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A variation on the variational principle and applications to entropy pairs

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Abstract. The variational principle states that the topological entropy of a topological dynamical system is equal to the sup of the entropies of invariant measures. It is proved that for any finite open cover there is an invariant measure such that the topological entropy of this cover is less than or equal to the entropies of all finer partitions. One consequence of this result is that for any dynamical system with positive topological entropy there exists an invariant measure whose set of entropy pairs is equal to the set of topological entropy pairs.

1. *Introduction*

Consider a topological dynamical system (X, T), where X is a compact metric space and T is a homeomorphism of X. In 1964 Adler, Konheim and McAndrew introduced topological entropy: first the entropy of an open cover, and then the entropy of the whole system as the sup of the topological entropies of covers. The problem of its relations with measure-theoretic entropy, defined in 1958 by Kolmogorov (recall that by definition the entropy of an invariant measure is the sup of the entropies of measurable partitions), was progressively solved by various authors during the first half of the seventies: the variational principle states that the topological entropy of (X, T) is equal to the supremum of the entropies of invariant measures.

This result was proved globally; the proofs do not involve any comparison between the topological entropy of a given open cover and the measure-theoretic entropies of some partitions connected to this cover. Recently, the authors and other people working in topological dynamics began to feel that some results of this kind might be helpful.

The main application we had in mind was to the study of entropy pairs. *Topological entropy pairs* were introduced in [**B**]: (x, y), $x \neq y$ is said to be an entropy pair if whenever U and V form an open cover of X and $x \in \mathrm{Int}(U^c)$, $y \in \mathrm{Int}(V^c)$, then $h_{\mathrm{top}}(T, (U, V)) > 0$. Whenever $h(X, T) > 0$ there exist entropy pairs. In [**GW1**] Glasner and Weiss proved that when an invariant measure μ on (X, T) is a K-measure with full topological support, then all pairs out of the diagonal are entropy pairs. In

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order to obtain a local version of this result, *entropy pairs with respect to a measure* μ were defined in [**BHM**]: (x, y) is a μ-entropy pair if any measurable partition (E_1, E_2) such that $x \in \mathrm{Int}(U_1)$ and $y \in \mathrm{Int}(E_2)$ has positive entropy. In analogy with topological entropy pairs, these pairs 'locate' measure-theoretic entropy. It was shown in [**BHM**] that any entropy pair with regard to a measure μ is a topological entropy pair. The converse question, whether for any entropy pair (x, y) one can find a measure μ such that (x, y) is also an entropy pair for μ, was left open in the general case. Clearly, a better knowledge of the relation between entropy of covers and entropy of partitions was likely to help solve this question.

The first theorem (Theorem 1) in this article is a variational principle for covers: given some open cover $\mathcal{U}$ of X, there exists a T-invariant measure μ such that for any measurable partition $\mathcal{P}$ of X, finer than $\mathcal{U}$, one has $h_\mu(T, \mathcal{P}) \geq h_{\mathrm{top}}(T, \mathcal{U})$. It is proved in the following way. First, with the help of a purely combinatorial lemma, one shows the existence of points of X such that the 'statistical entropy' of $\mathcal{P}$ given by the associated frequencies of words is sufficiently big. The required measure is then obtained as a limit of convex combinations of point masses of this type. Several applications of Theorem 1 are given.

First, we introduce a kind of 'entropy capacity' of compact sets: for a compact subset $K \subset X$ define $c(K) = \mathrm{Sup}\{h_{\mathrm{top}}(\mathcal{U}, K^c)\}$, where the sup is taken for all open covers $\mathcal{U}$ of K. It is shown that $c(K)$ is the sup of the entropies of all ergodic measures μ on X such that $\mu(K) > 0$.

The main application of Theorem 1 is a converse of the main result in [**BHM**]: given (X, T), one can always find an invariant measure μ such that any entropy pair is an entropy pair for μ (Theorem 3). With the further requirement that μ be ergodic this statement becomes false, as shown in Proposition 4. Nevertheless, in any neighborhood of an entropy pair there are entropy pairs for ergodic measures (Theorem 4).

Aided by this result we finally deduce another result that further emphasizes the symmetry between topological and measure-theoretic entropy pairs: if $\phi : (X, T) \to (Y, S)$ is a factor map, μ is invariant for T, $\nu = \phi\mu$, and (y, y') is a ν-entropy pair for (Y, T), then there is a μ-entropy pair (x, x') with image (y, y') under ϕ. This result, parallel to the corresponding statement in the topological setting [**B**], could not be proved in [**BHM**].

Finally, in the Appendix we prove a result concerning the topology of the set E of entropy pairs (Theorem 6), one consequence of which is that E has no isolated points.

2. *A combinatorial lemma*

2.1. *Notation.* Throughout this paper, $\phi : [0, 1] \to \mathbb{R}$ denotes the function

$$\phi(x) = -x \log x \quad \text{for } 0 < x \leq 1; \phi(0) = 0.$$

Let A be a finite set, called *the alphabet*. Sequences $\omega = \omega_1 \cdots \omega_n \in A^n$, for $n \geq 1$, are called *words of length n on the alphabet A*. Let n and k be two integers with $1 \leq k \leq n$. For every word ω of length n and every word α of length k on the same

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alphabet, we denote by $d_\alpha(\omega)$ the frequency of appearances of α in ω, i.e.

$$d_\alpha(\omega) = \frac{1}{n-k+1}\,\mathrm{Card}\{i;\, 1 \le i \le n-k+1,\, \omega_i\omega_{i+1}\cdots\omega_{i+k-1} = \alpha_1\alpha_2\cdots\alpha_{k-1}\}.$$

For every word ω of length n on the alphabet A, we let

$$H_k(\omega) = \sum_{\alpha\in A^k} \phi(d_\alpha(\omega)).$$

LEMMA 1. *For every $h > 0$, $\epsilon > 0$, every integer $k \ge 1$ and every sufficiently large integer n,*

$$\mathrm{Card}\{\omega \in A^n;\, H_k(\omega) \le kh\} \le \exp(n(h+\epsilon)).$$

Remark. It is equally true that, if $h \le \log \mathrm{Card}\, A$, for sufficiently large n,

$$\mathrm{Card}\{\omega \in A^n;\, H_k(\omega) \le kh\} \ge \exp(n(h-\epsilon)).$$

We do not prove this inequality here, since we have no use for it in what follows.

