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The integration of generalized hypergeometric functions†

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Let the generalized hypergeometric function of one variable be denoted by

$$ {}_AF_B[(a);(b);z] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^{A}(a_i)_m z^m}{\prod_{i=1}^{B}(b_i)_m m!} $$

where $(a)_m$ is the Pochhammer symbol ((1, 3)).

Then certain integrals involving products of generalized hypergeometric functions with several Bessel functions yield, in turn, generalized hypergeometric functions of two variables, which may be denoted by ((1, 2))

$$ F\left[\begin{matrix}(\alpha);(a);(a');x\\(\beta);(b);(b');y\end{matrix}\right] = \sum_{n,m=0}^{\infty} \frac{\prod_{i=1}^{N}(\alpha_i)_{n+m}\prod_{i=1}^{A}(a_i)_n\prod_{i=1}^{A'}(a_i')_m x^n y^m}{\prod_{i=1}^{M}(\beta_i)_{n+m}\prod_{i=1}^{B}(b_i)_n\prod_{i=1}^{B'}(b_i')_m n!\,m!}, $$

with convergence conditions as indicated.

Consider the integral

$$ I_1 = \int_0^{\infty} t^{\mu-1}\,{}_AF_B[(a);(b);xt]\,{}_{A'}F_{B'}[(a');(b');yt]\,K_\nu(zt)\,dt, $$

where for convergence, $A \leqslant B$ and $A' \leqslant B'$.

If one uses the expansion

$$ \begin{aligned} {}_AF_B[(a);(b);z] &= {}_{2A}F_{2B+1}\left[\tfrac{1}{2}(a),\tfrac{1}{2}(a)+\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}(b),\tfrac{1}{2}(b)+\tfrac{1}{2};\frac{z^2}{4^{1-A+B}}\right] \\ &\quad + z\frac{\prod_{i}^{A}(a_i)}{\prod_{i}^{B}(b_i)}\,{}_{2A}F_{2B+1}\left[\tfrac{1}{2}(a)+\tfrac{1}{2},\tfrac{1}{2}(a)+1;\tfrac{3}{2},\tfrac{1}{2}(b)+1,\tfrac{1}{2}(b)+\tfrac{1}{2};\frac{z^2}{4^{1-A+B}}\right], \end{aligned} \tag{1} $$

then the integral I_1 can be expressed as the sum of four integrals, each of a form like that of I_1, but with altered arguments. On substituting power series expansions of the hypergeometric functions, and upon interchange of the order of double summation and integration, one finds integrals of known form ((3), vol. II, p. 51)

$$ \int_0^{\infty} t^{\alpha-1}K_\nu(zt)\,dt = 2^{\alpha-2}z^{-\alpha}\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{2}\alpha-\tfrac{1}{2}\nu) \quad \text{for} \quad \Re(\alpha \pm \nu) > 0. $$

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Now α, in each of the four integrals, is $\mu+2n+2m$, $\mu+2n+2m+1$, $\mu+2n+2m+1$ and $\mu+2n+2m+2$, respectively. Utilizing the definition of the Pochhammer symbol,

$$\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+n+m) = (\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)_{n+m}\,\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu),$$

allows one to evaluate each of the four integrals of the expansion. The integral I_1, then becomes

