS+SP PM can be solved as below,

$$[F_{a1} \quad F_{a2} \quad F_{a3} \quad F_{a4} \quad F_{c1} \quad F_{c2}]^T = -(\mathbf{J}^T)^{-1}[F_x \quad F_y \quad F_z \quad T_x \quad T_y \quad T_z]^T \qquad (23b)$$

Based on $\boldsymbol{d} = \boldsymbol{B}_1^B - \boldsymbol{o}^B$, from Eqs. (2), (3a), (3b) (4) and (8), leads to

$$\delta_1 = \frac{1}{2r_1}\begin{bmatrix} x_l l_1 - y_l l_2 + 2X_o - L \\ x_m l_1 - y_m l_2 + 2Y_o + L \\ x_n l_1 - y_n l_2 + 2Z_o \end{bmatrix},$$

$$\boldsymbol{\delta}_2 = \frac{1}{2r_2}\begin{bmatrix} x_l l_1 + y_l l_2 + 2X_o - L \\ x_m l_1 + y_m l_2 + 2Y_o - L \\ x_n l_1 + y_n l_2 + 2Z_o \end{bmatrix},$$

$$\delta_3 = \frac{1}{2r_3}\begin{bmatrix} -x_l l_1 + y_l l_2 + 2X_o + L \\ -x_m l_1 + y_m l_2 + 2Y_o - L \\ -x_n l_1 + y_n l_2 + 2Z_o \end{bmatrix},$$

$$\boldsymbol{\delta}_4 = \frac{1}{2r_4}\begin{bmatrix} -x_l l_1 - y_l l_2 + 2X_o + L \\ -x_m l_1 - y_m l_2 + 2Y_o + L \\ -x_n l_1 - y_n l_2 + 2Z_o \end{bmatrix},$$

$$e_1^B = \frac{1}{2}\begin{bmatrix} x_l l_1 - y_l l_2 - L \\ x_m l_1 - y_m l_2 + L \\ x_n l_1 - y_n l_2 \end{bmatrix}, \quad \boldsymbol{e}_2^B = \frac{1}{2}\begin{bmatrix} x_l l_1 + y_l l_2 - L \\ x_m l_1 + y_m l_2 - L \\ x_n l_1 + y_n l_2 \end{bmatrix},$$

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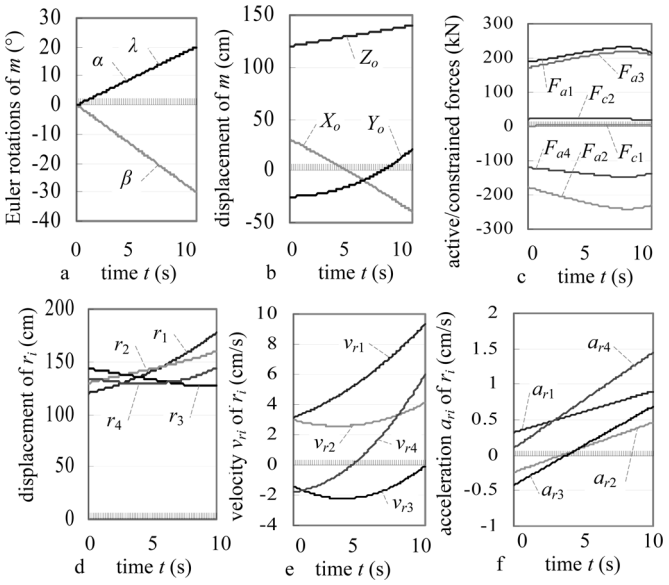


Fig. 4. The inverse kinematics and forces solved results of the 3SPS+SP PM.

$$e_3^B = \frac{1}{2}\begin{bmatrix} -x_l l_1 + y_l l_2 + L \\ -x_m l_1 + y_m l_2 - L \\ -x_n l_1 + y_n l_2 \end{bmatrix},$$

$$\boldsymbol{e}_4^B = \tfrac{1}{2}\begin{bmatrix} -x_l l_1 - y_l l_2 + L \\ -x_m l_1 - y_m l_2 + L \\ -x_n l_1 - y_n l_2 \end{bmatrix},$$

$$d = -\frac{1}{2c_\beta} \times \begin{bmatrix} 2Z_o c_\alpha s_\beta - l_1(-s_\alpha c_\beta s_\lambda + c_\alpha c_\lambda) - l_2(s_\alpha c_\beta c_\lambda + c_\alpha s_\lambda) \\ 2Z_o s_\alpha s_\beta + l_1(s_\alpha c_\beta c_\lambda + c_\alpha s_\lambda) + l_2(c_\alpha c_\beta c_\lambda - s_\alpha s_\lambda) \\ 2Z_o c_\beta \end{bmatrix}. \tag{24}$$

$\mathbf{J}$ and $(\mathbf{J}^T)^{-1}$ can be solved from Eqs. (21) and (24). The active forces F_{ai} $(i = 1, 2, 3, 4)$ and two constrained forces F_{cj} can be solved from Eqs. (21) and (23(b)).