One can regard this lemma as a finite version of the Shannon–McMillan theorem; in fact one can easily prove this theorem assuming the lemma.

Proof. Put $d = \mathrm{Card}(A)$.

(i) *Assume first that $k = 1$.* We have

$$\mathrm{Card}\{\omega \in A^n;\, H_1(\omega) \le h\} = \sum_{\underline{q}\in K} \frac{n!}{q_1!\cdots q_d!}, \tag{1}$$

where K is the set of $\underline{q} = (q_1, \ldots, q_d) \in \mathbb{N}^d$ such that

$$\sum_{i=1}^d q_i = n \quad \text{and} \quad \sum_{i=1}^d \phi\left(\frac{q_i}{n}\right) \le h.$$

By Stirling's formula, there exist two universal constants c and c' such that

$$c\left(\frac{m}{e}\right)^m \sqrt{m} \le m! \le c'\left(\frac{m}{e}\right)^m \sqrt{m} \quad \text{for every } m > 0.$$

From this we deduce the existence of a constant $C(d)$ such that for every $\underline{q} \in K$,

$$\frac{n!}{q_1!\cdots q_d!} \le C(d)\exp\left(n\sum_{i=1}^d \phi\left(\frac{q_i}{n}\right)\right) \le C(d)\exp(nh).$$

Now the sum (1) contains at most $(n+1)^d$ terms, so we have

$$\mathrm{Card}\{\omega \in A^n;\, H_1(\omega) \le h\} \le (n+1)^d C(d)\exp(nh) \le \exp(n(h+\epsilon))$$

for all sufficiently large n, as was to be proved.

(ii) *The case $k > 1$.* For every word ω of length $n \ge 2k$ on the alphabet A, and for $0 \le j < k$, we let n_j be the integral part of $(n-j)/k$, and let $\omega^{(j)}$ be the word

$$(\omega_{j+1}\cdots\omega_{j+k})(\omega_{j+k+1}\cdots\omega_{j+2k})\cdots(\omega_{j+(n_j-1)k+1}\cdots\omega_{j+n_jk})$$

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of length n_j on the alphabet $B = A^k$.

Now let α be a word of length k on the alphabet A; we also consider α as an element of B.

One easily verifies that, for every word ω of length n on the alphabet A,

$$\left| d_\alpha(\omega) - \frac{1}{k}\sum_{j=0}^{k-1} d_\alpha(\omega^{(j)}) \right| \leq \frac{k}{n-2k+1}.$$

The function ϕ being uniformly continuous, we see that for large enough n and for every word ω of length n on A,

$$\sum_{\alpha\in B} \left| \phi(d_\alpha(\omega)) - \phi\left(\frac{1}{k}\sum_{j=0}^{k-1} d\alpha(\omega^{(j)})\right)\right| < \frac{\epsilon}{2}$$

and by the convexity of ϕ,

$$\frac{1}{k}\sum_{j=0}^{k-1} H_1(\omega^{(j)}) = \frac{1}{k}\sum_{j=0}^{k-1}\sum_{\alpha\in B} \phi(d_\alpha(\omega^{(j)})) \leq \frac{\epsilon}{2} + \sum_{\alpha\in A^k} \phi(d_\alpha(\omega)) = \frac{\epsilon}{2} + H_k(\omega).$$

Thus, if $H_k(\omega) \leq kh$, there exists a j such that $H_1(\omega^{(j)}) \leq \epsilon/2 + kh$.

Now, given j and a word u of length n_j on the alphabet B, there exist $d^{n-n_jk} \leq d^{2k-2}$ words ω of length n on A such that $\omega^{(j)} = u$. Thus

$$\begin{aligned}\text{Card}\{\omega \in A^n;\, H_k(\omega) \leq kh\} &\leq d^{2k-2}\sum_{j=0}^{k-1} \text{Card}\left\{u \in B^{n_j};\, H_1(u) \leq \frac{\epsilon}{2} + kh\right\} \\ &\leq d^{2k-2}\sum_{j=0}^{k-1} \exp(n_j(\epsilon + kh))\end{aligned}$$

for large enough n, and as in the first part of the proof,

$$\leq d^{2k-2}k\exp\left(n\left(\frac{\epsilon}{k} + h\right)\right) \leq \exp(n(h+\epsilon)).$$

□

3. *A variational principle*

3.1. *Notation.* In what follows, (X, T) is a topological dynamical system. Denote by $M(X, T)$ the set of T-invariant probability measures on X, and by $M_E(X, T)$ the subset of ergodic measures.

We say that a partition $\mathcal{P}$ is finer than a cover $\mathcal{U}$ when every atom of $\mathcal{P}$ is contained in an element of $\mathcal{U}$.

If $\mathcal{P} = \{X_1, \ldots, X_d\}$ is a partition of X, $x \in X$ and $N \in \mathbb{N}$, one writes $\omega(\mathcal{P}, N, x)$ for the word of length N on the alphabet $\{1, \ldots, d\}$ defined by:

$$\text{for } 1 \leq n \leq N, \quad \omega(\mathcal{P}, N, x)_n = i \text{ if } T^{n-1}x \in X_i.$$

The following theorem is a variational principle for covers (for the variational principle itself see, for example, [**DGS**]).

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THEOREM 1. *Let (X, T) be a topological dynamical system, and let $\mathcal{U}$ be an open cover of X. Then there exists a measure $\mu \in M(X, T)$ such that $h_\mu(\mathcal{P}) \geq h_{\text{top}}(\mathcal{U})$ for all Borel partitions $\mathcal{P}$ finer than $\mathcal{U}$.*

LEMMA 2. *Let $\mathcal{U}$ be a cover of X, $h = h_{\text{top}}(\mathcal{U})$, $K \geq 1$ an integer, and $(\mathcal{P}_\ell; 1 \leq \ell \leq K)$ a finite sequence of partitions of X, all finer than $\mathcal{U}$.*

For every $\epsilon > 0$ and sufficiently large N, there exists an $x \in X$ such that

$$H_k(\omega(\mathcal{P}_\ell, N, x)) \geq k(h - \epsilon) \quad \textit{for every } k, \ell \textit{ with } 1 \leq k, \ell \leq K.$$

Proof. One can assume that all the partitions $\mathcal{P}_\ell$ have the same number of elements d. Let $A = \{1, \ldots, d\}$. For $1 \leq k, \ell \leq K$ and $N \geq K$, denote

$$\Omega(N, k) = \{\omega \in A^N; H_k(\omega) < k(h - \epsilon)\}.$$

By Lemma 1, for large enough N

$$\text{Card}(\Omega(N, k)) \leq \exp(N(h - \epsilon/2)) \quad \text{for all } k \leq K.$$

