Root distribution of complex coefficient polynomials

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The Routh Array is known to be a very convenient way to determine the number of polynomial roots that lie in the right half of the complex plane using only the polynomial coefficients. Extensive use of this is made particularly in control system stability studies [1]. We now show that the same real-coefficient Routh algorithm can be used to obtain this information for polynomials with complex coefficients, which are prominent in areas of engineering such as electrical inductance theory [2]. The method is easy to apply and is seen to be more straightforward than existing methods for this task [e.g. 3,4]. If required, it also enables the use of appropriate root-finding methods for real-coefficient polynomials to obtain the roots of complex-coefficient polynomials.

For the benefit of readers who are unfamiliar with the Routh Array algorithm, which is essentially the Euclid algorithm for long division, we give a brief description for the polynomial $f(z) = \sum_{j=0}^{n} a_j z^j$, where a_j ($j = 0, 1, \ldots, n$) are real. The array is formed as follows:

$$\begin{array}{c|cccccc} r_{1,1} & r_{1,2} & r_{1,3} & \cdot & \cdot & r_{1,\frac{1}{2}n+1} \\ r_{2,1} & r_{2,2} & r_{2,3} & \cdot & r_{2,\frac{1}{2}n} & 0 \\ r_{3,1} & r_{3,2} & r_{3,3} & \cdot & r_{3,\frac{1}{2}n} & \\ \cdot & \cdot & \cdot & & & \\ r_{n-1,1} & r_{n-1,2} & & & & \\ r_{n,1} & 0 & & & & \\ r_{n+1,1} & & & & & \end{array}$$

where n is assumed even (in the subsequent analysis, we will always have polynomials of even degree) and the first two rows contain the alternate coefficients of $f(z)$, i.e.

$$r_{1,j} = a_{n-2,j+2} \text{ and } r_{2,j} = a_{n-2,j+1}, \text{ for } j = 1, 2, \ldots, \left(\tfrac{1}{2}n + 1\right).$$

The rest of the table is computed by the recurrence relation

$$r_{k,j} = r_{k-2,j+1} - \frac{r_{k-2,1}}{r_{k-1,1}} r_{k-1,j+1}, \tag{1}$$

for $k = 3, 4, \ldots, n+1, j = 1, 2, \ldots, \text{int}\left(\tfrac{1}{2}(n-k)+1\right)$, where int ( ) is the integer part of (), with $r_{k,N} = 0$ for $N > \text{int}\left(\tfrac{1}{2}(n-k)+1\right)$. (We note that, if n is odd then the first two rows each contain $\left(\tfrac{1}{2}(n-1)+1\right)$ entries,

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and the algorithm is the same except that j is restricted to this maximum value.) The number of sign changes in the first column of the table, $r_{1,1}, r_{2,1}, r_{3,1}, \ldots, r_{n+1,1}$, then gives the number of right half-plane roots of $f(z)$ [1].

Suppose we have the n-th degree polynomial given by

$$P(z) = (a_0 + ib_0) + (a_1 + ib_1)z + \ \ldots \ + (a_n + ib_n)z^n = \sum_{k=0}^{n} (a_k + ib_k)z^k$$

where a_k and b_k $(k = 0, 1, \ldots, n)$ are real numbers.

Its *coefficient* complex conjugate is then

$$\bar{P}(z) = (a_0 - ib_0) + (a_1 - ib_1)z + \ \ldots \ + (a_n - ib_n)z^n = \sum_{k=0}^{n} (a_k - ib_k)z^k.$$

If $P(z)$ is written in its factored form as

$$P(z) = (a_n + ib_n) \prod_{j=1}^{n} (z - \alpha_j),$$

where α_j $(j = 1, 2, \ldots, n)$ are its roots, then $\bar{P}(z)$ is given by

$$\bar{P}(z) = (a_n - ib_n) \prod_{j=1}^{n} (z - \bar{\alpha}_j),$$

where its roots $\bar{\alpha}_j$ are the complex conjugates of those of $P(z)$.

