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# Linear rigidity of stationary stochastic processes

ALEXANDER I. BUFETOV†‡§¶, YOANN DABROWSKI‖ and YANQI QIU†

† *Aix-Marseille Université, Centrale Marseille, CNRS, I2M, UMR7373, 39 rue F. Joliot Curie, 13453 Marseille Cedex 13, France (e-mail: yqi.qiu@gmail.com)*

‡ *Steklov Institute of Mathematics, 8 Gubkina Street, Moscow, 119991, Russia*

§ *Institute for Information Transmission Problems, Bolshoy Karetny per. 19, 127994, Moscow, Russia*

¶ *National Research University Higher School of Economics, Myasnitskaya ul., 20, Moscow, 101000, Russia (e-mail: bufetov@mi.ras.ru)*

‖ *Université de Lyon, Université Lyon 1, Institut Camille Jordan, 43 bd. du 11 novembre 1918, 69622 Villeurbanne Cedex, France (e-mail: dabrowski@math.univ-lyon1.fr)*

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*Abstract*. We consider stationary stochastic processes $\{X_n : n \in \mathbb{Z}\}$ such that $X_0$ lies in the closed linear span of $\{X_n : n \neq 0\}$; following Ghosh and Peres, we call such processes linearly rigid. Using a criterion of Kolmogorov, we show that it suffices, for a stationary stochastic process to be linearly rigid, that the spectral density vanishes at zero and belongs to the Zygmund class $\Lambda_*(1)$. We next give a sufficient condition for stationary determinantal point processes on $\mathbb{Z}$ and on $\mathbb{R}$ to be linearly rigid. Finally, we show that the determinantal point process on $\mathbb{R}^2$ induced by a tensor square of Dyson sine kernels is *not* linearly rigid.

## 1. *Introduction*

This paper is devoted to rigidity of stationary determinantal point processes.

Recall that stationary determinantal point processes are strongly chaotic: they have the Kolmogorov property (Lyons [**11**]) and the Bernoulli property (Lyons and Steif [**12**]); and they satisfy the central limit theorem (Costin and Lebowitz [**2**], Soshnikov [**16**]). On the other hand, Ghosh [**5**] and Ghosh and Peres [**6**] proved, for the determinantal point processes such as the Dyson sine process and the Ginibre point process, that the number of particles in a finite window is measurable with respect to the completion of the sigma-algebra describing the configurations outside that finite window. Their argument is spectral: they construct, for any small $\varepsilon$, a compactly supported smooth function $\varphi_\varepsilon$ such

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that $\varphi_\varepsilon$ equals 1 in a fixed finite window and the linear statistic corresponding to $\varphi_\varepsilon$ has variance smaller than $\varepsilon$.

In the same spirit, we consider general stationary stochastic processes (in broad sense) $\{X_n : n \in \mathbb{Z}\}$ such that $X_0$ lies in the closed linear span of $X_n$, $n \neq 0$; following Ghosh and Peres, we call such processes linearly rigid. In 1941, Kolmogorov [**9**, **10**] gave a sufficient condition for linear rigidity: namely, that the spectral density of our process vanishes at zero and the integral of the inverse of the spectral density diverges. Such a condition is easy to verify for example for the sine process, since the spectral density $\omega$ in the neighbourhood of zero has the form $\omega(\theta) = |\theta|$. More generally, in order that a stationary stochastic process be rigid, we check that it suffices that the spectral density vanishes at zero and belongs to the Zygmund class $\Lambda_*(1)$. We next give a sufficient condition for stationary determinantal point processes on $\mathbb{Z}$ and on $\mathbb{R}$ to be rigid. Finally, we show that the determinantal point process on $\mathbb{R}^2$ induced by a tensor square of the Dyson sine kernel is *not* linearly rigid.

We now turn to more precise statements. Let $X = \{X_n : n \in \mathbb{Z}^d\}$ be a multi-dimensional time-stationary stochastic process of real-valued random variables defined on a probability space $(\Omega, \mathbb{P})$. Let $H(X) \subset L^2(\Omega, \mathbb{P})$ denote the closed subspace linearly spanned by $\{X_n : n \in \mathbb{Z}^d\}$ and let $\check{H}_0(X)$ denote the one linearly spanned by $\{X_n : n \in \mathbb{Z}^d \setminus \{0\}\}$.

*Definition 1.1.* The stochastic process $X$ is said to be linearly rigid if

$$X_0 \in \check{H}_0(X). \tag{1}$$

Let $\mathrm{Conf}(\mathbb{R}^d)$ be the set of locally finite configurations on $\mathbb{R}^d$. For a bounded Borel subset $B \subset \mathbb{R}^d$, we denote by $N_B : \mathrm{Conf}(\mathbb{R}^d) \to \mathbb{N} \cup \{0\}$ the function defined by

$$N_B(\mathcal{X}) := \text{the cardinality of } B \cap \mathcal{X}.$$

The space $\mathrm{Conf}(\mathbb{R}^d)$ is equipped with the Borel $\sigma$-algebra which is the smallest $\sigma$-algebra making all the $N_B$ measurable. Recall that a point process with phase space $\mathbb{R}^d$ is, by definition, a Borel probability measure on the space $\mathrm{Conf}(\mathbb{R}^d)$. For background on point processes, the reader is referred to Daley and Vere-Jones' book [**3**].