$$\begin{aligned} I_1 = \frac{2^{\mu-2}}{z\mu}\Bigg\{ & \Gamma\left(\frac{\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right) F\left[\begin{matrix}\frac{\mu+\nu}{2},\frac{\mu-\nu}{2}; \frac{1}{2}(a),\frac{1}{2}(a)+\frac{1}{2}; \frac{1}{2}(a'),\frac{1}{2}(a')+\frac{1}{2}; \frac{x^2}{4^{B-A}z^2} \\ ; \frac{1}{2},\frac{1}{2}(b),\frac{1}{2}(b)+\frac{1}{2}; \frac{1}{2},\frac{1}{2}(b'),\frac{1}{2}(b')+\frac{1}{2}; \frac{y^2}{4^{B'-A'}z^2}\end{matrix}\right] \\ & + \frac{2y}{z}\frac{\prod_i^{A'}(a_i')}{\prod_i^{B'}(b_i')}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right) \\ & \quad \times F\left[\begin{matrix}\frac{1+\mu+\nu}{2},\frac{1+\mu-\nu}{2}; \frac{1}{2}(a),\frac{1}{2}(a)+\frac{1}{2}; \frac{1}{2}(a')+\frac{1}{2},\frac{1}{2}(a')+1; \frac{x^2}{4^{B-A}z_2} \\ ; \frac{1}{2},\frac{1}{2}(b),\frac{1}{2}(b)+\frac{1}{2}; \frac{3}{2},\frac{1}{2}(b')+\frac{1}{2},\frac{1}{2}(b')+1; \frac{y^2}{4^{B'-A'}z^2}\end{matrix}\right] \\ & + \frac{2x}{z}\frac{\prod_i^{A}(a_i)}{\prod_i^{B}(b_i)}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right) \\ & \quad \times F\left[\begin{matrix}\frac{1+\mu+\nu}{2},\frac{1+\mu-\nu}{2}; \frac{1}{2}(a)+\frac{1}{2},\frac{1}{2}(a)+1; \frac{1}{2}(a'),\frac{1}{2}(a')+\frac{1}{2}; \frac{x^2}{4^{B-A}z^2} \\ ; \frac{3}{2},\frac{1}{2}(b)+\frac{1}{2},\frac{1}{2}(b)+1; \frac{1}{2},\frac{1}{2}(b'),\frac{1}{2}(b')+\frac{1}{2}; \frac{y^2}{4^{B'-A'}z^2}\end{matrix}\right] \\ & + \frac{2^2xy}{z^2}\frac{\prod_i^{A}(a_i)\prod_i^{A'}(a_i')}{\prod_i^{B}(b_i)\prod_i^{B'}(b_i')} \\ & \quad \times F\left[\begin{matrix}\frac{1+\mu+\nu}{2},\frac{1+\mu-\nu}{2}; \frac{1}{2}(a)+\frac{1}{2},\frac{1}{2}(a)+1; \frac{1}{2}(a')+\frac{1}{2},\frac{1}{2}(a')+1; \frac{x^2}{4^{B-A}z^2} \\ ; \frac{3}{2},\frac{1}{2}(b)+\frac{1}{2},\frac{1}{2}(b)+1; \frac{3}{2},\frac{1}{2}(b')+\frac{1}{2},\frac{1}{2}(b')+1; \frac{y^2}{4^{B'-A'}z^2}\end{matrix}\right]\Bigg\} \end{aligned} \tag{2}$$

$A < B$, $A' < B'$, $\Re(\mu \pm \nu) > 0$, $\Re z > 0$; and for $A = B$, $A' = B'$ in addition

$$\left|\frac{x^2}{4^{B-A}z^2}\right|^{\frac{1}{2}} + \left|\frac{y^2}{4^{B'-A'}z^2}\right|^{\frac{1}{2}} < 1.$$

Because of equation (1), this differs from an earlier result in ((6), p. 762).

For the specialization of parameters to $\nu = \pm\frac{1}{2}$, $\mu = \alpha+\frac{1}{2}$, that is for

$$K_{\pm\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \quad \text{as kernel}$$

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the expression for the integral I_1 reduces to an expansion formula for generalized hypergeometric functions of two variables

$$F\left[\begin{matrix}\alpha;\,(a),(a');\,x\\ ;\,(b),(b');\,y\end{matrix}\right]=F\left[\begin{matrix}\frac{\alpha}{2},\frac{\alpha+1}{2};\,\frac{1}{2}(a),\frac{1}{2}(a)+\frac{1}{2};\,\frac{1}{2}(a'),\frac{1}{2}(a')+\frac{1}{2};\quad x^2\\ ;\,\frac{1}{2},\frac{1}{2}(b),\frac{1}{2}(b)+\frac{1}{2};\,\frac{1}{2},\frac{1}{2}(b'),\frac{1}{2}(b')+\frac{1}{2};\,y^2\end{matrix}\right]$$

$$+\alpha y\frac{\prod_i^{A'}(a_i')}{\prod_i^{B'}(b_i')}F\left[\begin{matrix}\frac{1+\alpha}{2},\frac{\alpha}{2}+1;\,\frac{1}{2}(a),\frac{1}{2}(a)+\frac{1}{2},\frac{1}{2}(a')+\frac{1}{2},\frac{1}{2}(a')+1;\quad x^2\\ ;\,\frac{1}{2},\frac{1}{2}(b),\frac{1}{2}(b)+\frac{1}{2};\,\frac{3}{2},\frac{1}{2}(b')+\frac{1}{2},\frac{1}{2}(b')+1;\,y^2\end{matrix}\right]$$

