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Eigenvalue asymptotics for an elliptic boundary problem

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We consider an elliptic boundary problem in a bounded region $\Omega \subset \mathbb{R}^n$ wherein the spectral parameter is multiplied by a real-valued weight function with the property that it, together with its reciprocal, is essentially bounded in Ω. The problem is considered under limited smoothness assumptions and under an ellipticity with parameter condition. Then, fixing our attention upon the operator induced on $L_2(\Omega)$ by the boundary problem under null boundary conditions, we establish results pertaining to the asymptotic behaviour of the eigenvalues of this operator under weaker smoothness assumptions than have hitherto been supposed.

1. Introduction

In this paper we are concerned with the boundary problem

$$A(x, D)u(x) - \lambda\omega(x)u(x) = f(x) \quad \text{in } \Omega, \tag{1.1}$$

$$B_j(x, D)u(x) = g_j(x) \quad \text{on } \Gamma \text{ for } j = 1, \ldots, m, \tag{1.2}$$

where $A(x, D)$ is a linear differential operator of order $2m$ defined in the bounded region $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, with boundary Γ, $B_j(x, D)$ is a linear differential operator of order $m_j < 2m$ defined on Γ, and $\omega(x)$ is a real-valued function defined on Ω with the property that it, together with its reciprocal, is essentially bounded in Ω. Our assumptions concerning the problem (1.1), (1.2) will be made precise in § 2. However, let us mention at this point that the problem is considered under limited smoothness assumptions and under an ellipticity with parameter condition. Then, by fixing our attention on the operator induced on $L_2(\Omega)$ by the boundary problem under null boundary conditions, we establish results pertaining to the asymptotic behaviour of the eigenvalues of this operator under weaker smoothness assumptions than have hitherto been supposed.

General results pertaining to the asymptotic behaviour of the eigenvalues of operators induced by boundary problems of the form (1.1), (1.2) (with the $g_j = 0$)

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under limited smoothness assumptions are to be found, for example, in [3,7] for the self-adjoint case (i.e. when the induced operator cited above is self-adjoint), and in [6,8] (where the analogue of (1.1), (1.2) for systems is considered), [11,14] for the non-self-adjoint case. However, an inspection of the works cited above shows that, in spite of their various approaches to the problem of establishing the eigenvalue asymptotics, these authors all use some variation of the following result.

RESULT 1.1. Let T_1, T_2 be bounded operators in $L_2(\Omega)$ such that their ranges are contained in the Sobolev space $W_p^\ell(\Omega)$, where $\ell \in \mathbb{N}$, $2 \leqslant p < \infty$ and $\ell p > n$. Also let T_2^* denote the adjoint of T_2. Then T_1 and T_2 are Hilbert–Schmidt operators and $T = T_1 T_2^*$ is an integral operator of trace class whose kernel $K(x,y)$ is continuous in $\bar{\Omega} \times \bar{\Omega}$ and in this set satisfies the estimate

$$|K(x,y)| \leqslant c \prod_{j=1}^{2} \|T_j\|^{\tau}_{L_2(\Omega) \to W_p^\ell(\Omega)} \|T_j\|^{1-\tau}_{L_2(\Omega) \to L_p(\Omega)}, \tag{1.3}$$

where $\tau = n/\ell p$ and the operator norms indicated in (1.3) are defined in the text preceding proposition 3.2, below.

This result is of extreme important vis-à-vis the problem of eigenvalue asymptotics, and we will demonstrate this by describing one method of how it can be applied to the boundary problem (1.1), (1.2). Accordingly, let $R(\lambda)$ denote the resolvent of the induced Hilbert space operator cited above (we suppose that the resolvent set of this operator is not empty and that λ lies in some suitable subset), take $\ell = 2m$ in result 1.1 and choose $k \in \mathbb{N}$ so that $2mk > n/2$. Hence, if we take $T_1 = T_1(\lambda) = R(\lambda)^k$, $T_2 = T_2(\lambda) = (R(\lambda)^*)^k$ and show that, for fixed λ, $T_1(\lambda)$ and $T_2(\lambda)$ satisfy the hypotheses of result 1.1, then it follows from that result and [9] that

$$\operatorname{tr} T(\lambda) = \int_\Omega K(x,x,\lambda)\, \mathrm{d}x, \tag{1.4}$$