Given a stationary point process on $\mathbb{R}^d$ and $\lambda > 0$, we introduce the stationary stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}^d}$ by the formula

$$N_n^{(\lambda)}(\mathcal{X}) := \text{the cardinality of } \mathcal{X} \cap (n\lambda + [-\lambda/2, \lambda/2)^d). \tag{2}$$

*Definition 1.2.* A stationary point process $\mathbb{P}$ on $\mathbb{R}^d$ is called *linearly rigid* if, for any $\lambda > 0$, the stationary stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}^d}$ is linearly rigid, i.e.

$$N_0^{(\lambda)} \in \check{H}_0(N^{(\lambda)}).$$

The above definition is motivated by the definition due to Ghosh and Peres of rigidity of point processes on $\mathbb{R}^d$; see [**5**] and [**6**]. Given a Borel subset $C \subset \mathbb{R}^d$, we will denote by

$$\mathcal{F}_C = \sigma(\{N_B : B \subset C,\ B \text{ bounded Borel}\})$$

the $\sigma$-algebra generated by all random variables of the form $N_B$, where $B \subset C$ ranges over all bounded Borel subsets of $C$. Let $\mathbb{P}$ be a point process on $\mathbb{R}$, i.e. $\mathbb{P}$ is a Borel probability on $\mathrm{Conf}(\mathbb{R}^d)$, and denote the $\mathbb{P}$-completion of $\mathcal{F}_C$ by $\mathcal{F}_C^{\mathbb{P}}$.

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*Definition 1.3.* (Ghosh [**5**], Ghosh and Peres [**6**]) A point process $\mathbb{P}$ on $\mathbb{R}^d$ is called *number rigid* if, for any bounded Borel set $B \subset \mathbb{R}^d$ with Lebesgue-negligible boundary $\partial B$, the random variable $N_B$ is $\mathcal{F}^{\mathbb{P}}_{\mathbb{R}^d \setminus B}$-measurable.

*Remark 1.1.* Of course, in the above definition, it suffices to take Borel sets $B$ of the form $[-\gamma, \gamma)^d$ for $\gamma > 0$; cf. [**6**].

A linear rigid stationary point process on $\mathbb{R}^d$ is of course rigid in the sense of Ghosh and Peres. Observe that proofs for rigidity in [**5**, **6**] and [**1**] in fact establish linear rigidity. We would like also to mention a notion of insertion–deletion tolerance studied by Holroyd and Soo in [**7**], which is in contrast to the notion of the rigidity property.

## 2. *The Kolmogorov criterion for linear rigidity*

In this note, the Fourier transform of a function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i2\pi x \cdot \xi}\, dx.$$

Denote by $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the $d$-dimensional torus. In what follows, we identify $\mathbb{T}^d$ with $[-1/2, 1/2)^d$. The Fourier coefficients of a measure $\mu$ on $\mathbb{T}^d$ are given, for any $k \in \mathbb{Z}^d$, by the formula

$$\hat{\mu}(k) = \int_{\mathbb{T}^d} e^{-i2\pi k \cdot \theta}\, d\mu_X(\theta) \quad \text{where } k \cdot \theta := k_1\theta_1 + \cdots + k_d\theta_d.$$

Denote by $\mu_X$ the spectral measure of $X$, i.e.

$$\forall k \in \mathbb{Z}^d, \quad \mathbb{E}(X_0 X_k) = \mathbb{E}(X_n X_{n+k}) = \int_{\mathbb{T}^d} e^{-i2\pi k \cdot \theta}\, d\mu_X(\theta) = \hat{\mu}_X(k). \tag{3}$$

Recall that we have the following natural isometric isomorphism:

$$H(X) \simeq L^2(\mathbb{T}^d, \mu_X), \tag{4}$$

by assigning to $X_n \in H(X)$ the function $\theta \mapsto e^{i2\pi n \cdot \theta} \in L^2(\mathbb{T}^d, \mu_X)$.

Let $\mu_X = \mu_a + \mu_s$ be the Lebesgue decomposition of $\mu_X$ with respect to the normalized Lebesgue measure $m(d\theta) = d\theta_1 \cdots d\theta_d$ on $\mathbb{T}^d$, i.e. $\mu_a$ is absolutely continuous with respect to $m$ and $\mu_s$ is singular to $m$. Set

$$\omega_X(\theta) := \frac{d\mu_a}{dm}(\theta).$$

LEMMA 2.1. (The Kolmogorov criterion) *We have*

$$\operatorname{dist}(X_0, \check{H}_0(X)) = \left( \int_{\mathbb{T}^d} \omega_X^{-1}\, dm \right)^{-1/2},$$

*where by* $\operatorname{dist}(X_0, \check{H}_0(X))$ *we mean the least* $L^2$*-distance between the random variable* $X_0$ *and the linear space* $\check{H}_0(X)$ *and the right-hand side is to be interpreted as zero if* $\int_{\mathbb{T}^d} \omega_X^{-1}\, dm = \infty$.

COROLLARY 2.2. *The stationary stochastic process* $X = (X_n)_{n \in \mathbb{Z}^d}$ *is linearly rigid if and only if*

$$\int_{\mathbb{T}^d} \omega_X^{-1}\, dm = \infty.$$

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Lemma 2.1 is due to Kolmogorov [**9**, **10**]. For the reader's convenience, we include its proof.

*Proof of Lemma 2.1.* We follow the argument of Lyons and Steif [**12**], cf. also [**15**]. By the Lebesgue decomposition of $\mu$, we may take a subset $A \subset \mathbb{T}^d$ of full Lebesgue measure $m(A) = 1$ such that $\mu_a(A) = 1$ and $\mu_s(A) = 0$.