The constrained bending moment T_c due to F_{cj} are solved as follows:

$$\boldsymbol{T}_{c1} = r_1 F_{c1}, \quad \boldsymbol{T}_{c2} = r_1 F_{c2}, \quad \boldsymbol{T}_c = \boldsymbol{T}_{c1} + \boldsymbol{T}_{c2},$$
$$T_c = |\boldsymbol{T}_c| = \sqrt{T_{cx}^2 + T_{cy}^2 + T_{cz}^2} \tag{25}$$

6. Analytic solved Examples

In the 3SPS+SP PM, set $L = 100$, $l_1 = 40$, $l_2 = 50$ cm; $\boldsymbol{F} = [-20\ -30\ -60]$ kN; $\boldsymbol{T} = [-30\ -30100]$ kN · cm.

By using Matlab,[12] when given four independent pose parameters (Z_o, α, β, λ) vs. time t (Fig. 4(a) and (b)), the position components (X_o, Y_o) of m are solved (see Fig. 4(b)); four active forces F_{ai} $(i = 1, 2, 3, 4)$ and two passive forces (F_{c1}, F_{c2}) are solved, (see Fig. 4(c)); the extension, velocity and acceleration of four legs r_i are solved (Fig. 4(d)–(f)). The forward velocity and acceleration of m are solved (see Fig. 5(a)–(d)). The analytical solved results have been verified by its force simulation mechanism.

7. Conclusions

The 3SPS+SP parallel manipulator (PM) can effectively remove the redundant self-motion and enlarge the workspace. It has 4 dofs and includes three SPS-type legs and one SP-type leg. Each of SPS-type legs has large capability of load-bearing. The SP-type leg bears an active force along its axis and a bending moment.

The displacement, velocity, acceleration, workspace and active/constrained forces of the 3SPS+SP PM can be solved by derived analytical equations and verified by its simulation mechanism. By replacing a SP-type leg with a X-type leg, forty types of 3SPS+X PMs can be synthesized. The analytic equations of the kinematics and statics of 3SPS+SP PM can be applied for the family of the 3SPS+X PMs.

The reachable workspace of the 3SPS+SP PM includes four upper boundary surfaces with five vertices, and four lower boundary surfaces with five vertices. Each of the upper or the lower boundary surfaces is formed by a group of curves. In the reachable workspace, no interference among four legs occurs.

The 3SPS+SP PM has some potential applications for the 4-dof parallel kinematic machines, such as parallel machine tool, parallel sensor, surgical manipulator, tunnel borer, barbette of warship, and satellite surveillance platform.

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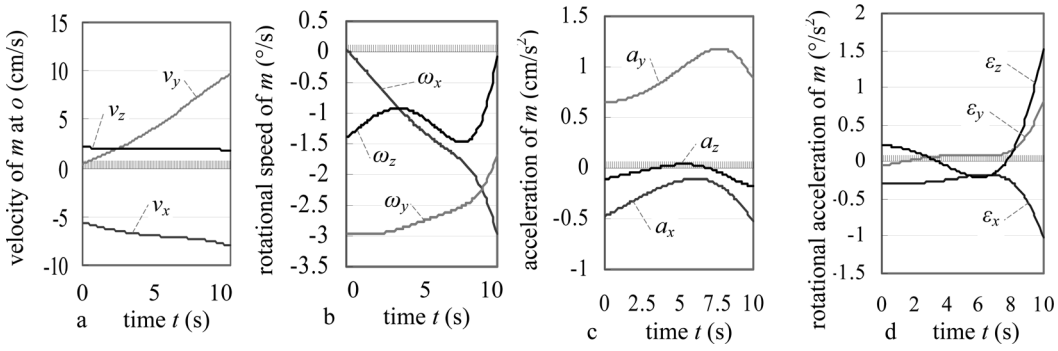


Fig. 5. The forward velocity and acceleration of m for the 3SPS+SP PM.

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