Let us choose such an N which, moreover, satisfies $K^2 < \exp(N\epsilon/2)$. For $1 \leq k, \ell \leq K$, let

$$Z(k, \ell) = \{x \in X; \omega(\mathcal{P}_\ell, N, x) \in \Omega(N, k)\}.$$

The set $Z(k, \ell)$ is the union of $\text{Card}(\Omega(N, k))$ elements of $(\mathcal{P}_\ell)_0^{N-1}$. Now this partition is finer than the cover $\mathcal{U}_0^{N-1}$, hence $Z(k, \ell)$ is covered by

$$\text{Card}(\Omega(N, k)) \leq \exp(N(h - \epsilon/2))$$

elements of $\mathcal{U}_0^{N-1}$. Finally, $\bigcup_{1 \leq k, \ell \leq K} Z(k, \ell)$ is covered by $K^2 \exp(N(h - \epsilon/2)) < \exp(Nh)$ elements of $\mathcal{U}_0^{N-1}$. As every subcover of $\mathcal{U}_0^{N-1}$ has at least $\exp(Nh)$ elements,

$$\bigcup_{1 \leq k, \ell \leq K} Z(k, \ell) \neq X.$$

This completes the proof of the lemma. □

3.2. *Proof of theorem 1.* Let $\mathcal{U} = \{U_1, \ldots, U_d\}$ be an open cover of X. It is clearly sufficient to consider Borel partitions $\mathcal{P}$ of X of the form

$$\mathcal{P} = \{X_1, \ldots, X_d\} \quad \text{with } X_i \subset U_i \text{ for every } i. \tag{2}$$

(i) *Assume first that X is Cantor (0-dimensional).* The family of partitions finer than $\mathcal{U}$, consisting of clopen sets and satisfying (2) is countable; let $(\mathcal{P}_\ell; \ell \geq 1)$ be an enumeration of this family.

According to Lemma 2, there exists a sequence of integers (N_K) tending to $+\infty$ and a sequence (x_K) of elements of X such that

$$H_k(\omega(\mathcal{P}_\ell, N_K, x_K)) \geq k\left(h - \frac{1}{K}\right) \quad \text{for every } 1 \leq k, \ell \leq K. \tag{3}$$

Write

$$\mu_K = \frac{1}{N_K} \sum_{i=0}^{K-1} \delta_{T^i x_K}.$$

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Replacing the sequence (μ_K) by a subsequence (this means replacing the sequence (N_K) by a subsequence, and the sequence (x_K) by the corresponding subsequence: preserving the property (3)), one can assume that the sequence of measures (μ_K) converges weak* to a probability measure μ. This measure μ is clearly T-invariant. Fix $k, \ell \geq 1$, and let E be an atom of the partition $(\mathcal{P}_\ell)_0^{k-1}$, with name $\alpha \in \{1, \ldots, d\}^k$. For every K one has

$$|\mu_K(E) - d_\alpha(\omega(\mathcal{P}_\ell, N_K, x_K))| \leq \frac{2k}{K}.$$

Now as E is clopen,

$$\mu(E) = \lim_{K\to\infty} \mu_K(E) = \lim_{K\to\infty} d_\alpha(\omega(\mathcal{P}_\ell, N_K, x_K)),$$

hence

$$\phi(\mu(E)) = \lim_{K\to\infty} \phi(d_\alpha(\omega(\mathcal{P}_\ell, N_K, x_K))),$$

and, summing over $\alpha \in \{1, \ldots, d\}^k$, one gets

$$H_\mu((\mathcal{P}_\ell)_0^{k-1}) = \lim_{K\to\infty} H_k(\omega(\mathcal{P}_\ell, N_K, x_K)) \geq kh.$$

Finally, by sending k to infinity one obtains $h_\mu(\mathcal{P}_\ell) \geq h$.

Now, as X is Cantor, the family of partitions $(\mathcal{P}_\ell)$ is dense in the collection of Borel partitions of X satisfying (2), with respect to the distance associated with $L^1(\mu)$. Thus, $h_\mu(\mathcal{P}) \geq h$ for every partition of this kind.

(ii) *The general case.* Let us recall a well-known fact: there exists a dynamical system (Y, S), where Y is Cantor, and a continuous surjective map $\pi : Y \to X$ with $\pi \circ S = T \circ \pi$.

(Proof: as X is a compact metric space, it is easy to construct a Cantor set K and a continuous surjective $f : K \to X$.

Put

$$Y = \{\underline{y} \in K^{\mathbb{Z}};\ f(y_{n+1}) = Tf(y_n) \text{ for every } n \in \mathbb{Z}\}$$

and let $\pi : Y \to X$ be defined by $\pi(\underline{y}) = f(y_0)$.

Y is a closed subset of $K^{\mathbb{Z}}$—where the latter is equipped with the product topology—and is invariant under the shift S on $K^{\mathbb{Z}}$. It is immediate to check that π satisfies the required conditions.)

Let $\mathcal{V} = \pi^{-1}(\mathcal{U}) = (\pi^{-1}(U_1), \ldots, \pi^{-1}(U_d))$ be the pre-image of $\mathcal{U}$ under π. One has $h_{\text{top}}(\mathcal{V}) = h_{\text{top}}(\mathcal{U}) = h$. By the above remark, there exists $\nu \in M(Y, S)$ such that $h_\nu(\mathcal{Q}) \geq h$ for every Borel partition $\mathcal{Q}$ of Y finer than $\mathcal{V}$. Let $\mu = \nu \circ \pi^{-1}$ be the measure which is the image of ν under π. One has $\mu \in M(X, T)$ and, for every Borel partition $\mathcal{P}$ of X finer than $\mathcal{U}$, $\pi^{-1}(\mathcal{P})$ is a Borel partition of Y which is finer than $\mathcal{V}$ with

$$h_\mu(\mathcal{P}) = h_\nu(\pi^{-1}(\mathcal{P})) \geq h.$$

This completes the proof of the theorem. □

COROLLARY 1. *Let (X, T) be a topological dynamical system, let $\mathcal{U}$ be an open cover of X, and let $\mathcal{P}$ be a Borel partition finer than $\mathcal{U}$. Then, there exists a T-invariant ergodic measure μ on X such that $h_\mu(\mathcal{P}) \geq h_{\text{top}}(\mathcal{U})$.*

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Proof. By Theorem 1 there exists $\mu \in M(X,T)$ with $h_\mu(\mathcal{P}) \geq h_{\text{top}}(\mathcal{U})$. Let $\mu = \int_\Omega \mu_\omega \, dm(\omega)$ be its ergodic decomposition. The corollary now follows from the formula, ([**DGS**, Theorem 13.3])

$$\int h_\omega(\mathcal{P}) \, dm(\omega) = h_\mu(\mathcal{P}).$$

□

4. *Measures of positive entropy charging a closed subset*

4.1. *Notation.* For K a closed subset of X, let R_K be the collection of covers of K consisting of open subsets of X, and let P_K be the collection of finite Borel partitions of K.