Denote

$$L_0 = \overline{\text{span}}^{L^2(\mathbb{T}^d, \mu_X)}[e^{i2\pi n\cdot\theta} : n \neq 0].$$

By the isometric isomorphism (4), it suffices to show that

$$\text{dist}(1, L_0) = \left(\int_{\mathbb{T}^d} \omega_X^{-1}\, dm\right)^{-1/2}, \tag{5}$$

where 1 is the constant function taking value 1. Write

$$1 = p + h \quad \text{such that } p \perp L_0,\ h \in L_0.$$

Modifying, if necessary, the values of $p$ and $h$ on a $\mu$-negligible subset, we may assume that

$$1 = p(\theta) + h(\theta) \quad \text{for all } \theta \in \mathbb{T}^d.$$

Since $p \perp L_0$, we have

$$0 = \langle p, e^{i2\pi n\cdot\theta}\rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} p(\theta) e^{-i2\pi n\cdot\theta}\, d\mu(\theta) \quad \text{for any } n \in \mathbb{Z}^d\backslash\{0\}. \tag{6}$$

Let $\xi \in \mathbb{C}$ denote

$$\xi = \int_{\mathbb{T}^d} p(\theta)\, d\mu(\theta).$$

Then, by (6), all the Fourier coefficients of the complex measure $p \cdot d\mu$ coincide with the corresponding Fourier coefficients of $\xi dm$ (the multiple of Lebesgue measure $dm$ by $\xi$); consequently, we have

$$p \cdot d\mu = \xi\, dm.$$

It follows that $p$ must vanish almost everywhere with respect to the singular component $\mu_s$ of $\mu$, and $p(\theta)\omega_X(\theta) = \xi$ for $m$-almost every $\theta \in \mathbb{T}^d$. Thus, we have

$$\|p\|_{L^2(d\mu)} = \|p\|_{L^2(d\mu_a)} \tag{7}$$

and

$$h(\theta) = 1 - \xi\omega_X(\theta)^{-1} \quad \text{for } m\text{-almost every } \theta \in \mathbb{T}^d. \tag{8}$$

*Case 1.* $\int_{\mathbb{T}^d} \omega_X^{-1}\, dm < \infty$. Define a function $f : \mathbb{T}^d \to \mathbb{C}$ by $f = \omega_X^{-1}\chi_A$. Then $f \in L^2(d\mu) \ominus L_0$. Indeed,

$$\|f\|^2_{L^2(d\mu)} = \int_{\mathbb{T}^d} \omega_X^{-2}\chi_A\, d\mu = \int_{\mathbb{T}^d} \omega_X^{-2}\, d\mu_a = \int_{\mathbb{T}^d} \omega_X^{-1}\, dm < \infty.$$

And, for all $n \in \mathbb{Z}^d\backslash 0$,

$$\langle f, e^{i2\pi n\cdot\theta}\rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} \omega_X(\theta)^{-1}\chi_A(\theta) e^{-i2\pi n\cdot\theta}\, d\mu(\theta) = \int_{\mathbb{T}^d} e^{-i2\pi n\cdot\theta}\, dm(\theta) = 0.$$

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It follows that $f \perp h$, i.e.

$$0 = \langle h,\, f\rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} h\omega_X^{-1}\chi_A\, d\mu = \int_{\mathbb{T}^d} h\, dm.$$

By (8), we get

$$\int_{\mathbb{T}^d} (1 - \xi\omega_X^{-1})\, dm = 0$$

and hence

$$\xi = \left(\int_{\mathbb{T}^d} \omega_X^{-1}\, dm\right)^{-1}.$$

It follows that

$$\operatorname{dist}(1,\, L_0)^2 = \|p\|^2_{L^2(d\mu)} = \|p\|^2_{L^2(d\mu_a)} = \xi^2 \int_{\mathbb{T}^d} \omega_X^{-2}\omega_X\, dm = \xi.$$

This shows the desired equality (5).

*Case 2.* $\int_{\mathbb{T}^d} \omega_X^{-1}\, dm = \infty$. We claim that $\xi = 0$. If the claim were verified, then we would get the desired identity in this case:

$$\operatorname{dist}(1,\, L_0) = 0.$$

So, let us turn to the proof of the claim. We argue by contradiction. If $\xi \neq 0$, then $p \neq 0$ and

$$\|p\|^2_{L^2(d\mu)} = \|p\|^2_{L^2(d\mu_a)} = \xi^2\|\omega_X^{-1}\|^2_{L^2(d\mu_a)} = \xi^2 \int_{\mathbb{T}^d} \omega_X^{-1}\, dm = \infty.$$

This contradicts the fact that $p \in L^2(d\mu)$. □

*Remark 2.1.* If the spectral measure $\mu_X$ is absolutely continuous and given by $\mu_X(dz) = \omega(z)\, dm(z)$, then, for any $n \in \mathbb{N}$, the following are equivalent.

(i) $\sum_{l=-n}^{n} X_l \in \overline{\operatorname{span}}^{H(X)}\{X_k : k \in \mathbb{Z},\, |k| \geq n+1\}$.

(i)′ $\sum_{l=-n}^{n} z^l \in \overline{\operatorname{span}}^{L^2_\omega}\{z^l : k \in \mathbb{Z},\, |k| \geq n+1\}$.

(ii) For any $w_1, w_2, \ldots, w_n \in \mathbb{C}\backslash\{1\}$,

$$\int_{\mathbb{T}} \frac{\prod_{l=1}^{n} |(z - w_l)(z - \bar{w}_l)|^2}{\omega(z)}\, dm(z) = \infty.$$

(ii)′ For any $w_1, w_2, \ldots, w_n \in \mathbb{T}\backslash\{1\}$,

$$\int_{\mathbb{T}} \frac{\prod_{l=1}^{n} |(z - w_l)(z - \bar{w}_l)|^2}{\omega(z)}\, dm(z) = \infty.$$

Indeed, (i) and (i)′ are equivalent. Assume that (i)′ is satisfied; let us show (ii). If (ii) is violated, then there exist $w_1, w_2, \ldots, w_n \in \mathbb{C}\backslash\{1\}$ such that

$$\int_{\mathbb{T}} \frac{\prod_{l=1}^{n} |(z - w_l)(z - \bar{w}_l)|^2}{\omega_X(z)}\, dm(z) < \infty.$$

Define

$$h(z) := \frac{\prod_{l=1}^{n}(z - w_l)(z - \bar{w}_l)}{z^n\omega_X(z)} = \frac{\sum_{l=-n}^{n} a_l z^l}{\omega_X(z)}.$$

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Then $h \in L^2_\omega(\mathbb{T}) \ominus \overline{\operatorname{span}}^{L^2_\omega}\{z^k : k \in \mathbb{Z},\ |k| \geq n+1\}$. We have

$$\left(\sum_{l=-n}^{n} z^l,\, h(z)\right)_{L^2_\omega} = \sum_{l=-n}^{n} a_l = \prod_{l=1}^{n} |1 - w_l|^2 \neq 0.$$

This contradicts (i)′; hence, (i)′ implies (ii).

