be any numbers satisfying $A < 1 < B$, and $A \ln k \leqslant \frac{p_k}{k} \leqslant B \ln k$ then holds for all large k, depending on how close A and B are to 1. Thus, Ruiz made a confusing derivation of (1), making an unnecessary use of his claimed inequality. However, it is easy to show that the result (1) is a trivial consequence of the PNT. In the following proposition, we show this fact by a general establishment.

Proposition: Suppose that $\sum_{k=1}^{n} \ln A(k) \sim A(n)$ as $n \to \infty$. Then

$$\lim_{n \to \infty} \sqrt[A(n)]{A(1)A(2)\ldots A(n)} = e.$$

Proof: We have

$$\ln \sqrt[A(n)]{A(1)A(2)\ldots A(n)} = \frac{1}{A(n)} \sum_{k=1}^{n} \ln A(k) \to 1 \text{ as } n \to \infty,$$

from which the required result follows on exponentiation.

Setting $A(n) = p_n$, the n-th prime number, equation (1) follows from the PNT in the form $\sum_{k=1}^{n} \ln p_k \sim p_n$ as $n \to \infty$.

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105.28 Periodic Möbius sequences

Introduction

When teaching A level students about sequences, some questions arose about periodic sequences generated by iterative formulae of the type $x_{r+1} = \frac{ax_r + b}{cx_r + d}$ where a, b, c and d are integers. A student of one of my colleagues thought they had found one with period 5, but we were unable to reproduce such a sequence. This led me to investigate what periods were possible with integer (or equivalently rational) coefficients. This is probably

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reproducing previously known results but I have not been able to find any references to them.

A Möbius transformation in the extended complex plane is one of the form $z \to \dfrac{az + b}{cz + d}$, where $ad - bc \neq 0$.

Because of this I will call such sequences *Möbius sequences*. Möbius transformations are often used in the extended complex plane where we can cope with a zero denominator. But as our main interest is in real periodic sequences, we will assume x_1 is such that $cx_r + d \neq 0$ for all r (for a given iterative formula, call all such x_1 *valid* and any other numbers *invalid*). By inverting the transformation and applying repeatedly to the (at most one) value of x for which $cx + d$ is zero shows that for a particular transformation the set of invalid x_1 is at most countable and so the set of valid complex (or real) numbers x_1 is uncountable. We will use this at one point.

Matrix representation

I will use matrices to investigate the sequences. For matrices A, B define the equivalence relation ~ by

$$A \sim B \Leftrightarrow A = \mu B \text{ for some complex number } \mu \neq 0.$$

For $v \neq 0$, let $\begin{pmatrix} u \\ v \end{pmatrix}$ represent the number $\dfrac{u}{v}$. Then $\begin{pmatrix} x_{r+1} \\ 1 \end{pmatrix} \sim M \begin{pmatrix} x_r \\ 1 \end{pmatrix}$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so we can generate the sequence by M. If A and B are 2 by 2 matrices, $A \sim B$ implies that they generate the same sequence (for a given x_1). Using suitable rotation matrices we can get sequences of any period, but as we shall see this is not the case if we restrict a, b, c and d to be rational.

All subsequent matrices are 2 by 2 unless indicated otherwise. Define the *period* of a matrix to mean the least positive integer n such that $M^n \sim I$ where I is the identity matrix (if such an integer exists). The period of M is thus the period of each sequence of valid numbers generated by M. Define a fixed point of M as a value of x_1 such that the sequence generated by M is constant.

We will use of the eigenvalues of a matrix. The eigenvalues of a square matrix M are the roots of the characteristic equation $\det(M - \lambda I) = 0$. For a 2 by 2 matrix this gives $\lambda^2 - \operatorname{tr}(M)\lambda + \det(M) = 0$ where $\operatorname{tr}(M)$ is the trace of the matrix, which is the sum of the entries on the leading diagonal. So if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1, λ_2 then their sum is $\operatorname{tr}(M) = a + d$ and their product is $\det(M) = ad - bc$. If the (real or complex) eigenvalues are distinct then the matrix can be diagonalised as $M = PDP^{-1}$ where D is a matrix which has the eigenvalues on the leading diagonal and zeros elsewhere. Then $M^r = P D^r P^{-1}$ where D^r has the r-th powers of the eigenvalues on the leading diagonal and zeros elsewhere.