Given a closed subset K of X, it is easy to establish necessary and sufficient conditions so that:

$$\text{There exists a measure } \mu \in M_E(X,T) \text{ with } \mu(K) > 0.$$

For example, this property is equivalent to:

$$\exists \epsilon > 0, \forall N > 0 \ \exists x \in X \text{ such that } \frac{1}{N} \text{Card}\{n; 0 \leq n < N, T^n x \in K\} \geq \epsilon.$$

But it is more difficult to decide whether there exists such a measure with some additional properties: for example, a measure of positive entropy.

If $\mathcal{U}$ is a cover of K, denote by $(\mathcal{U}, K^c)$ the cover of X formed by the elements of $\mathcal{U}$ and the set K^c; we use the same convention for the partitions of K. Let

$$c(K) = \text{Sup}\{h_{\text{top}}(\mathcal{U}, K^c); \mathcal{U} \in R_K\},$$

and, for $\mu \in M(X,T)$,

$$c_\mu(K) = \text{Sup}\{h_\mu(\mathcal{P}, K^c); \mathcal{P} \in P_K\}.$$

One can easily verify that the function c is monotone and subadditive ($K \subset L \Rightarrow c(K) \leq c(L)$ and $c(K \cup L) \leq c(K) + c(L)$). This suggests that it can serve as a kind of 'capacity' function on the closed subsets of a dynamical system. The following theorem gives an explicit formula for c in terms of the invariant measures of the system.

THEOREM 2. *Let (X,T) be a topological dynamical system and let K be a closed subset of X. In order that*

$$\textit{there exists } \mu \in M_E(X,T) \textit{ with } h_\mu(T) > 0 \textit{ and } \mu(K) > 0 \tag{4}$$

it is necessary and sufficient that

$$\textit{there exists an open cover } \mathcal{U} \textit{ of } K \textit{ with } h_{\text{top}}(\mathcal{U}, K^c) > 0. \tag{5}$$

More precisely,

$$c(K) = \text{Sup}\{h_\mu(T); \mu \in M_E(X,T), \mu(K) > 0\}.$$

Remark. In condition (5) one can restrict to covers of K consisting of two open sets.

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Proof. By Corollary 1, one immediately gets

$$\mathrm{Sup}\{c_\mu(K);\, \mu \in M_E(X,T)\} \geq c(K)$$

and the theorem is proved in two steps.

(i) *For every* $\mu \in M_E(X,T)$ *one has*

$$c_\mu(K) = \begin{cases} 0 & \text{if } \mu(K) = 0 \\ h_\mu(T) & \text{if } \mu(K) > 0. \end{cases}$$

In fact, one clearly has $c_\mu(K) \leq h_\mu(T)$. The case when $\mu(K) = 0$ being trivial, we now suppose that $\mu(K) > 0$. Let $\epsilon > 0$. As the measure μ is ergodic, one can choose n large enough so that $\mu\left(\bigcup_{i=0}^{n} T^{-i}K\right) > 1 - \epsilon$. Choose $\delta > 0$ such that

$$d(x,y) < \delta \Longrightarrow d(T^{-i}x, T^{-i}y) < \epsilon \quad \text{for } i = 0, \ldots, n$$

and $\mathcal{P}_\epsilon$ is a partition of K, each element of which has a diameter $< \delta$.

All the atoms of $\mathcal{Q}_\epsilon = (\mathcal{P}_\epsilon, K^c)_0^n$ have a diameter $< \epsilon$, with the exception of at most one of them which has measure $< \epsilon$. We, therefore, have

$$h_\mu(T) = \mathrm{Sup}_\epsilon\, h_\mu(\mathcal{Q}_\epsilon) = \mathrm{Sup}_\epsilon\, h_\mu(\mathcal{P}_\epsilon, K^c) \leq c_\mu(K).$$

(ii) *For every* $\mu \in M(X,T)$, $c_\mu(K) \leq c(K)$. Let $(K_1, \ldots, K_d)$ be a Borel partition of K, $\mathcal{P} = (K_1, \ldots, K_d, K^c)$, and $\epsilon > 0$.

For $1 \leq i \leq d$, let F_i be a closed subset of K_i, and put $F = \bigcup_i F_i$; we assume that these closed subsets have been chosen so that the partition $\mathcal{Q} = (F_1, \ldots, F_d, F^c)$ satisfies $|h_\mu(\mathcal{Q}) - h_\mu(\mathcal{P})| < \epsilon$, and so that $h_\mu(K \setminus F, F \cup K^c) < \epsilon$.

Let V be a neighborhood of K with $h_\mu(V \setminus K, K \cup V^c) < \epsilon$; let $\mathcal{R}$ denote the partition $(V \setminus F, F \cup V^c)$ and $\mathcal{S}$ the partition $(V \setminus F, F_1, \ldots, F_d, V^c)$.

As

$$(K \setminus F, F \cup K^c) \vee (V \setminus K, K \cup V^c) \succ \mathcal{R}$$

one has $h_\mu(\mathcal{R}) \leq 2\epsilon$.

Also let $\mathcal{U} = (U_1, \ldots, U_d)$ be an open cover of K, with $F_i \subset U_i \subset V$ for all i, and $U_i \cap F_j = \emptyset$ for $i \neq j$.