Conversely, let us assume (ii) and show (i)′. If (i)′ is not satisfied, then there exists a function $g \in L^2_\omega \ominus \overline{\operatorname{span}}^{L^2_\omega}\{z^l : k \in \mathbb{Z},\ |k| \geq n+1\}$ such that $g \neq 0$ and the scalar product $(\sum_{l=-n}^{n} z^l,\, g)_{L^2_\omega} \neq 0$. We have

$$0 = \int_{\mathbb{T}} g(z) z^k \omega(z)\, dm(z) \quad \text{for any } k \in \mathbb{Z},\ |k| \geq n+1.$$

This implies that there exist $(c_{-n}, \ldots, c_n)$ such that

$$g(z)\omega(z) = \sum_{-n}^{n} c_l z^l.$$

Hence, $g(z) = \sum_{-n}^{n} c_l z^l / \omega(z)$ and

$$\sum_{-n}^{n} c_l = \left(g, \sum_{-n}^{n} z^l\right)_{L^2_\omega} \neq 0.$$

Since $\omega(z) = \omega(z^{-1})$, if we denote $\check{g}(z) := g(z^{-1})$, then $\check{g} \in L^2_\omega(\mathbb{T})$. Thus, $\Re(g + \check{g})$ and $\Im(g + \check{g})$ are functions in $L^2_\omega(\mathbb{T})$. We have

$$\Re(g + \check{g})(z) = \frac{\sum_{-n}^{n} \Re(c_l)(z^l + z^{-l})}{\omega(z)} \quad \text{and} \quad \Im(g + \check{g})(z) = \frac{\sum_{-n}^{n} \Im(c_l)(z^l + z^{-l})}{\omega(z)}.$$

Since $\sum_{-n}^{n} c_l \neq 0$, we may assume without loss of generality that $\sum_{-n}^{n} \Re(c_l) \neq 0$. Define $P(z)$ as the polynomial given by $P(z) = z^n \sum_{-n}^{n} \Re(c_l)(z^l + z^{-l})$ and let $m = \deg P \leq n$; then there exist $w_1, \ldots, w_m$ such that

$$P(z) = \Re(c_m) \prod_{l=1}^{m} (z - w_l)(z - \bar{w}_l).$$

Since $P(1) = \sum_{-n}^{n} \Re(c_l) \neq 0$, we know that $w_1, \ldots, w_m$ are all different from 1. Now, using the fact that $\Re(g + \check{g}) \in L^2_\omega$, we deduce that

$$\int_{\mathbb{T}} \frac{\prod_{l=1}^{m} |(z - w_l)(z - \bar{w}_l)|^2}{\omega(z)} < \infty,$$

which of course violates (ii). This contradiction shows that (ii) implies (i)′.

The equivalence between (ii) and (ii)′ is obvious.

Denote by $\operatorname{Cov}(U, V)$ the covariance between two random variables $U$ and $V$: $\operatorname{Cov}(U, V) = \mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)$.

If $X = (X_n)_{n \in \mathbb{Z}^d}$ is a stochastic process such that

$$\sum_{n \in \mathbb{Z}^d} |\operatorname{Cov}(X_0, X_n)| < \infty, \tag{9}$$

then we may define a continuous function on $\mathbb{T}^d$ by the formula

$$\omega_X(\theta) := \sum_{n \in \mathbb{Z}^d} \operatorname{Cov}(X_0, X_n) e^{i 2\pi n \cdot \theta}. \tag{10}$$

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LEMMA 2.3. *Let $X = (X_n)_{n\in\mathbb{Z}^d}$ be a stationary stochastic process satisfying condition (9). Then we have the following explicit Lebesgue decomposition of $\mu_X$:*

$$\mu_X = (\mathbb{E}X_0)^2 \cdot \delta_0 + \omega_X \cdot m, \tag{11}$$

*where $\delta_0$ is the Dirac measure on the point $0 \in \mathbb{T}^d$ and $\omega_X$ is the function on $\mathbb{T}^d$ defined by (10).*

*Proof.* Note that, under the assumption (9), the function $\omega_X(\theta)$ is well defined and continuous on $\mathbb{T}^d$. For proving the decomposition (11), it suffices to show that the Fourier coefficients of $\mu_X$ coincide with those of $\nu_X := (\mathbb{E}X_0)^2 \cdot \delta_0 + \omega_X \cdot m$. But, if $n \in \mathbb{Z}^d$, then

$$\hat{\nu}_X(n) = (\mathbb{E}X_0)^2 + \mathrm{Cov}(X_0,\, X_n) = \mathbb{E}(X_0 X_n) = \hat{\mu}_X(n).$$

The lemma is completely proved. □

## 3. *A sufficient condition for linear rigidity*

THEOREM 3.1. *Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary stochastic process. If*

$$\sup_{N\geq 1}\left(N \sum_{|n|\geq N} |\mathrm{Cov}(X_0,\, X_n)|\right) < \infty \tag{12}$$

*and*

$$\sum_{n\in\mathbb{Z}} \mathrm{Cov}(X_0,\, X_n) = 0, \tag{13}$$

*then $X$ is linearly rigid.*

*Remark 3.1.* The condition (12) is a sufficient condition such that the spectral density $\omega_X$ is a function in the Zygmund class $\Lambda_*(1)$; see below for the definition. The condition (13) implies in particular that $\omega_X$ vanishes at the point $0 \in \mathbb{T}$.