where 'tr' denotes the trace and where we have written $T(\lambda)$ and $K(x,x,\lambda)$ in place of T and $K(x,x)$, respectively, in order to show their dependence on λ. Furthermore, if we can establish estimates for the norms of $R(\lambda)$ and $R(\lambda)^*$ of the form indicated in (1.3) for the T_j, then we can use these estimates to establish analogous estimates for the $T_j(\lambda)$ and then appeal to (1.3), with the T_j replaced by $T_j(\lambda)$, to arrive at the asymptotic behaviour of $\int_\Omega K(x,x,\lambda)\, \mathrm{d}x$ as $|\lambda| \to \infty$. By appealing to this last result as well as to the fact that $\operatorname{tr} T(\lambda)$ is just a sum of terms involving the eigenvalues of the induced operator under consideration, we can make use of (1.4) to derive information concerning the asymptotic behaviour of the eigenvalues.

We have just described one method for dealing with the problem of eigenvalue asymptotics and, as was shown, this method involves the use of $R(\lambda)$ and $R(\lambda)^*$. We remark at this point that, in all of the works cited above, the operators $R(\lambda)$ and $R(\lambda)^*$ are also used, although their method of use may not necessarily coincide with that described here. It is important to note that the method described here actually works, in that all the conditions cited above for it to work can easily be shown to be satisfied from a consideration of the mapping properties of $R(\lambda)$ and $R(\lambda)^*$. The mapping properties of $R(\lambda)$ are deduced from a consideration of estimates, in suitable Sobolev spaces, for solutions of the boundary problem (1.1), (1.2), and these

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estimates can be established under the so-called 'minimal' smoothness assumptions (see assumption 2.1, below, and [6, § 1]). On the other hand, in order to ascertain the mapping properties of $R(\lambda)^*$ we must impose further smoothness assumptions upon Γ and upon the coefficients of $A(x, D)$ and the $B_j(x, D)$ to ensure that the boundary problem formally adjoint to (1.1), (1.2) exists, has the same 'minimal' smoothness properties described above for the problem (1.1), (1.2), and that $R(\lambda)^*$ is precisely the resolvent of the Hilbert space operator induced by this formally adjoint problem under null boundary conditions. Then the mapping properties of $R(\lambda)^*$ can be deduced from a consideration of estimates for solutions of this formally adjoint boundary problem. We observe from [6, § 2.5] that the further smoothness assumptions required to deal with $R(\lambda)^*$ are quite considerable.

Hence, in light of what was said above, and this is what is novel about our work, we will now show that all the extra smoothness assumptions required because of the use of $R(\lambda)^*$ are totally unnecessary and, in fact, it is totally unnecessary to work with $R(\lambda)^*$ at all. Indeed, we shall achieve our goal by making use of the following modification of result 1.1 (see [4, theorem 12.21, p. 205]).

RESULT 1.2. Let T_1, T_2 satisfy the hypotheses of result 1.1. Then T_1 and T_2 are Hilbert–Schmidt operators and $T = T_1T_2$ is an integral operator of trace class whose kernel $K(x, y)$ has the property that $K(x, x)$ is integrable on Ω and $\operatorname{tr} T = \int_\Omega K(x, x)\,dx$.

In applying result 1.2 to our work, we take ℓ and k as above, let $T_j = T_j(\lambda) = R(\lambda)^k$ for $j = 1, 2$, so that now $T = T(\lambda) = R(\lambda)^{2k}$ and as before we write $K(x, y, \lambda)$ in place of $K(x, y)$, so that again we have (1.4). Then under 'minimal' smoothness assumptions, we shall derive estimates for the various norms of $R(\lambda)$ arising when $R(\lambda)$ is considered as a mapping from various function spaces over Ω into other spaces. These estimates will then be used to arrive at the asymptotic behaviour of $\int_\Omega K(x, x, \lambda)\,dx$ as $|\lambda| \to \infty$. Then, as explained above, we will use the latter result, together with (1.4), to derive information concerning the asymptotic behaviour of the eigenvalues of the induced operator cited above under 'minimal' smoothness assumptions.

We remark at this point that, for the special case where the boundary conditions (1.2) are of Dirichlet type, the asymptotics of $\operatorname{tr} R(\lambda)^{2k}$ and of the eigenvalues of the induced operator have also been established in [6] under the 'minimal' smoothness assumptions. However, the method employed therein for this special case is totally different to the method described above. Indeed, this special case is dealt with by approximating the boundary problem (1.1), (1.2) by a smooth boundary problem for which the relevant results are known and then employing a limiting argument.