For every element A of the partition $\mathcal{R}$ and every element B of the cover $(\mathcal{U}, K^c)$, $A \cap B$ is a subset of an element of the partition $\mathcal{S}$. This immediately implies the same property for $\mathcal{R}_0^n$, $(\mathcal{U}, K^c)_0^n$ and $\mathcal{S}_0^n$. We therefore conclude:

$$\forall A \in \mathcal{R}_0^n, \quad \mathrm{Card}\{B \in \mathcal{S}_0^n;\, B \subset A\} \leq N((\mathcal{U}, K^c)_0^n)$$

therefore

$$H_\mu(\mathcal{S}_0^n) \leq H_\mu(\mathcal{R}_0^n) + \log(N((\mathcal{U}, K^c)_0^n))$$

and

$$h_\mu(\mathcal{S}) \leq h_\mu(\mathcal{R}) + h_{\mathrm{top}}(\mathcal{U}, K^c) \leq 2\epsilon + h_{\mathrm{top}}(\mathcal{U}, K^c) \leq 2\epsilon + c(K).$$

Finally,

$$h_\mu(\mathcal{P}) \leq \epsilon + h_\mu(\mathcal{Q}) \leq \epsilon + h_\mu(\mathcal{S}) \leq 3\epsilon + c(K),$$

and the result follows. This completes the proof of the theorem. □

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We have, in fact, shown that

$$\begin{aligned} c(K) &= \mathrm{Sup}\{c_\mu(K);\ \mu \in M(X,T)\} = \mathrm{Sup}\{c_\mu(K);\ \mu \in M_E(X,T)\} \\ &= \mathrm{Sup}\{h_\mu(T);\ \mu \in M_E(X,T);\ \mu(K) > 0\}. \end{aligned}$$

Theorem 2 therefore provides a topological interpretation of the sup of the entropies of ergodic measures charging some compact subset. In view of this remark, it is interesting to distinguish two classes of dynamical systems with positive entropy: (1) those for which for every compact K, either $c(K) = 0$, or $c(K) = h_{\mathrm{top}}(T)$ and (2) those containing a closed subset $K \subset X$ with $0 < c(K) < h_{\mathrm{top}}(T)$. The first family contains all uniquely ergodic systems with positive entropy. The other class contains some minimal systems—the reader can check that in the minimal system constructed in §4 of [**GW3**] there exists a closed subset K with $c(K) = \log 2 < h_{\mathrm{top}}(T) = \log 3$—and some that are not [**BH**, Proposition 4.2].

4.2. *Big sets and small sets.*

Definition. Let K be a closed subset of X. When $c(K) > 0$ we say that K is *big*, otherwise K is *small*.

Immediately we get the following proposition.

PROPOSITION 1. *A universally null set K (i.e. $\mu(K) = 0$ for every $\mu \in M(X,T)$) is* small. *When (X,T) is uniquely ergodic and has positive entropy, the converse is also true.*

For a further application of these notions, see the Appendix.

5. *Topological and metric entropy pairs*

5.1. *Definitions and summary of results on entropy pairs.* Topological entropy pairs have been introduced and studied in [**B**]; we remind the reader of their definition and of some of their properties.

Let (X,T) be a topological dynamical system.

Definition. Let a, b be two distinct points of X. One calls (a,b) an *entropy pair* of (X,T) if, for every closed neighborhood A of a and every closed neighborhood B of b with $A \cap B = \emptyset$, one has $h_{\mathrm{top}}(A^c, B^c) > 0$.

We denote by $E(X,T)$ the set of entropy pairs of (X,T).

PROPOSITION 2. ([**B**])

(i) *$E(X,T) = \emptyset$ if and only if $h_{\mathrm{top}}(X,T) = 0$.*

(ii) *Let Δ denote the diagonal in $X \times X$, then $E(X,T) \cup \Delta$ is a closed invariant subset of $X \times X$.*

(iii) *Let A and B be two disjoint closed subset in X. If $h_{\mathrm{top}}(A^c, B^c) > 0$, then $A \times B$ contains an entropy pair.*

The entropy pairs for a measure have been introduced in [**BHM**]; we recall their definition and some of their basic properties.

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Let (X, T) be a topological dynamical system, and $\mu \in M(X, T)$. Denote by $\mathcal{B}_X$ the completion for μ of the σ-algebra of Borel subsets of X.

Let A and B be two disjoint subsets in X. We say that the partition $\mathcal{P} = (Q, Q^c)$ *separates* A and B if $A \subset Q$ and $B \subset Q^c$.

Definition. Let (X, T) be a topological dynamical system, and $\mu \in M(X, T)$. If a, b are two distinct points in X, we say that (a, b) is a *μ-entropy pair* if $h_\mu(K, K^c) > 0$ for every $K \in \mathcal{B}_X$ such that $a \in \mathrm{Int}(K)$ and $b \in \mathrm{Int}(K^c)$.

We will write $E_\mu(X, T)$ for the set of μ-entropy pairs in (X, T).

PROPOSITION 3. ([**BHM**])
(i) *$E_\mu(X, T) = \emptyset$ if and only if $h_\mu(X, T) = 0$.*
(ii) *$E_\mu(X, T) \cup \Delta$ is a closed and invariant subset of $X \times X$.*
(iii) *Let A and B be two disjoint closed subsets. If $h_\mu(Q, Q^c) > 0$ for every Borel partition $\mathcal{P} = (Q, Q^c)$ separating A and B, then there exists a μ-entropy pair (a, b) with $a \in A$ and $b \in B$.*

5.2. *The existence of a measure μ for which $E(X, T) = E_\mu(X, T)$.* In [**BHM**] it was shown that, for every measure $\mu \in M(X, T)$, one has $E_\mu(X, T) \subset E(X, T)$, and that $E(X, T) = E_\mu(X, T)$ when (X, T) is uniquely ergodic. However, the question whether in the general case, if for every $(a, b) \in E(X, T)$ there exists a measure $\mu \in M(X, T)$ such that $(a, b) \in E_\mu(X, T)$, was left open. The following result answers that question in the affirmative and shows, moreover, that it is possible to find a single measure good for all entropy pairs.

THEOREM 3. *Let (X, T) be a topological dynamical system. There exists a measure $\mu \in M(X, T)$ such that $E(X, T) = E_\mu(X, T)$.*

Proof. Let $((a_n, b_n); n \geq 1)$ be a dense sequence of points in $E(X, T)$. For every $n \geq 1$ and $r \geq 1$, let $C_{n,r}$ (respectively $D_{n,r}$) be the closed ball with center a_n (respectively b_n) and radius $d(x, y)/3r$. For every n and every r, $h_{\mathrm{top}}(C_{n,r}^c, D_{n,r}^c) > 0$, since (a_n, b_n) is an entropy pair. Thus, by Theorem 2, there exists a measure $\mu_{n,r} \in M(X, T)$ with $h_{\mu_{n,r}}(Q, Q^c) > 0$ for every Borel partition (Q, Q^c) separating $C_{n,r}$ and $D_{n,r}$. Let

$$\mu = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} 2^{-n-r} \mu_{n,r}.$$

For $(a, b) \in E(X, T)$, we now show that $(a, b) \in E_\mu(X, T)$.