We shall apply a result of Móricz [**14**, Theorem 3] on absolutely convergent Fourier series and Zygmund class functions. Recall that a continuous 1-periodic function $\varphi$ defined on $\mathbb{R}$ is said to be in the *Zygmund class* $\Lambda_*(1)$ if there exists a constant $C$ such that

$$|\varphi(x+h) - 2\varphi(x) + \varphi(x-h)| \leq Ch \tag{14}$$

for all $x \in \mathbb{R}$ and for all $h > 0$.

THEOREM 3.2. (Móricz [**14**]) *If $\{c_n\}_{n\in\mathbb{Z}} \in \mathbb{C}$ is such that*

$$\sup_{N\geq 1}\left(N \sum_{|n|\geq N} |c_n|\right) < \infty, \tag{15}$$

*then the function $\varphi(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{i2\pi n\theta}$ is in the Zygmund class $\Lambda_*(1)$.*

*Proof of Theorem 3.1.* First, in view of (10), our assumption (13) implies that

$$\omega_X(0) = 0.$$

Next, by Theorem 3.2, under the assumption (12), we have

$$\omega_X \in \Lambda_*(1).$$

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Since all Fourier coefficients of $\omega_X$ are real, we have

$$\omega_X(\theta) = \omega_X(-\theta).$$

Consequently, there exists $C > 0$ such that

$$\omega_X(\theta) = \frac{\omega_X(\theta) + \omega_X(-\theta)}{2} = \frac{\omega_X(\theta) + \omega_X(-\theta) - 2\omega_X(0)}{2} \leq C|\theta|,$$

whence

$$\int_{\mathbb{T}} \omega_X^{-1}\, dm = \infty,$$

and the stochastic process $X = (X_n)_{n\in\mathbb{Z}}$ is linearly rigid by the Kolmogorov criterion. □

4. *Applications to stationary determinantal point processes*

In this section, we first give a sufficient condition for linear rigidity of stationary determinantal point processes on $\mathbb{R}$ and then give an example of a very simple stationary, but not linearly rigid, determinantal point process on $\mathbb{R}^2$. We briefly recall the main definitions. Let $B \subset \mathbb{R}^d$ be a bounded Borel subset. Let $K_B : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the operator of convolution with the Fourier transform $\widehat{\chi_B}$ of the indicator function $\chi_B$. In other words, the kernel of $K_B$ is

$$K_B(x, y) = \widehat{\chi_B}(x - y). \tag{16}$$

In particular, if $d = 1$ and $B = (-1/2, 1/2)$, then we find the well-known Dyson sine kernel

$$K_{\text{sine}}(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}.$$

Note that we always have $K_B(x, x) = K_B(0, 0)$.

Denote by $\mathbb{P}_{K_B}$ the determinantal point process induced by $K_B$. For background on the determinantal point processes, the reader is referred to [**8**, **11**, **13**, **16**].

PROPOSITION 4.1. *Let $\mathbb{P}_{K_B}$ be the stationary determinantal point process on $\mathbb{R}^d$ induced by the kernel $K_B$ in (16). For any $\lambda > 0$, denote by $N^{(\lambda)} = (N_n^{(\lambda)})_{n\in\mathbb{Z}^d}$ the stationary stochastic process associated to $\mathbb{P}_{K_B}$ as in (2). Then*

$$\sum_{n\in\mathbb{Z}^d} |\mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})| < \infty \tag{17}$$

*and*

$$\sum_{n\in\mathbb{Z}^d} \mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) = 0. \tag{18}$$

*Proof.* Fix a number $\lambda > 0$; for simplifying the notation, let us denote $N_n^{(\lambda)}$ by $N_n$. Denote, for any $n \in \mathbb{Z}^d$,

$$Q_n = n\lambda + [-\lambda/2, \lambda/2)^d.$$

By definition of a determinantal point process, we have

$$\mathbb{E}(N_n) = \mathbb{E}(N_0) = \int_{Q_0} K_B(x, x)\, dx = \lambda^d K_B(0, 0).$$

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If $n \neq 0$, we have

$$\begin{aligned}\mathbb{E}(N_0 N_n) &= \iint \chi_{Q_0}(x)\chi_{Q_n}(y) \begin{vmatrix} K_B(x,x) & K_B(x,y) \\ K_B(y,x) & K_B(y,y) \end{vmatrix} dx\, dy \\ &= \lambda^{2d} K_B(0,0)^2 - \iint_{Q_0 \times Q_n} |K_B(x,y)|^2\, dx\, dy,\end{aligned}$$

whence

$$\operatorname{Cov}(N_0, N_n) = -\iint_{Q_0 \times Q_n} |K_B(x,y)|^2\, dx\, dy. \tag{19}$$

We also have

$$\begin{aligned}\mathbb{E}(N_0^2) &= \mathbb{E}\Big[\sum_{x,y \in \mathcal{X}} \chi_{Q_0}(x)\chi_{Q_0}(y)\Big] \\ &= \mathbb{E}\Big[\sum_{x \in \mathcal{X}} \chi_{Q_0}(x)\Big] + \mathbb{E}\Big[\sum_{x,y \in \mathcal{X}, x \neq y} \chi_{Q_0}(x)\chi_{Q_0}(y)\Big] \\ &= \int_{Q_0} K_B(x,x)\, dx + \iint \chi_{Q_0}(x)\chi_{Q_0}(y) \begin{vmatrix} K_B(x,x) & K_B(x,y) \\ K_B(y,x) & K_B(y,y) \end{vmatrix} dx\, dy \\ &= \lambda^d K_B(0,0) + \lambda^{2d} K_B(0,0)^2 - \iint_{Q_0 \times Q_0} |K_B(x,y)|^2\, dx\, dy,\end{aligned}$$

whence

$$\operatorname{Cov}(N_0, N_0) = \operatorname{Var}(N_0) = \lambda^d K_B(0,0) - \iint_{Q_0 \times Q_0} |K_B(x,y)|^2\, dx\, dy. \tag{20}$$