Finally, let us outline the contents of this paper. In § 2 we introduce some terminology, assumptions and definitions that we require for our work and also present some known results concerning the boundary problem (1.1), (1.2) that are needed in the remainder of the paper. Section 3 is devoted to establishing the asymptotic behaviour of $\operatorname{tr} R(\lambda)^{2k}$ under 'minimal' smoothness assumptions, and this result is used in § 4 to establish results pertaining to the asymptotic behaviour of the eigenvalues.

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2. Preliminaries

In this section we introduce some terminology, definitions and assumptions concerning the boundary problem (1.1), (1.2) and present some known results concerning this problem. These results will then be used to derive our main results.

Accordingly, we let $x = (x_1, \dots, x_n) = (x', x_n)$ denote a generic point in $\mathbb{R}^n$ and use the notation $D_j = -i\partial/\partial x_j$, $D = (D_1, \dots, D_n)$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ for $\xi = (\xi_1, \dots, \xi_n) = (\xi', \xi_n) \in \mathbb{R}^n$, where $\alpha = (\alpha_1, \dots, \alpha_n) = (\alpha', \alpha_n)$ is a multi-index whose length $\sum_{j=1}^n \alpha_j$ is denoted by $|\alpha|$. For $1 < p < \infty$, $0 \leqslant s < \infty$ and G an open set in $\mathbb{R}^n$, we let $W_p^s(G)$ denote the Sobolev space of order s related to $L_p(G)$ and let $\|\cdot\|_{s,p,G}$ denote the norm in this space (we refer the reader to [15, ch. 2 and 4] for details). Furthermore, recalling that m is a fixed integer with $2m$ denoting the order of $A(x, D)$, we shall use norms depending upon a parameter $\lambda \in \mathbb{C} \setminus \{0\}$, namely

$$|||u|||_{s,p,G} = \|u\|_{s,p,G} + |\lambda|^{s/2m}\|u\|_{0,p,G} \quad \text{for } u \in W_p^s(G).$$

Assume for the moment that the boundary Γ of Ω (see (1.1), (1.2)) is of class $C^{2m-1,1}$ and let s be an integer satisfying $1 \leqslant s \leqslant 2m$. Then the functions $u \in W_p^s(\Omega)$ have boundary values $v = u_{|\Gamma}$; and the space of these boundary values can be identified with the Sobolev space $W_p^{s-1/p}(\Gamma)$ and we denote by $\|\cdot\|_{s-1/p,p,\Gamma}$ the norm in this space (see [6, § 2] and [12, p. 20] for details). Furthermore, we shall also use norms depending upon a parameter $\lambda \in \mathbb{C} \setminus \{0\}$, namely

$$|||v|||_{s-1/p,p,\Gamma} = \|v\|_{s-1/p,p,\Gamma} + |\lambda|^{(s-1/p)/2m}\|v\|_{0,p,\Gamma} \quad \text{for } v \in W_p^{s-1/p}(\Gamma).$$

Lastly, in the following we let $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$.

Let us now turn to the boundary problem (1.1), (1.2).

Assumption 2.1.

(1) Γ is of class $C^{2m-1,1}$.

(2) The coefficients of $A(x, D) = \sum_{|\alpha| \leqslant 2m} a_\alpha(x) D^\alpha$ are complex-valued functions in $L_\infty(\Omega)$, while the top-order coefficients $a_\alpha(x)$ ($|\alpha| = 2m$) are continuous in $\bar{\Omega}$.

(3) The coefficients of

$$B_j(x, D) = \sum_{|\alpha| \leqslant m_j} b_{j\alpha}(x) D^\alpha$$

are complex-valued functions defined on Γ satisfying $b_{j\alpha} \in C^{2m-1-m_j,1}(\Gamma)$ for $|\alpha| \leqslant m_j$.

Definition 2.2.

(i) We say that $\omega(x)$ satisfies condition 1, above, if $\omega(x)$ is defined and continuous and does not vanish in $\bar{\Omega}$.

(ii) We say that $\omega(x)$ satisfies condition 2 if there exists a sequence of non-empty subregions $\{\Omega_k\}_1^N$, $N \in \mathbb{N}$, of Ω whose closures are pairwise disjoint and also contained in Ω such that the following conditions are satisfied:

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(a) if Γ_k denotes the boundary of Ω_k for $k = 1, \dots, N$, then Γ_k is of class $C^{2m-1,1}$;

(b) if $\Omega_0 = \Omega \setminus \bigcup_{k=1}^{N} \bar{\Omega}_k$, then $\omega(x)$ extends by continuity from Ω_k to a continuous, non-vanishing function in $\bar{\Omega}_k$ for $k = 0, \dots, N$.