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I will also use the adjugate of a matrix at one point, in preference to the inverse. This is purely for convenience as if the matrix has integer entries then so does its adjugate, whereas the inverse may not. Recall that

$$\operatorname{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

More generally, for a square matrix the adjugate is the transpose of the matrix of cofactors (so for a non-singular matrix it is the product of the determinant and the inverse).

Fixed points

Can M generate periodic sequences for certain x_1 only? If $M \sim I$ then it has period 1 for all x_1. Otherwise it has at most two fixed points given by the roots of $cx^2 - (a-d)x - b = 0$ and then the sequence has period 1 for these x_1 only. For period $n > 1$ we need a fixed point of M^n which is not a fixed point of M. Clearly the fixed points of M are also fixed points of M^n, so if x_1 is a fixed point of M^n but not of M, then M cannot have 2 distinct fixed points.

Proposition 1: If a Möbius sequence generated by M has period $n > 1$ for a particular x_1, then the Möbius sequences generated by M will have period n for all valid x_1 and so $M^n \sim I$.

Proof: Suppose, for a contradiction, that the sequence generated by M has period $n > 1$ for a particular x_1, but not for all valid x_1. Then x_1 is a fixed point of M^n but not a fixed point of M. So, as noted above, M must have at most 1 fixed point.

Suppose $c = 0$. Then clearly $d \neq 0$. If $a \neq \pm d$ then a simple induction shows

$$M^n = \begin{pmatrix} a^n & \frac{b}{a-d}(a^n - d^n) \\ 0 & d^n \end{pmatrix},$$

which has a single fixed point $b/(d-a)$ for all n, including 1, contradicting our assumption. If $a = d$ the sequence will have period 1 for all x_1 (if $b = 0$) or will not be periodic for any x_1 (if $b \neq 0$). If $a = -d$ then $M \neq I$ but $M^2 \sim I$ so the sequence will have period 2 for all x_1.

If $c \neq 0$ we require $cx^2 - (a-d)x - b = 0$ to have equal roots and so $(a-d)^2 + 4bc = 0$ and hence $(a+d)^2 = 4(ad - bc)$ in which case the characteristic equation of M has zero discriminant and so M has a repeated eigenvalue. Denoting this by λ, then $a + d = 2\lambda$ and we can write $M = \begin{pmatrix} \lambda + \delta & b \\ c & \lambda - \delta \end{pmatrix}$. Then because $\det(M) = \lambda^2$, we get $\delta^2 + bc = 0$. Then a simple induction shows

$$M^n = \lambda^{n-1}\begin{pmatrix} \lambda + n\delta & nb \\ nc & \lambda - n\delta \end{pmatrix}.$$

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The fixed points of this satisfy $n(cx^2 - 2\delta x - b) = 0$ and hence they are independent of n. So the fixed points of M^n are the same as the fixed points of M which means the period is 1, contradicting our assumption.

So if a Möbius sequence has period $n > 1$ then $\begin{pmatrix} x_1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ 1 \end{pmatrix} \sim M^n \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$ for all valid x_1. But as observed earlier there are uncountably many valid x_1 (so there are certainly more than 2) but M^n cannot have more than 2 fixed points unless $M^n \sim I$.

Periodic Möbius sequences

Proposition 2: If M has repeated eigenvalues and is periodic, then M has period 1.

Proof: As in proposition 1, writing $M = \begin{pmatrix} \lambda + \delta & b \\ c & \lambda - \delta \end{pmatrix}$, gives $M^n = \lambda^{n-1} \begin{pmatrix} \lambda + n\delta & nb \\ nc & \lambda - n\delta \end{pmatrix}$. So, as $n > 0$, if $M^n \sim 1$ then $b = c = \delta = 0$. Hence $M = \lambda I$ and so the period is 1.

Theorem: For any integer $n > 1$, M has period n if, and only if, its eigenvalues are of the form $\lambda \exp(\pm k\pi i/n)$ where λ is a non-zero complex number and k is an integer coprime to n.