Now recall that $K_B$ is an orthogonal projection. Thus, we have

$$K_B(0,0) = K_B(x,x) = \int |K_B(x,y)|^2\, dy = \sum_{n \in \mathbb{Z}^d} \int_{Q_n} |K_B(x,y)|^2\, dy. \tag{21}$$

The identities (19), (20) and (21) imply that

$$\begin{aligned}\sum_{n \in \mathbb{Z}^d} \operatorname{Cov}(N_0, N_n) &= \lambda^d K_B(0,0) - \int_{Q_0} dx \sum_{n \in \mathbb{Z}^d} \int_{Q_n} |K_B(x,y)|^2\, dy \\ &= \lambda^d K_B(0,0) - \lambda^d K_B(0,0) = 0.\end{aligned}$$

Moreover, the above series converge absolutely. Proposition 4.1 is completely proved. □

COROLLARY 4.2. *The spectral density $\omega_{N^{(\lambda)}}$ of the stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}^d}$ is a continuous non-negative function on $\mathbb{T}^d = [-\frac{1}{2}, \frac{1}{2}]^d$ and vanishes only at $(0, \ldots, 0)$.*

*Proof.* By Lemma 2.3, the spectral density $\omega_{N^{(\lambda)}}$ of the stochastic process $N^{(\lambda)}$ is given by

$$\omega_{N^{(\lambda)}}(\theta_1, \ldots, \theta_d) = \sum_{n \in \mathbb{Z}^d} \operatorname{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) e^{i2\pi(n_1\theta_1 + \cdots + n_d\theta_d)}. \tag{22}$$

By (17), the series in (22) converges uniformly and absolutely on $\mathbb{T}^d$. It follows that $\omega_{N^{(\lambda)}}$ is a continuous function on $\mathbb{T}^d$.

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Now the equality (18) implies that $\omega_{N^{(\lambda)}}(0, \dots, 0) = 0$. Moreover, for any $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d \setminus \{(0, \dots, 0)\}$, we have

$$\left| \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) e^{i2\pi(n_1\theta_1 + \cdots + n_d\theta_d)} \right| < \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})|.$$

By (19), we have

$$|\mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})| = -\mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) \quad \text{for any } n \in \mathbb{Z}^d \setminus \{0\}.$$

Note that if $\theta = (\theta_1, \dots, \theta_d) \neq (0, \dots, 0)$, then

$$\begin{aligned} \omega_{N^{(\lambda)}}(\theta_1, \dots, \theta_d) &\geq \mathrm{Cov}(N_0^{(\lambda)}, N_0^{(\lambda)}) - \left| \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) e^{i2\pi(n_1\theta_1 + \cdots + n_d\theta_d)} \right| \\ &> \mathrm{Cov}(N_0^{(\lambda)}, N_0^{(\lambda)}) - \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})| \\ &= \sum_{n \in \mathbb{Z}^d} \mathrm{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) = 0. \end{aligned}$$

This shows that $\omega_{N^{(\lambda)}}$ vanishes only at $(0, \dots, 0)$. □

4.1. *Stationary determinantal point processes on* $\mathbb{R}$.

THEOREM 4.3. *Assume that* $B \subset \mathbb{R}$ *satisfies*

$$\sup_{R>0} \left( R \int_{|\xi| \geq R} |\widehat{\chi_B}(\xi)|^2 \, d\xi \right) < \infty. \tag{23}$$

*Then the stationary determinantal point process* $\mathbb{P}_{K_B}$ *is linearly rigid.*

*Proof.* By definition of linear rigidity, we need to show that for any $\lambda > 0$, the stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}}$ is linearly rigid. As in the proof of Proposition 4.1, we denote $N_n^{(\lambda)}$ by $N_n$. By Theorem 3.1, it suffices to show that

$$\sup_{N \geq 1} \left( N \sum_{|n| \geq N} |\mathrm{Cov}(N_0, N_n)| \right) < \infty \tag{24}$$

and

$$\sum_{n \in \mathbb{Z}} \mathrm{Cov}(N_0, N_n) = 0. \tag{25}$$

By Proposition 4.1, the identity (25) holds in the general case. It remains to prove (24). By (19), we have

$$\begin{aligned} \sup_{N \geq 1} \left( N \sum_{|n| \geq N} |\mathrm{Cov}(N_0, N_n)| \right) &= \sup_{N \geq 1} N \int_{x \in Q_0} \int_{y \in \bigcup_{|n| \geq N} Q_n} |\widehat{\chi}_B(x - y)|^2 \, dx \, dy \\ &= \sup_{N \geq 1} N \int_{-\lambda/2}^{\lambda/2} \int_{|y| \geq (N - 1/2)\lambda} |\widehat{\chi}_B(x - y)|^2 \, dx \, dy \\ &\leq \sup_{N \geq 1} N \int_{-\lambda/2}^{\lambda/2} \int_{|\xi| \geq (N-1)\lambda} |\widehat{\chi}_B(\xi)|^2 \, dx \, dy \\ &= \sup_{N \geq 1} \lambda N \int_{|\xi| \geq (N-1)\lambda} |\widehat{\chi}_B(\xi)|^2 \, d\xi < \infty, \end{aligned}$$

where, in the last inequality, we used our assumption (23). Theorem 4.3 is proved completely. □

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*Remark 4.1.* When $B$ is a finite union of finite intervals on the real line, the rigidity of the stationary determinantal point process $\mathbb{P}_{K_B}$ is due to Ghosh [**5**].