Let U (respectively V) be an open neighborhood of a (respectively b), with $U \cap V = \emptyset$. By density, there exists n such that $a_n \in U$ and $b_n \in V$, and there exists r with $C_{n,r} \subset U$ and $D_{n,r} \subset V$. Every partition $\mathcal{P} = (Q, Q^c)$ which separates U and V also separates $C_{n,r}$ and $D_{n,r}$, thus $h_{\mu_{n,r}}(\mathcal{P}) > 0$, whence

$$h_\mu(\mathcal{P}) \geq 2^{-n-r} h_{\mu_{n,r}}(\mathcal{P}) > 0,$$

which proves our assertion. □

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5.3. *A counter-example.* One cannot always choose the measure whose existence is guaranteed in Theorem 3 to be ergodic.

PROPOSITION 4. *There exist a topological dynamical system and an entropy pair for that system, which is not a metric entropy pair for any ergodic measure.*

Proof. Let (X, T) be a topological dynamical system of uniform positive entropy (a u.p.e. system) ([**B**], [**GW1**]) and let (Y, S) be the symbolic system constructed by Weiss in [**W**]. Recall that:

(i) this system is of zero topological entropy, and it contains a dense subset of periodic points;

(ii) it is topologically transitive;

(iii) no ergodic measure has all of Y as its topological support.

Consider the Cartesian product $(X \times Y, T \times S)$. Since (X, T) is of uniform positive entropy and since (Y, S) satisfies (i), by [**BL**],

$$E(X \times Y, T \times S) = \{((x, y), (x', y)); x, x' \in X, x \neq x', y \in Y\}.$$

In particular, if y_0 is a transitive point of Y, and x, x' are two distinct points in X, $((x, y_0), (x', y_0))$ is an entropy pair for $(X \times Y, T \times S)$.

Now let $\mu \in M_E(X \times Y, T \times S)$ and let ν be its projection on Y. By (iii)—since ν is ergodic and as y_0 is transitive—it follows that y_0 does not belong to the support of ν, and there exists a neighborhood U of y_0 with $\nu(U) = 0$. Let A be a closed neighborhood of x which does not contain x', and let $\mathcal{P}$ be the partition $(A \times U, (A \times U)^c)$; $\mathcal{P}$ separates the points (x, y_0) and (x', y_0), and $h_\mu(\mathcal{P}) = 0$ since $\mu(A \times U) \leq \nu(U) = 0$. Thus $((x, y_0), (x', y_0))$ is not a μ-entropy pair. □

6. *Entropy pairs and the ergodic decomposition*

6.1. *Preliminaries.*

Definition. Let $\mu \in M(X, T)$ and let A, B be two disjoint Borel subsets of X. We say that (A, B) *satisfy* the property $S(\mu)$ if $h_\mu(Q, Q^c) > 0$ for every Borel partition (Q, Q^c) separating A and B.

Thus, by the definitions and results mentioned above:

(i) if a, b are two distinct points in X, $(a, b) \in E_\mu(X, T)$ if and only if the property $S(\mu)$ holds for every pair of disjoint neighborhoods A of a and B of b;

(ii) if A and B are two closed disjoint subsets satisfying the property $S(\mu)$, then $(A \times B) \cap E_\mu(X, T) \neq \emptyset$.

We denote by $\mathcal{P}_\mu$ the Pinsker σ-algebra of the measure-theoretical dynamical system $(X, \mathcal{B}_X, T, \mu)$. The following result appeared in the proof of Proposition 7 in [**BHM**].

LEMMA 3. *Let A, B be two disjoint Borel subsets of X. Then (A, B) satisfy the property $S(\mu)$ if and only if*

$$\mathbb{E}(\mathbf{1}_A \mid \mathcal{P}_\mu)(x)\, \mathbb{E}(\mathbf{1}_B \mid \mathcal{P}_\mu)(x) \text{ is not } \mu \text{ a.e. equal to zero.}$$

The following result is due to A. Maass.

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LEMMA 4. *Let A, B be two disjoint Borel subsets of X, satisfying the property $S(\mu)$. Then*

$$\mathrm{Inf}\{h_\mu(Q, Q^c);\ (Q, Q^c) \text{ separates } A \text{ and } B\} > 0.$$

Proof. In the context of the proof of Proposition 7 in [**BHM**], denote by $\mathcal{P}$ the partition $(A, B, (A \cup B)^c)$ and by $\mathcal{Q}$ the partition $(A \cup B, (A \cup B)^c)$. For every $m > 0$,

$$\begin{aligned} h_\mu(\mathcal{P}, T^m) - h_\mu(\mathcal{Q}, T^m) &= H_\mu(\mathcal{P} \mid \mathcal{P}_{T_-^m}) - H_\mu(\mathcal{P} \mid \mathcal{Q}_{T_-^m}) \\ &= H_\mu(\mathcal{P} \mid \mathcal{P}_{T_-^m} \vee \mathcal{P}_\mu) - H_\mu(\mathcal{P} \mid \mathcal{Q}_{T_-^m} \vee \mathcal{P}_\mu). \end{aligned}$$

Hence,

$$\lim_{m\to\infty} (h_\mu(\mathcal{P}, T^m) - h_\mu(\mathcal{Q}, T^m)) = H_\mu(\mathcal{P} \mid \mathcal{P}_\mu) - H_\mu(\mathcal{Q} \mid \mathcal{P}_\mu)$$

and the latter is > 0 by Lemma 3 (see Proposition 7 in [**BHM**]). There exists, therefore, an m such that $h_\mu(\mathcal{P}, T^m) > h_\mu(\mathcal{Q}, T^m)$.