Multiplying these two polynomials together will then yield a polynomial of degree $2n$ given by

$$F(z) = P(z)\bar{P}(z) = (a_n^2 + b_n^2) \prod_{j=1}^{n} (z - \alpha_j)(z - \bar{\alpha}_j)$$

$$= (a_n^2 + b_n^2) \prod_{j=1}^{n} \left(z^2 - 2 \operatorname{Re}\{\alpha_j\} + |\alpha_j|^2\right)$$

which consists entirely of *real* coefficients and so may be written as

$$F(z) = c_{2n}z^{2n} + c_{2n-1}z^{2n-1} + \ \ldots \ + c_1 z + c_0 \ ^*$$

where

$$c_k = \begin{cases} \displaystyle\sum_{j=0}^{k} (a_j a_{k-j} + b_j b_{k-j}) & k = 0, 1, 2, \ldots, n \\ \displaystyle\sum_{j=k-n}^{n} (a_j a_{k-j} + b_j b_{k-j}) & k = n+1, n+2, \ldots, 2n. \end{cases}$$

A key point to notice here is that the $2n$ roots of $F(z)$ will be the n roots

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* $F(z)$ can also be formed by $P(z)\bar{P}(z) = |P(z)|^2 = \left(\sum_{k=0}^{n} a_k z^k\right)^2 + \left(\sum_{k=0}^{n} b_k z^k\right)^2$.

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of $P(z)$ and the n roots of $\bar{P}(z)$, which are the conjugates of those of $P(z)$. We also note that any *real* root of $P(z)$ will be a *repeated* root at the same value in $F(z)$. Hence, as all of the real parts of the roots of $P(z)$ will be preserved in $F(z)$, the proportion of right to left half-plane roots will be preserved in both polynomials, i.e. if $P(z)$ has l left half-plane and r right half-plane roots ($l + r = n$) then $F(z)$ will have $2l$ and $2r$ left and right half-plane roots respectively.

As a consequence of this property, we see that the number of sign changes in the first column of the Routh Array for $F(z)$ will be *double* that for $P(z)$. So halving this number gives the number of right half-plane roots of $P(z)$.

Example

Consider the third degree polynomial

$$P(z) = z^3 + (-3 + 5i)z^2 + (4 - 44i)z + (-4 + 68i)$$

giving

$$\bar{P}(z) = z^3 + (-3 - 5i)z^2 + (4 + 44i)z + (-4 - 68i)$$

and

$$F(z) = P(z)\bar{P}(z) = z^6 - 6z^5 + 42z^4 - 472z^3 + 2656z^2 - 6016z + 4640.$$

Forming the Routh Array for $F(z)$ using the recurrence relation (1) gives

1	42	2656	4640
−6	−472	−6016	0
−36.667	1653.333	4640	
−742.545	−6775.273	0	
1987.894	4640		
−5042.076	0		
4640			

There are *four* sign changes in the first column of the array giving four right half-plane roots of $F(z)$, hence two right half-plane roots of $P(z)$. In fact the roots of $F(z)$ are at $(4 \pm 2i)$, $(-3 \pm 7i)$, 2 and 2, those of $P(z)$ are at $(4 + 2i)$, $(-3 - 7i)$ and 2.

It should be mentioned that sometimes the Routh-test breaks down because of zeros appearing in denominator elements in the algorithm (e.g. in the rare case of an even distribution of roots about the imaginary axis, the first element of the second row would be $r_{2,1} = 0$). However, such problems are well-documented [1, 5] and are usually overcome in standard ways such as *reversing* the coefficients in $F(z)$, effectively giving the root distribution of the reciprocal roots, or by using shifted imaginary axis values [5] or even by replacing the offending row alternant polynomial with its derivative [1].

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If the actual roots of $P(z)$ are required, we have the useful bonus of being able to use an appropriate root-finding method for real coefficient polynomials on $F(z)$ [e.g. 5, 6]. Although the number of roots to be found is double those actually required, they will always occur as complex conjugate pairs and/or multiple (usually double) real roots. We see that usually only one of a complex pair need be tested to satisfy $P(z) = 0$, and any repeated real root of $F(z)$ is automatically retained in $P(z)$ with half its multiplicity and so no further test is needed for these. In the rare cases of repeated complex root pairs in $F(z)$*, only one test is required for the pair if we find that $P(z) \neq 0$ for our choice, then the conjugate is accepted with half its multiplicity. However, if the first chosen root satisfies $P(z) = 0$ we would have to test its conjugate also because it is possible that the conjugate pair are both roots of $P(z)$.

To test which of a complex pair is a root of $P(z)$, we can use the simplicity of the Routh Array once again rather than direct substitution of complex values. This is done by splitting $P(z)$ into its real and imaginary components, i.e.

$$P(z) = P_1(z) + iP_2(z),$$

where $P_1(z) = \sum_{k=0}^{n} a_k z^k$ and $P_2(z) = \sum_{k=0}^{n} b_k z^k$.