4.2. *Tensor product of sine kernels.* In higher dimension, the situation becomes quite different. Let

$$S = I \times I = (-1/2, 1/2) \times (-1/2, 1/2) \subset \mathbb{R}^2.$$

Then the associated kernel $K_S$ has a tensor form: $K_S = K_{\text{sine}} \otimes K_{\text{sine}}$, that is, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, we have

$$K_S(x, y) = K_{\text{sine}}(x_1, y_1) K_{\text{sine}}(x_2, y_2) = \frac{\sin(\pi(x_1 - y_1))}{\pi(x_1 - y_1)} \frac{\sin(\pi(x_2 - y_2))}{\pi(x_2 - y_2)}.$$

PROPOSITION 4.4. *The determinantal point process $\mathbb{P}_{K_S}$ is not linearly rigid. More precisely, let $N^{(1)} = (N_n^{(1)})_{n \in \mathbb{Z}^2}$ be the stationary stochastic process given as in Definition 1.2; then*

$$N_0^{(1)} \notin \check{H}_0(N^{(1)}).$$

To prove the above result, we need to introduce some extra notation. First, we define the multiple Zygmund class $\Lambda_*$ as follows. A continuous function $\varphi(x, y)$ periodic in each variable with period 1 is said to be in the multiple Zygmund class $\Lambda_*(1, 1)$ if for the double difference operator $\Delta_{2,2}$ of second order in each variable, applied to $\varphi$, there exists a constant $C > 0$ such that for all $x = (x_1, x_2) \in (-1/2, 1/2) \times (-1/2, 1/2)$ and $h_1, h_2 > 0$, we have

$$|\Delta_{2,2}\varphi(x_1, x_2; h_1, h_2)| \le C h_1 h_2, \tag{26}$$

where

$$\begin{aligned}\Delta_{2,2}\varphi(x_1, x_2; h_1, h_2) &:= \varphi(x_1 + h_1, x_2 + h_2) + \varphi(x_1 - h_1, x_2 + h_2) \\ &+ \varphi(x_1 + h_1, x_2 - h_2) + \varphi(x_1 - h_1, x_2 - h_2) - 2\varphi(x_1 + h_1, x_2) \\ &- 2\varphi(x_1 - h_1, x_2) - 2\varphi(x_1, x_2 + h_2) - 2\varphi(x_1, x_2 - h_2) + 4\varphi(x_1, x_2).\end{aligned}$$

The following result is due to Fülöp and Móricz [**4**, Theorem 2.1 and Remark 2.3].

THEOREM 4.5. (Fülöp–Móricz) *If $\{c_{jk}\}_{j,k \in \mathbb{Z}} \in \mathbb{C}$ is such that*

$$\sup_{N \ge 1, M \ge 1} \left( MN \sum_{|j| \ge N, |k| \ge M} |c_{jk}| \right) < \infty, \tag{27}$$

*then the function*

$$\varphi(\theta_1, \theta_2) = \sum_{j,k \in \mathbb{Z}} c_{jk} e^{i2\pi(j\theta_1 + k\theta_2)}$$

*is in the Zygmund class $\Lambda_*(1, 1)$.*

Let us turn to the study of the density function $\omega_{N^{(1)}}$.

LEMMA 4.6. *There exists $c > 0$ such that for any $\theta_1, \theta_2 \in [-1/2, 1/2]$, we have*

$$\omega_{N^{(1)}}(\theta_1, \theta_2) \ge c(|\theta_1| + |\theta_2|). \tag{28}$$

*Proof.* To make notation lighter, in this proof we simply write $\omega$ for $\omega_{N^{(1)}}$.

For any $n = (n_1, n_2) \in \mathbb{Z}^2$, let us denote $S_n = S \times (n + S)$, where

$$n + S := (-1/2 + n_1, 1/2 + n_1) \times (-1/2 + n_2, 1/2 + n_2).$$

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By the same argument as in the proof of Proposition 4.1, we obtain that for any $n = (n_1, n_2) \in \mathbb{Z}^2 \setminus \{0\}$,

$$\widehat{\omega}(n) = -\int_{S_n} |K_S(x, y)|^2 \, dx \, dy \tag{29}$$

and

$$\widehat{\omega}(0) = K_S(0, 0) - \int_{S_0} |K_S(x, y)|^2 \, dx \, dy.$$

The following properties can be easily checked.

- $\sum_{n \in \mathbb{Z}^2} \widehat{\omega}(n) = 0$.
- $\widehat{\omega}(\varepsilon_1 n_1, \varepsilon_2 n_2) = \widehat{\omega}(n_1, n_2)$, where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.
- There exist $c, C > 0$ such that

$$\frac{c}{(1 + n_1^2)(1 + n_2^2)} \leq |\widehat{\omega}(n_1, n_2)| \leq \frac{C}{(1 + n_1^2)(1 + n_2^2)}.$$

For instance, $\sum_{n \in \mathbb{Z}^2} \widehat{\omega}(n) = 0$ follows from Proposition 4.1. These properties combined with Theorem 4.5 yield that:

- $\omega(0, 0) = 0$;
- $\omega(\varepsilon_1 \theta_1, \varepsilon_2 \theta_2) = \omega(\theta_1, \theta_2)$ for any $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $\theta_1, \theta_2 \in (-1/2, 1/2)$;
- the function $\omega(\theta_1, \theta_2)$ is in the multiple Zygmund class $\Lambda_*(1, 1)$.