Now let $\mathcal{R}$ be a partition separating A and B. We have $\mathcal{R} \vee \mathcal{Q} \succ \mathcal{P}$, hence $h_\mu(\mathcal{R}, T^m) \geq h_\mu(\mathcal{P}, T^m) - h_\mu(\mathcal{Q}, T^m)$ and, therefore,

$$h_\mu(\mathcal{R}) \geq \frac{1}{m} h_\mu(\mathcal{R}, T^m) \geq \frac{1}{m}(h_\mu(\mathcal{P}, T^m) - h_\mu(\mathcal{Q}, T^m)) > 0.$$

□

THEOREM 4. *Let $\mu \in M(X, T)$, and let $\mu = \int_\Omega \mu_\omega \, dm(\omega)$ be its ergodic decomposition.*

(i) *For m-almost every ω, $E_{\mu_\omega}(X, T) \subset E_\mu(X, T)$.*

(ii) *If $(a, b) \in E_\mu(X, T)$, then, for every neighborhood V of (a, b),*

$$m\{\omega;\ V \cap E_{\mu_\omega}(X, T) \neq \emptyset\} > 0.$$

Thus for an appropriate choice of Ω,

$$\mathrm{cls}\left(\bigcup\{E_{\mu_\omega};\ \omega \in \Omega\}\right) = E_\mu.$$

Proof. (i) Let A, B be two open subsets with $\bar{A} \cap \bar{B} = \emptyset$ and $(\bar{A} \times \bar{B}) \cap E_\mu(X, T) = \emptyset$. $\bar{A}$, $\bar{B}$ do not satisfy the property $S(\mu)$ and, therefore, there exists a partition $\mathcal{Q} = (Q, Q^c)$ separating $\bar{A}$ and $\bar{B}$ with $h_\mu(\mathcal{Q}) = 0$. Then ([**DGS**, Theorem 13.3])

$$\int h_\omega(\mathcal{Q}) \, dm(\omega) = h_\mu(\mathcal{Q}) = 0.$$

Hence, for m-almost every ω, $h_\omega(\mathcal{Q}) = 0$. It follows from the definition of entropy pairs that $A \times B$ does not contain a μ_ω-entropy pair and $(A \times B) \cap E_{\mu_\omega}(X, T) = \emptyset$.

Since $E_\mu(X, T) \cup \Delta$ is closed in $X \times X$, its complement can be written as a countable union of sets of the form $A \times B$, where A and B are two open sets of the above type. Also, by definition of entropy pairs, $E_{\mu_\omega}(X, T) \cap \Delta = \emptyset$ for all ω, and we conclude that for almost every ω,

$$E_{\mu_\omega}(X, T) \cap (E_\mu(X, T))^c = \emptyset.$$

(ii) With no loss in generality, we can consider the case where $V = A \times B$, with A (respectively B) a closed neighborhood of a (respectively b), and $A \cap B = \emptyset$. Thus (A, B) satisfy the property $S(\mu)$.

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Put

$$K = \{\omega \in \Omega;\ (A, B) \text{ satisfy } S(\mu_\omega)\}.$$

By Proposition 3,

$$K \subset \{\omega \in \Omega;\ (A \times B) \cap E_{\mu_\omega}(X, T) \neq \emptyset\},$$

and it suffices to show that $m(K) > 0$.

Suppose $m(K) = 0$. As the space $(X, \mathcal{B}_X)$ is a Lebesgue space, there exists a countable family $(\mathcal{P}_i;\ i \in I)$ of partitions separating A and B which, for every probability measure ν on X, is $L^1(\nu)$-dense in the collection of partitions separating A and B. Let (Lemma 4)

$$c = \mathrm{Inf}\{h_\mu(Q, Q^c);\ (Q, Q^c) \text{ separates } A \text{ and } B\} > 0$$

and $0 < \epsilon < c$.

For $\omega \notin K$, there exists a partition $\mathcal{P}$ separating A and B with $h_{\mu_\omega}(\mathcal{P}) = 0$. Hence by density, there exists $i \in I$ with $h_{\mu_\omega}(\mathcal{P}_i) \leq \epsilon$. There exists, therefore, a partition $\{\Omega_n\}$ of $\Omega \setminus K$, with $m(\Omega_n) > 0$ for every n, and a sequence of partitions $\mathcal{P}_n = (Q_n, Q_n^c)$ of X separating A and B, such that for every n

$$h_{\mu_\omega}(\mathcal{P}_n) < \epsilon \quad \text{for every } \omega \in \Omega_n.$$

For every n write

$$t_n = m(\Omega_n), \quad \mu_n = \frac{1}{t_n} \int_{\Omega_n} \mu_\omega \, dm(\omega).$$

We have

$$h_{\mu_n}(\mathcal{P}_n) = \frac{1}{t_n} \int_{\Omega_n} h_{\mu_\omega}(\mathcal{P}_n)\, dm(\omega) \leq \epsilon,$$
$$\sum t_n = 1 \quad \text{and} \quad \mu = \sum t_n \mu_n.$$

The measures μ_n are mutually singular, and there exists a sequence of Borel subsets X_n of X such that, for every n, $\mu_n(X_n) = 1$ and $\mu_n(X_k) = 0$ for every $k \neq n$.

Let $E = \cap_n Q_n$ and put $Q = E \cup \bigcup_n (Q_n \cap X_n)$. Then $A \subset E \subset Q \subset B^c$ and for every k,

$$(Q_k \triangle Q) \subset \left(\bigcup_{n \neq k} X_n \right) \cup X_k^c.$$

Denoting $\mathcal{Q} = (Q, Q^c)$, we see that $\mathcal{Q}$ is a Borel partition separating A and B and for every n, $\mathcal{Q} = \mathcal{P}_n \bmod \mu_n$. In particular, $h_{\mu_n}(\mathcal{Q}) = h_{\mu_n}(\mathcal{P}_n) \leq \epsilon$ and by linearity, $h_\mu(\mathcal{Q}) \leq \epsilon$. This, however, contradicts the way ϵ was chosen and the proof is complete. □

7. *Metric entropy pairs and factors*

THEOREM 5. *Let $\pi : (X, T) \to (Y, S)$ be a homomorphism of topological dynamical systems, let μ be a T-invariant measure on X, and let ν be its image under π. Then, for every ν-entropy pair $(a, b) \in E_\nu(Y, S)$ there exists a μ-entropy pair $(a', b') \in E_\mu(X, T)$ with $\pi(a') = a$ and $\pi(b') = b$.*

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Proof. (i) *Assume first that* (X, T, μ) *is ergodic.* Let $\mathcal{Y} = \pi^{-1}(\mathcal{B}_Y)$ and $\mathcal{Z} = \pi^{-1}(\mathcal{P}_\nu)$. Then $\mathcal{Z} = \mathcal{Y} \cap \mathcal{P}_\mu$ and from Theorem 1.2 in [**GW2**] we know that the factors $\mathcal{Y}$ and $\mathcal{P}_\mu$ are relatively independent over $\mathcal{Z}$.