Let us henceforth denote by $a_0(x,\xi)$ and $b_{j0}(x,\xi)$ the principal symbols of $A(x,D)$ and $B_j(x,D)$, respectively, $j = 1, \dots, m$.

DEFINITION 2.3. Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin. Then the boundary problem (1.1), (1.2) will be called elliptic with parameter in $\mathcal{L}$ if assumption 2.1 holds and the following conditions are satisfied:

(i) either $\omega(x)$ satisfies condition 1 and $a_0(x,\xi) - \lambda\omega(x) \neq 0$ for $(x,\xi) \in \bar{\Omega} \times \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ if $|\xi| + |\lambda| \neq 0$, or it satisfies condition 2 and, for $k = 0, \dots, N$, $a_0(x,\xi) - \lambda\omega(x) \neq 0$ for $(x,\xi) \in \bar{\Omega}_k \times \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ if $|\xi| + |\lambda| \neq 0$, where $\omega(x)$ is defined by continuity on the boundary of Ω_k;

(ii) let x^0 be a point of Γ and let $\omega(x^0)$ be defined by continuity if $\omega(x)$ satisfies condition 2. Assume that the boundary problem (1.1), (1.2) is rewritten in a coordinate system associated with x^0: it is obtained from the original system by a rotation, after which the positive x_n-axis has the direction of the interior normal to Γ at x^0. Then, for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \mathcal{L}$, the boundary problem on the half-line

$$\begin{aligned} a_0(x^0,\xi',D_n)v(t) - \lambda\omega(x^0)v(t) &= 0 \qquad \text{for } t = x_n > 0, \\ b_{j0}(x^0,\xi',D_n)v(t) &= 0 \qquad \text{at } t = 0 \text{ for } j = 1, \dots, m, \\ v(t) &\to 0 \quad \text{as } t \to \infty, \end{aligned}$$

has only the trivial solution if $|\xi'| + |\lambda| \neq 0$.

From [6, theorem 2.1] we have the following result.

THEOREM 2.4. *Suppose that the boundary problem (1.1), (1.2) is elliptic with parameter in $\mathcal{L}$. There then exists a $\lambda_0 = \lambda_0(p) > 0$ such that, for $\lambda \in \mathcal{L}$ with $|\lambda| \geqslant \lambda_0$, the boundary problem has a unique solution $u \in W_p^{2m}(\Omega)$ for any $f \in L_p(\Omega)$ and $g_j \in W_p^{2m-m_j-1/p}(\Gamma)$, and the a priori estimate*

$$|||u|||_{2m,p,\Omega} \leqslant c\left(\|f\|_{0,p,\Omega} + \sum_{j=1}^{m} |||g_j|||_{2m-m_j-1/p,p,\Gamma} \right)$$

holds, where the constant c does not depend upon f, the g_j and λ.

Let A_p denote the operator on $L_p(\Omega)$ that acts as $A(x,D)$ and has domain

$$D(A_p) = \{u \in W_p^{2m}(\Omega) \mid B_j(x,D)u(x) = 0 \text{ on } \Gamma \text{ for } j = 1, \dots, m\},$$

and let V_p denote the operator of multiplication induced on $L_p(\Omega)$ by $\omega(x)$. It then follows from theorem 2.4 that $V_p^{-1}A_p$ is a closed operator on $L_p(\Omega)$ with non-empty resolvent set containing those $\lambda \in \mathcal{L}$ for which $|\lambda| \geqslant \lambda_0$ and if we let $R_p(\lambda)$ denote the resolvent of $V_p^{-1}A_p$ then, for $\lambda \in \mathcal{L}$ with $|\lambda| \geqslant \lambda_0$, we have

$$|||R_p(\lambda)f|||_{2m,p,\Omega} \leqslant c\|f\|_{0,p,\Omega} \tag{2.1}$$

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for any $f \in L_p(\Omega)$, where the constant c does not depend upon f and λ. Hence, it follows from the Rellich–Kondrachov theorem that $V_p^{-1}A_p$ has compact resolvent and a discrete spectrum, while we know from [2, p. 131] that the eigenvalues and principal vectors of $V_p^{-1}A_p$ do not depend upon p. Because of the above facts, and since we will be dealing with the eigenvalue asymptotics, we shall mainly be concerned from now on with the operator $V^{-1}A$ acting on $L_2(\Omega)$, where we have written V for V_2 and A and A_2. For brevity we shall also write $R(\lambda)$ for $R_2(\lambda)$ for λ in the resolvent set of $V^{-1}A$. We note for later use, and this is a consequence of the foregoing results, that $R_p(\lambda) = R(\lambda)|_{L_p(\Omega)}$ for $p > 2$.