Hence, there exists $C > 0$ such that

$$|\omega(\theta_1, \theta_2) - \omega(\theta_1, 0) - \omega(0, \theta_2)| \leq C|\theta_1 \theta_2|. \tag{30}$$

LEMMA 4.7. *There exists $c_1 > 0$ such that*

$$\omega(\theta_1, 0) \geq c_1|\theta_1| \quad \textit{and} \quad \omega(0, \theta_2) \geq c_1|\theta_2|. \tag{31}$$

Let us postpone the proof of Lemma 4.7 and proceed to the proof of Lemma 4.6. The inequalities (30) and (31) imply that

$$\omega(\theta_1, \theta_2) \geq c_1(|\theta_1| + |\theta_2|) - C|\theta_1 \theta_2|.$$

Now, if $|\theta_1|$ is small enough such that $2C|\theta_1| \leq c_1$, then we have

$$\omega(\theta_1, \theta_2) \geq \frac{c_1}{2}(|\theta_1| + |\theta_2|).$$

If $2C|\theta_1| \geq c_1$, by Corollary 4.2, the function $\omega(\theta_1, \theta_2)$ is continuous on $[-1/2, 1/2]^2$ and vanishes only at $(0, 0)$. Consequently,

$$\inf_{|\theta_1| \geq c_1/2C} \omega(\theta_1, \theta_2) = c_2 > 0.$$

It follows, by using the elementary fact that $|\theta_1| + |\theta_2| \leq 1$, that

$$\inf_{|\theta_1| \geq c_1/2C} \omega(\theta_1, \theta_2) = c_2 \geq \frac{c_2}{2}(|\theta_1| + |\theta_2|).$$

Taking $c = \min(c_1/2, c_1/2)$, we get the desired inequality (28). □

Now let us turn to the proof of Lemma 4.7.

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*Proof of Lemma 4.7.* By symmetry, it suffices to prove that there exists $c > 0$ such that $\omega(\theta_1, 0) \geq c|\theta_1|$. To this end, let us denote $\omega_1(\theta_1) := \omega(\theta_1, 0)$. Then $\omega_1(0) = 0$ and there exists $c > 0$ such that if $k \neq 0$, then

$$\widehat{\omega}_1(k) < 0 \quad \text{and} \quad |\widehat{\omega}_1(k)| \geq c/(1 + k^2).$$

Indeed, we have

$$\omega_1(\theta_1) = \sum_{k\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} \widehat{\omega}(k, n_2) e^{i2\pi k\theta_1}.$$

If $k \neq 0$, then, by (29), we have $\widehat{\omega}(k, n_2) < 0$ and hence

$$|\widehat{\omega}_1(k)| = \sum_{n_2\in\mathbb{Z}} |\widehat{\omega}(k, n_2)| \geq \sum_{n_2\in\mathbb{Z}} \frac{c}{(1 + n_2^2)(1 + k^2)} \geq \frac{c'}{1 + k^2}. \tag{32}$$

We claim that $\omega_1(0) = 0$. Indeed, by definition, we have

$$\omega_1(0) = \sum_{k\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} \widehat{\omega}(k, n_2) = \omega(0, 0) = 0,$$

where, in the last equality, we used Corollary 4.2 that claims that $\omega(0, 0) = 0$. Now we have

$$\sum_{k\in\mathbb{Z}} \widehat{\omega}_1(k) = \omega_1(0) = 0.$$

It follows that

$$\begin{aligned}\omega_1(\theta_1) &= \sum_{k\in\mathbb{Z}} \widehat{\omega}_1(k) e^{i2\pi k\theta_1} = \sum_{k\in\mathbb{Z}} \widehat{\omega}_1(k) \left( \frac{e^{i2\pi k\theta_1} + e^{-i2\pi k\theta_1}}{2} - 1 \right) \\ &= \sum_{k\in\mathbb{Z}, k\neq 0} -\widehat{\omega}_1(k)(1 - \cos(2\pi k\theta_1)) = \sum_{k\in\mathbb{Z}, k\neq 0} |\widehat{\omega}_1(k)|(1 - \cos(2\pi k\theta_1)).\end{aligned}$$

Since $|\widehat{\omega}_1(k)|(1 - \cos(2\pi k\theta_1))$ is non-negative for any $k \in \mathbb{Z}$, we have

$$\omega_1(\theta_1) \geq \sum_{j=1}^{\infty} |\widehat{\omega}_1(2j - 1)|(1 - \cos(2\pi(2j - 1)\theta_1)).$$

The inequality (32) implies that there exists $c'' > 0$ such that $|\widehat{\omega}_1(2j - 1)| \geq (c''/(2j - 1)^2)$; hence, we obtain that

$$\omega_1(\theta_1) \geq c'' \sum_{j=1}^{\infty} \frac{1}{(2j - 1)^2} (1 - \cos(2\pi(2j - 1)\theta_1)).$$

Combining with the Fourier series of the absolute value function (the Fourier coefficient of the absolute value function on $(-\frac{1}{2}, \frac{1}{2})$ can be computed explicitly),

$$|\alpha| = \frac{1}{4} - \frac{2}{\pi^2} \sum_{j=1}^{\infty} \frac{\cos(2\pi(2j - 1)\alpha)}{(2j - 1)^2} \quad \text{for } \alpha \in (-1/2, 1/2);$$

$$\sum_{j=1}^{\infty} \frac{1}{(2j - 1)^2} = \frac{\pi^2}{8} \quad \text{(take } \alpha = 0 \text{ in the above series)},$$

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we obtain that

$$\omega_1(\theta_1) \geq c''\left(\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2} - \sum_{j=1}^{\infty}\frac{\cos(2\pi(2j-1)\theta_1)}{(2j-1)^2}\right)$$
$$= c''\left(\frac{\pi^2}{8} + \frac{\pi^2}{2}\left(|\theta_1| - \frac{1}{4}\right)\right) = c''\frac{\pi^2}{2}|\theta_1|.$$

The proof of Lemma 4.7 is complete. □

*Proof of Proposition 4.4.* By Lemma 2.1, it suffices to show that

$$\int_{\mathbb{T}^2} \omega_{N^{(1)}}^{-1}\, dm < \infty. \tag{33}$$

By Lemma 4.6, the inequality (33) follows from the following elementary inequality:

$$\int_{|\theta_1|<1/2,|\theta_2|<1/2} \frac{1}{|\theta_1| + |\theta_2|}\, d\theta_1\, d\theta_2 < \infty.$$

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