Now to test if $z = \alpha + i\beta$ satisfies $P(z) = 0$, we require the remainder term of $\dfrac{P(z)}{z - (\alpha + i\beta)}$ to be zero. To avoid complex number division we can express this quotient as

$$\frac{P(z)}{z - (\alpha + i\beta)} = \left(\frac{P_1(z) + iP_2(z)}{z^2 - 2\alpha z + \alpha^2 + \beta^2}\right)(z - \alpha + i\beta)$$

and use the Routh recurrence relation (1), which is essentially the Euclid algorithm for long division of polynomials where each row $j (\geqslant 3)$ gives the coefficients of the remainder polynomials when that of row $(j - 2)$ is divided by that of row $(j - 1)$ from highest powers. Hence, if $P_1(z)$ and $P_2(z)$ are divided separately by $z^2 - 2\alpha z + \alpha^2 + \beta^2$ (where these coefficients are put into every *even* numbered row of the array for successive division) to give the *linear* remainder terms $c_1 z + c_0$ and $d_1 z + d_0$ respectively, we then have a total remainder given by

$$\begin{aligned} R(z) &= \left(\frac{c_1 z + c_0 + i(d_1 z + d_0)}{z^2 - 2\alpha z + \alpha^2 + \beta^2}\right)(z - \alpha + i\beta) \\ &= \frac{(c_1 + id_1)z + c_0 + id_0}{z - (\alpha + i\beta)} \\ &= \frac{(c_1 + id_1)[z - (\alpha + i\beta) + (\alpha + i\beta)] + c_0 + id_0}{z - (\alpha + i\beta)} \end{aligned}$$

* We note that the roots $(\alpha + i\beta)^2$ or $(\alpha \pm i\beta)$ in $P(z)$ will give roots $(\alpha \pm i\beta)^2$ in $F(z)$,

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$$= c_1 + id_1 + \frac{(c_1\alpha + c_0 - d_1\beta) + i(d_1\alpha + d_0 + c_1\beta)}{z - (\alpha + i\beta)}.$$

From this we see that if $\alpha + i\beta$ is a root then the *constant* remainder term must be zero, i.e.

$$c_1\alpha + c_0 - d_1\beta = 0 \qquad \text{and} \qquad d_1\alpha + d_0 + c_1\beta = 0. \tag{2}$$

To illustrate this test consider the example given previously for the polynomial

$$P(z) = z^3 + (-3 + 5i)z^2 + (4 - 44i)z + (-4 + 68i).$$

To test the root, say, at $\alpha + i\beta = 4 + 2i$ we have $-2\alpha = -8$, $\alpha^2 + \beta^2 = 20$ and using (1) for the real and imaginary components $P_1(z)$ and $P_2(z)$, with all even-numbered rows being the dividing quadratic's coefficients, gives

$P_1(z)$				$P_2(z)$			
1	−3	4	−4	0	5	−44	68
1	−8	20	0	1	−8	20	0
5	−16	−4		5	−44	68	
1	−8	20		1	−8	20	
24	−104			−4	−32		
(c_1)	(c_0)			(d_1)	(d_0)		

Substituting into (2), these values give:

$$c_1\alpha + c_0 - d_1\beta = 24 \times 4 - 104 - (-4) \times 2 = 0$$

and

$$d_1\alpha + d_0 + c_1\beta = -4 \times 4 - 32 + 24 \times 2 = 0,$$

verifying the correct root of $P(z)$. We note that, if instead, the other half of this complex pair is tested ($\beta = -2$) we get the values, $c_1\alpha + c_0 - d_1\beta = -16$ and $d_1\alpha + d_0 + c_1\beta = -96$, which we would obviously reject and accept its conjugate because we know that one of the pair is a root of $P(z)$.

Conclusions

A relatively easy method has been presented that uses the well-known Routh Array to determine the number of right half-plane roots of complex coefficient polynomials, using real number calculations. If required, it also enables the use of readily available root-finding methods for real-coefficient polynomials to obtain the pertinent roots (along with their conjugate values), as opposed to more specialised methods [e.g. 3, 7]. To test which complex roots are correct, a simple application of the Routh division procedure is outlined to achieve this task.

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Nemo (continued from page 58)

3. The Errols are formidably Brittanic. They are both, for example, *both* economists. Why *both*, I ask myself? One of them must feel perfectly redundant. They make love to two places of decimals only. Their children have all the air of vulgar fractions.

4. The first little mathematical gentleman
   Stepping, wee mite, in his loose trousers
   Under all the eternal dome of mathematical law.
   Fives, and tens,
   Threes and fours and twelves,
   All the *volte face* of decimals.
   The whirligig of dozens and the pinnacle of seven.

5. He had taught himself a language down here – if only to know it by sight, and to have formed his own crude ideas of its pronunciation, could be called learning it. He had worked at fractions and decimals, and tried a little algebra, but he was, and had been as a boy, a poor hand at figures.

6. He sighed as he thought of those happy days in which he used to fear that his mind and body would both give way under the pressure of decimal coinage.

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