To terminate the work of this section, we now state two well-known results which will be used in the remainder of the paper. Accordingly, let $\ell \in \mathbb{N}$ and let Ω' denote either Ω or $\mathbb{R}^n$. Also let $C^{0,\gamma}(\overline{\Omega'})$, $0 < \gamma < 1$, denote the space of uniformly Hölder continuous functions on Ω' of exponent γ (see [1, p. 10]) and, recalling from above that p always denotes a number satisfying $1 < p < \infty$, let q denote a number satisfying $1 < q \leqslant \infty$. Then the first of these aforementioned results is a version of the Sobolev embedding theorem (cf. [1, p. 98], [6, § 2], [13, p. 68]) which states that if

$$0 < \tau = \frac{n}{\ell}\left(\frac{1}{p} - \frac{1}{q}\right) < 1,$$

then there is a continuous embedding of $W_p^\ell(\Omega')$ into $L_q(\Omega')$ and, if $q = \infty$, of $W_p^\ell(\Omega')$ into $C^{0,\gamma}(\overline{\Omega'})$ for some γ in $(0, 1)$ while, for $u \in W_p^\ell(\Omega')$, the estimate

$$\|u\|_{0,q,\Omega'} \leqslant c\|u\|_{\ell,p,\Omega'}^{\tau}\|u\|_{0,p,\Omega'}^{1-\tau} \tag{2.2}$$

holds, where the constant c depends upon n, p, q, ℓ and Ω', but not upon u. The second of the aforementioned results is the interpolation theorem (cf. [1, theorem 4.17, p. 74], [6, § 2]), which states that if $u \in W_p^\ell(\Omega')$, $j \in \mathbb{N}$ with $j < \ell$, and

$$|u|_{j,p,\Omega'} = \left(\int_{\Omega'} \sum_{|\alpha|=j} |D^\alpha u|^p \, dx\right)^{1/p},$$

then

$$|u|_{j,p,\Omega'} \leqslant c\|u\|_{\ell,p,\Omega'}^{\tau}\|u\|_{0,p,\Omega'}^{1-\tau}, \tag{2.3}$$

where $\tau = j/\ell$ and the constant c does not depend upon u.

3. Trace asymptotics

Recalling the definition of the resolvent $R(\lambda)$ introduced in § 2, it is our intention now to deal with the fundamental problem, discussed in § 1, of establishing asymptotic behaviour of the trace of powers of $R(\lambda)$ under the limited smoothness assumptions concerning the boundary problem (1.1), (1.2) that were cited in § 2. To this end, let $0 < \theta \leqslant \pi$ and let $\mathcal{L}_\theta$ denote the closed sector in the λ-plane with vertex at the origin defined by the inequalities $\theta \leqslant |\arg \lambda| \leqslant \pi$. We then come to the main result of this section.

THEOREM 3.1. *Let the boundary problem (1.1), (1.2) be elliptic with parameter in $\mathcal{L}_\theta$. Let q be an even integer satisfying $2mq > n$. Then $R(\lambda)^q$ is an integral*

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operator of trace class. Furthermore, if we let $\operatorname{tr} R(\lambda)^q$ *denote the trace of* $R(\lambda)^q$, *then*

$$\operatorname{tr} R(\lambda)^q = c_q(-\lambda)^{(n/2m)-q} + o(|\lambda|^{(n/2m)-q}) \quad \text{as } |\lambda| \to \infty, \lambda \in \mathcal{L}_\theta, \tag{3.1}$$

where

$$c_q = \int_\Omega c_q(x)\,\mathrm{d}x \quad \text{and} \quad c_q(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\mathrm{d}\xi}{[\omega(x)^{-1}a_0(x,\xi)+1]^q} \tag{3.2}$$

and the holomorphic function $(-\lambda)^{\cdots}$ *is defined outside* $\bar{\mathbb{R}}_+$ *and is equal to* $|\lambda|^{\cdots}$ *when* $\lambda \in \mathbb{R}_-$.