Proof: If M has distinct eigenvalues λ_1 and λ_2 then we can diagonalise M as $M = PDP^{-1}$. Then

$$M^r = \mu I \Leftrightarrow PD^rP^{-1} = \mu I \Leftrightarrow D^r = \mu I \Leftrightarrow \lambda_1^r = \lambda_2^r = \mu.$$

For $M^r \sim I$ we want $\mu \neq 0$ (and hence also $\lambda_1, \lambda_2 \neq 0$) so

$$M^r \sim I \Leftrightarrow \left(\frac{\lambda_1}{\lambda_2}\right)^r = 1. \qquad (*)$$

Suppose the eigenvalues are $\lambda \exp(\pm k\pi i/n)$ with $\lambda \neq 0$ and k coprime to n. Then they are distinct so we can use (*). So $(\lambda_1/\lambda_2)^r = \exp(\pm 2rk\pi i/n) = 1$, which is true if, and only if, n divides r (as k is coprime to n) and so the period is n.

Conversely, suppose M has period $n > 1$. Then $\lambda_1 \neq \lambda_2$ (by Proposition 2). So using (*), we have $(\lambda_1/\lambda_2)^n = 1$. Hence $\lambda_1/\lambda_2 = \exp(2k\pi i/n)$ for some integer k. Note that k must be coprime to n, as otherwise let $r = n/(n, k) < n$ and then $(\lambda_1/\lambda_2)^r = \exp(2rk\pi i/n) = 1$ so the period will be less than n. Let $\lambda = \lambda_2 \exp(k\pi i/n) \neq 0$. Then we have $\lambda_1 = \lambda \exp(k\pi i/n)$ and $\lambda_2 = \lambda \exp(-k\pi i/n)$.

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Notes:

(1) For $n > 2, \lambda_1/\lambda_2 = \exp(2k\pi i/n) \notin \mathbb{R}$ so the eigenvalues cannot both be real. So if M is real the eigenvalues will be conjugates and hence λ will be real.

(2) For $n = 2$ we have $\lambda_2 = -\lambda_1$ but it can be convenient to use the general form for all $n > 1$.

Periodic rational Möbius sequences

The main result can be now deduced as a corollary to the Theorem.

Corollary: If a Möbius sequence with rational coefficients has period n then $n \in \{1, 2, 3, 4, 6\}$.

Proof: Suppose $a, b, c, d \in \mathbb{Q}$. By Proposition 1, if $n > 1$ then $M^n \sim 1$. So by the Theorem, $a + d = \lambda_1 + \lambda_2 = 2\lambda\cos(k\pi/n)$ and $ad - bc = \lambda_1\lambda_2 = \lambda^2$, hence

$$\cos\left(\frac{2k\pi}{n}\right) = 2\cos^2\left(\frac{k\pi}{n}\right) - 1 = \frac{(a+d)^2}{2(ad-bc)} - 1 \in \mathbb{Q}.$$

But as k is coprime to n then, by Niven's theorem, $\cos(2k\pi/n)$ is only rational for $n \in \{1, 2, 3, 4, 6\}$.

Note: The first reference to Niven's theorem which I found quoted the result as being only about sine [1], but in its original form by Niven [2] it states the only possible rational values for all trigonometric ratios of rational multiples of 2π. Also note that an earlier elementary proof is given by Olmsted [3], which in fact primarily proves the results for cosine and tangent.

Classification using trace and determinant

The proof of the corollary hints at a useful calculation. For a non-singular matrix M, define

$$\tau(M) = \frac{(\mathrm{tr}(M))^2}{\det(M)}.$$

Proposition 3: If the period of M is n,

$$n = 2 \Leftrightarrow \tau(M) = 0$$

$$n = 3 \Leftrightarrow \tau(M) = 1$$

$$n = 4 \Leftrightarrow \tau(M) = 2$$

$$n = 6 \Leftrightarrow \tau(M) = 3.$$

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Notes:

(1) The period is $\dfrac{\pi}{\cos^{-1}\left(\frac{1}{2}\sqrt{\tau(M)}\right)}$ whenever this gives an integer.