$$+\alpha x\frac{\prod_i^{A}(a_i)}{\prod_i^{B}(b_i)}F\left[\begin{matrix}\frac{1+\alpha}{2},\frac{\alpha}{2}+1;\,\frac{1}{2}(a)+\frac{1}{2},\frac{1}{2}(a)+1;\,\frac{1}{2}(a'),\frac{1}{2}(a')+\frac{1}{2};\,x^2\\ ;\,\frac{3}{2},\frac{1}{2}(b)+\frac{1}{2},\frac{1}{2}(b)+1;\,\frac{1}{2},\frac{1}{2}(b),\frac{1}{2}(b)+\frac{1}{2};\,y^2\end{matrix}\right]$$

$$+\alpha(\alpha+1)xy\frac{\prod_i^{A}(a_i)\prod_i^{A'}(a_i')}{\prod_i^{B}(b_i)\prod_i^{B'}(b_i')}F\left[\begin{matrix}1+\frac{\alpha}{2},\frac{\alpha+3}{2};\,\frac{1}{2}(a)+\frac{1}{2},\frac{1}{2}(a)+1;\,\frac{1}{2}(a')+\frac{1}{2},\frac{1}{2}(a')+1;\quad x^2\\ ;\,\frac{3}{2},\frac{1}{2}(b)+\frac{1}{2},\frac{1}{2}(b)+1;\,\frac{3}{2},\frac{1}{2}(b)+\frac{1}{2},\frac{1}{2}(b)+1;\,y^2\end{matrix}\right].$$

So $$\int_0^\infty t^{\alpha-\frac{1}{2}}\,{}_AF_B[(a);\,(b);\,xt]\,{}_{A'}F_{B'}[(a');\,(b');\,yt]\,K_{\frac{1}{2}}(zt)\,dt=F\left[\begin{matrix}\alpha;\,(a),(a');\,x/z\\ ;\,(b),(b');\,y/z\end{matrix}\right]$$

which is a result calculated by Slater ((5), p. 54) in the form

$$\int_0^\infty e^{-zt}t^{\lambda-1}\,{}_AF_B[(a);\,(b);\,xt]\,{}_{A'}F_{B'}[(a');\,(b');\,yt]\,dt$$
$$=\frac{\Gamma(\lambda)}{z^\lambda}\sum_{m=0}^{\infty}\frac{((a))_m(\lambda)_m}{m!\,((b))_m}\left(\frac{x}{z}\right)^m{}_{A'+1}F_{B'}[(a'),\lambda+m;\,(b');\,y/z].$$

Another specialization is obtained for $A=B=A'=B'=1$, with $a=\alpha$, $b=\alpha+1$, $a'=\beta$, $b'=\beta+1$.

Then the integral becomes

$$\int_0^\infty t^{\mu-1}\gamma(\alpha,xt)\,\gamma(\beta,yt)\,K_\nu(zt)\,dt$$
$$=\frac{\alpha\beta 2^{\mu+\alpha+\beta-2}}{z^{\mu+\alpha+\beta}}\left\{\Gamma\left(\frac{\mu+\alpha+\beta+\nu}{2}\right)\Gamma\left(\frac{\mu+\alpha+\beta-\nu}{2}\right)\right.$$
$$\times F\left[\begin{matrix}\frac{\mu+\alpha+\beta+\nu}{2},\frac{\mu+\alpha+\beta-\nu}{2};\,\frac{\alpha}{2};\,\frac{\beta}{2};\,x^2/z^2\\ ;\,\frac{1}{2},\frac{\alpha}{2}+1;\,\frac{1}{2},\frac{\beta}{2}+1;\,y^2/z^2\end{matrix}\right]$$
$$-2\left(\frac{\beta}{\beta+1}\right)\frac{y}{z}\Gamma\left(\frac{1+\mu+\alpha+\beta+1}{2}\right)\Gamma\left(\frac{1+\mu+\alpha+\beta-\nu}{2}\right)$$
$$\left.\times F\left[\begin{matrix}\frac{1+\mu+\alpha+\beta+\nu}{2},\frac{1+\mu+\alpha+\beta-\nu}{2};\,\frac{\alpha}{2};\,\frac{\beta+1}{2};\,x^2/z^2\\ ;\,\frac{1}{2},\frac{\alpha}{2}+1;\,\frac{3}{2},\frac{\beta+3}{2};\,y^2/z^2\end{matrix}\right]\right.$$