This means that for every function $f \in L^1(\mu)$ which is $\mathcal{Y}$-measurable, one has $\mathbb{E}(f \mid \mathcal{Z}) = \mathbb{E}(f \mid \mathcal{P}_\mu)$. In particular, for every $g \in L^1(\nu)$, $g \circ \pi$ is $\mathcal{Y}$-measurable and

$$\mathbb{E}(g \circ \pi \mid \mathcal{P}_\mu) = \mathbb{E}(g \circ \pi \mid \mathcal{Z}) = \mathbb{E}(g \circ \pi \mid \pi^{-1}(\mathcal{P}_\nu)) = \mathbb{E}(g \mid \mathcal{P}_\nu) \circ \pi.$$

Now let (a, b) be a ν-entropy pair, A a closed neighborhood of a, B a closed neighborhood of b with $A \cap B = \emptyset$, $A' = \pi^{-1}(A)$ and $B' = \pi^{-1}(B)$. By an application of the preceding remark with $g = \mathbf{1}_A$ and $g = \mathbf{1}_B$, one obtains

$$\begin{aligned}\int \mathbb{E}(\mathbf{1}_{A'} \mid \mathcal{P}_\mu)(x)\mathbb{E}(\mathbf{1}_{B'} \mid \mathcal{P}_\mu)(x)\, d\mu(x) &= \int \mathbb{E}(\mathbf{1}_A \mid \mathcal{P}_\nu)(\pi(x))\mathbb{E}(\mathbf{1}_B \mid \mathcal{P}_\nu)(\pi(x))\, d\mu(x) \\ &= \int \mathbb{E}(\mathbf{1}_A \mid \mathcal{P}_\nu)(y)\mathbb{E}(\mathbf{1}_B \mid \mathcal{P}_\nu)(y)\, d\nu(y).\end{aligned}$$

As (a, b) is a ν-entropy pair, (A, B) satisfy the property $S(\mu)$ and it follows from Lemma 3 that the last integral and, therefore, also the first one, do not vanish. We conclude that $h_\mu(\mathcal{P}) > 0$ for every partition $\mathcal{P} = (D', D'^c)$ of X such that $A' \subset D'$ and $B' \subset D'^c$.

Thus, there exists a μ-entropy pair $(a', b') \in E_\mu(X, T)$ with $a' \in A'$ and $b' \in B'$, i.e. $\pi(a') \in A$ and $\pi(b') \in B$.

As this is true for all closed neighborhoods A (respectively B) of a (respectively b), and as $E_\mu(X, T) \cup \Delta$ is closed, there exists a μ-entropy pair (a', b') with $\pi(a') = a$ and $\pi(b') = b$, and our proof is complete.

(ii) *The general case.* Let (a, b) be a ν-entropy pair, and let A, B be two neighborhoods of a and b respectively.

Let $\int_\Omega \mu_\omega\, dm(\omega)$ be the ergodic decomposition of μ, and let ν_ω be the image of μ_ω under π. For m-almost every ω, ν_ω appears in the ergodic decomposition of ν.

By Theorem 4(ii), for m-almost every ω there exists a ν_ω-entropy pair (c, d) with $c \in A$ and $d \in B$. By the above, there exists a μ_ω-entropy pair (c', d') with $\pi(c') = c$ and $\pi(d') = d$, hence $\pi(c') \in A$ and $\pi(d') \in B$. By Theorem 4(i), for almost every ω, (c', d') is a μ-entropy pair. Since $E_\mu(X, T) \cup \Delta$ is closed, there exists a μ-entropy pair (a', b') with $\pi(a') = a$ and $\pi(d') = b$. □

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Appendix. On the topology of $E(X, T)$

Let (X, T) be a topological dynamical system. The results of §4 enable us to obtain some information about the topology of $E(X, T)$.

LEMMA 5. *Let* A *and* B *be two disjoint closed subsets of* X *with* $h_{\text{top}}(A^c, B^c) > 0$ *and let*

$$A' = \{x \in A; \exists y \in B, (x, y) \in E(X, T)\}$$

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and

$$B' = \{y \in B; \exists x \in A, (x, y) \in E(X, T)\}.$$

Then $h_{\text{top}}(A'^c, B'^c) = h_{\text{top}}(A^c, B^c)$. *In particular,* A' *and* B' *are big sets (see §4.2).*

Proof. Since $E(X, T) \cup \Delta$ is closed, and as $A \cap B = \emptyset$, $(A \times B) \cap E(X, T)$ is closed. Since A' is the first projection of this set, A' is closed. B' is also closed.

Let $\{V_n\}$ be a decreasing sequence of open neighborhoods of A', with $\overline{V_1} \cap B = \emptyset$ and $\cap \overline{V_n} = A'$. For every n, $(A \setminus V_n) \times B$ does not contain entropy pairs, hence $h_{\text{top}}(A^c \cup V_n, B^c) = 0$. As

$$((\overline{V_n})^c, B^c) \vee (A^c \cup V_n, B^c) \succ (A^c, B^c)$$

one has

$$h_{\text{top}}((\overline{V_n})^c, B^c) = h_{\text{top}}((\overline{V_n})^c, B^c) + h_{\text{top}}(A^c \cup V_n, B^c) \geq h_{\text{top}}(A^c, B^c).$$

By semi-continuity, $h_{\text{top}}(A'^c, B^c) \geq h_{\text{top}}(A^c, B^c)$. Replacing A by B and B by A', one similarly gets $h_{\text{top}}(A'^c, B'^c) \geq h_{\text{top}}(A^c, B^c)$, and the opposite inequality is clear. □

THEOREM 6. *Let* W *be a neighborhood of an entropy pair* (x, y). *Then* $W \cap E(X, T)$ *is not of the form* $\{x\} \times K$ *for any subset* K *of* X. *In particular,* $E(X, T)$ *does not have isolated points.*

Proof. With no loss in generality, we can assume that $W = (A \times B)$, where A is a closed neighborhood of x, B a closed neighborhood of y and $A \cap B = \emptyset$. As (x, y) is an entropy pair, $h_{\text{top}}(A^c; B^c) > 0$. A' and B' being defined as in Lemma 5, we have $h_{\text{top}}(A'^c, B'^c) > 0$, hence A' does not reduce to a single point. □

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