(2) If the period is 1 then $\tau(M) = 4$. However the converse is false, for example take $M = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof: If $n \in \{2, 3, 4, 6\}$ then, by the Theorem, M has eigenvalues $\lambda \exp\left(\pm\dfrac{\pi i}{n}\right)$ as for these n we can take $k = \pm 1$. Then, as $\lambda \neq 0$,

$$\tau(M) = \frac{(2\lambda \cos\frac{\pi}{n})^2}{\lambda^2} = 4\cos^2\frac{\pi}{n}$$

which gives $0, 1, 2, 3$ for $n = 2, 3, 4, 6$ respectively.

For the converses, if $\tau(M) \in \{0, 1, 2, 3\}$ then $\det(M) \neq 0$ so M has eigenvalues $\lambda_1, \lambda_2 \neq 0$ and

$$\tau(M) = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1\lambda_2}$$

and hence

$$\lambda_1^2 + (2 - \tau(M))\lambda_1\lambda_2 + \lambda_2^2 = 0.$$

Solving this for $\tau(M) = 0, 1, 2$ and 3, we get $\dfrac{\lambda_1}{\lambda_2} = \exp\left(\pm\dfrac{2\pi i}{n}\right)$

where $n = 2, 3, 4$ and 6 respectively.

Hence the eigenvalues can be written as $\lambda \exp\left(\pm\dfrac{\pi i}{n}\right)$ for $\lambda \neq 0$ so, by the Theorem, M has period n.

General solution for Möbius sequences of period 2, 3, 4 *or* 6

The use of rotation matrices or eigenvalues gives a simple way to generate periodic matrices but if we want rational or integer coefficients then we could do with another approach. So we have the following more practical result.

Proposition 4:

(1) M has period 2 if, and only if, $\operatorname{tr}(M) = 0$ and $\det(M) \neq 0$.

(2) Any matrix of period $n > 1$ can be used to generate all other matrices of period n. In particular every matrix of period $n \in \{3, 4, 6\}$ is equivalent to $\operatorname{adj}(A)\begin{pmatrix} 1 & -1 \\ 1 & d_n \end{pmatrix}A$ for some non-singular matrix A

where $d_3 = 0, d_4 = 1, d_6 = 2$.

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Proof:

(1) follows directly from Proposition 3.

(2) If M, N are matrices both with period $n > 2$ then from the Theorem we can write $M = P(rD_1)P^{-1}$ and $N = Q(sD_2)Q^{-1}$, where r, s are non-zero numbers and $D_1 = D^{k_1}$, $D_2 = D^{k_2}$ with

$$D = \begin{pmatrix} \exp\left(\frac{\pi i}{n}\right) & 0 \\ 0 & \exp\left(-\frac{\pi i}{n}\right) \end{pmatrix}$$

and k_1, k_2 are integers each coprime to n.

Hence there is an integer k (coprime to n) such that $kk_2 \equiv k_1 \pmod n$, giving $D_2^k = \pm D_1$. So

$$N^k = s^k Q D_2^k Q^{-1} = \pm s^k Q D_1 Q^{-1} = \pm r^{-1} s^k Q P^{-1} M P Q^{-1} = \pm r^{-1} s^k R M R^{-1}$$

where $R = QP^{-1}$. So $M = \pm r s^{-k} R^{-1} N^k R \sim R^{-1} N^k R$, where k is coprime to n. So we can use one matrix N with period n to generate all others.

For $n \in \{3, 4, 6\}$ we can take $k = \pm 1$, so if M has period 3, 4 or 6 then $M \sim R^{-1} N R$ or $M \sim R^{-1} N^{-1} R$.

It is easy to check that $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ have periods 3, 4 and 6 respectively. So we can use these matrices for N in these cases.

For $N = \begin{pmatrix} 1 & -1 \\ 1 & d \end{pmatrix}$, let $S = \begin{pmatrix} -1 & 0 \\ d-1 & 1 \end{pmatrix}$ then $S^{-1} N S = (1+d) N^{-1}$ so $R^{-1} N^{-1} R = \frac{1}{1+d} (SR)^{-1} N (SR)$. So for $k = 1$, take $A = R$ and for $k = -1$, take $A = SR$ and then in both cases $M \sim A^{-1} N A \sim \operatorname{adj}(A) N A$.

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105.29 A geometric memory

I fell in love with Euclidean geometry when I was about ten and have loved it ever since. I remember at the age of twelve doing a term examination which contained the following problem.

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