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$$-2\left(\frac{\alpha}{\alpha+1}\right)\frac{x}{z}\Gamma\left(\frac{1+\mu+\alpha+\beta+\nu}{2}\right)\Gamma\left(\frac{1+\mu+\alpha+\beta-\nu}{2}\right)$$
$$\times F\left[\begin{matrix}\frac{1+\mu+\alpha+\beta+\nu}{2}, \frac{1+\mu+\alpha+\beta-\nu}{2}; \frac{\alpha+1}{2}, \frac{\beta}{2}; x^2/z^2 \\ ; \frac{3}{2}, \frac{\alpha+3}{2}; \frac{1}{2}, \frac{\beta}{2}+1; y^2/z^2\end{matrix}\right]$$
$$+\frac{4\alpha\beta xy}{(\alpha+1)(\beta+1)z^2}\Gamma\left(1+\frac{\mu+\nu+\alpha+\beta}{2}\right)\Gamma\left(1+\frac{\mu+\alpha+\beta-\nu}{2}\right)$$
$$\times F\left[\begin{matrix}1+\frac{\mu+\alpha+\beta+\nu}{2}, 1+\frac{\mu+\alpha+\beta-\nu}{2}; \frac{\alpha+1}{2}, \frac{\beta+1}{2}; x^2/z^2 \\ ; \frac{3}{2}, \frac{\alpha+3}{2}; \frac{3}{2}, \frac{\beta+3}{2}; y^2/z^2\end{matrix}\right]\Bigg\}.$$

Of a more simple form is the integral

$$I_2 = \int_0^\infty t^{\mu-1}\,{}_AF_B[(a);(b);x^2t^2]\,{}_{A'}F_{B'}[(a');(b');y^2t^2]\,K_\nu(zt)\,K_\alpha(zt)\,dt.$$

Using the results of ((4), p. 334), upon interchanging the order of integration and summation one finds

$$I_2 = \frac{2^{\mu-1}\Gamma\left(\frac{\mu+\alpha+\nu}{2}\right)\Gamma\left(\frac{\mu+\nu-\alpha}{2}\right)\Gamma\left(\frac{\mu-\nu+\alpha}{2}\right)\Gamma\left(\frac{\mu-\alpha-\nu}{2}\right)}{\Gamma(\mu)}$$
$$F\left[\begin{matrix}\frac{\mu+\alpha+\nu}{2}, \frac{\mu+\nu-\alpha}{2}, \frac{\mu+\alpha-\nu}{2}, \frac{\mu-\alpha-\nu}{2}; (a); (a'); (x/z)^2 \\ \frac{\mu}{2}, \frac{\mu+1}{2}; (b); (b'); (y/z)^2\end{matrix}\right]$$

for $\Re z > 0$ and $\Re\mu > |\Re\alpha| + |\Re\nu|$, $A < B-1$, $A' < B'-1$; and

$$\left|\left(\frac{x}{z}\right)^2\right|^{\frac{1}{2}} + \left|\left(\frac{y}{z}\right)^2\right|^{\frac{1}{2}} < 1 \quad \text{for} \quad A = B-1,\ A' = B'-1.$$

Closely related is the form

$$I_3 = \int_0^\infty t^{\mu-1}\,{}_AF_B\left[(a);(b);\left(\frac{xt}{2}\right)^4\right]\,{}_{A'}F_{B'}\left[(a');(b');\left(\frac{yt}{2}\right)^4\right]J_\nu(zt)\,K_\nu(zt)\,dt,$$

which can be given for certain cases. Namely,

$$I_3 = \frac{(-1)^k 2^{\mu-2}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu}{4}+\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{4}-\frac{\nu}{2}\right)}{z^\mu\pi} F\left[\begin{matrix}\frac{\mu}{4}, \frac{\mu+2}{4}, \frac{\mu}{2}+\frac{\nu}{2}; (a); (a'); -4\left(\frac{x}{z}\right)^4 \\ ; (b); (b'); -4\left(\frac{y}{z}\right)^4\end{matrix}\right]$$

for $\Re\mu > -2\Re\nu, 0$, $|\arg z| < \frac{1}{4}\pi$, $\frac{1}{2}\mu - \nu = 2k+1$, $k = 0, 1, 2, \ldots$, $B > A+3$, $B' > A'+3$; and $r^{\frac{1}{4}} + s^{\frac{1}{4}} < 1$ for $B = A+3$, $B' = A'+3$, where

$$r = \left|-4\left(\frac{x}{z}\right)^4\right|, \quad s = \left|-4\left(\frac{y}{z}\right)^4\right|.$$

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