In order to prove the theorem some preparation is required. Accordingly, let Ω' denote either Ω or $\mathbb{R}^n$ and let T be a bounded linear operator in $L_2(\Omega')$. In addition, if X and Y denote subspaces of Banach spaces that are also contained in $L_2(\Omega')$ such that the restriction of T to X is a continuous operator from X into Y, then we will henceforth use the notation $\|T\|_{X\to Y}$ to denote the norm of this latter operator.

PROPOSITION 3.2. *Let* T *be a bounded linear operator from* $L_2(\Omega')$ *into* $W_p^\ell(\Omega')$, *where* $\ell \in \mathbb{N}$, $p \geqslant 2$ *and* $\ell p > n$. *Then* T *is an integral operator whose kernel* $K(x,y)$ *is measurable in* $\Omega' \times \Omega'$ *and has the following properties:*

(i) $K(x^0,\cdot) \in L_2(\Omega')$ *for every* $x^0 \in \Omega'$;

(ii) $K(x,\cdot)$ *is a continuous function of* x *in* Ω' *with values in* $L_2(\Omega')$; *and*

(iii) $$\left(\int_{\Omega'} |K(x,y)|^2\,\mathrm{d}y\right)^{1/2} \leqslant c\|T\|^{\tau}_{L_2(\Omega')\to W_p^\ell(\Omega')}\|T\|^{1-\tau}_{L_2(\Omega')\to L_p(\Omega')}$$

for $x \in \Omega'$, *where* $\tau = n/\ell p$ *and the constant* c *depends only upon* ℓ, n, p *and* Ω'.

Lastly, if $\Omega' = \Omega$, *then* T *is a Hilbert–Schmidt operator.*

Proof. For the case $\Omega' = \Omega$, the proposition has been proved in [6, § 5] for general p and in [2, 3] for the case $p = 2$. Hence, we need only concentrate on the case $\Omega' = \mathbb{R}^n$. However, in view of the Sobolev embedding theorem (see (2.2) and the text preceding it), a scrutiny of the proofs given in [3, 6] shows that for this case all the assertions of the proposition, except that concerning the measurability of $K(x,y)$, also hold. Thus, directing our attention to the question of measurability, let us observe from the Sobolev embedding theorem that we may consider T as a continuous mapping of $L_2(\mathbb{R}^n)$ into $L_\infty(\mathbb{R}^n)$ and accordingly we will denote by T^* the adjoint of T when T is considered in this sense. Then T^* induces a bounded linear transformation of $L_1(\mathbb{R}^n)$ into $L_2(\mathbb{R}^n)$, and hence we know from [10, problem 59, p. 519] that $T^*|_{L_1(\mathbb{R}^n)}$ is an integral operator whose kernel is measurable on $\mathbb{R}^n \times \mathbb{R}^n$. It follows immediately from this last result and the results of [10, problem 59, p. 519] that the kernel $K(x,y)$ of T is measurable, and this completes the proof of the proposition. □

We will now use proposition 3.2 to derive the following result (recall that Ω' denotes either Ω or $\mathbb{R}^n$).

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PROPOSITION 3.3. *Let $\ell \in \mathbb{N}$ be fixed and let $\{T_{k+1-j}\}_{j=1}^{k}$, $k \in \mathbb{N}$, be a sequence of bounded operators in $L_2(\Omega')$ with the property that, for every $p \in [2, \infty)$, the restriction of each of the T_j to $X_p(\Omega') = L_2(\Omega') \cap L_p(\Omega')$ maps into $L_2(\Omega') \cap W_p^\ell(\Omega')$ and is bounded as an operator from $X_p(\Omega')$ into $W_p^\ell(\Omega')$, where $X_p(\Omega')$ is equipped with the norm $\|\cdot\|_{0,p,\Omega'}$. Suppose also that $\ell k > \frac{1}{2}n$ and let $p_1 = 2$ if $k = 1$, while, if $k > 1$, then let the numbers $p_1, \dots, p_k$ be chosen so that*

$$2 = p_1 < p_2 < \cdots < p_k, \quad \text{with } \frac{n}{\ell}\left(\frac{1}{p_j} - \frac{1}{p_{j+1}}\right) = \tau_j < 1 \text{ for } j = 1, \dots, k-1 \text{ and } p_k > n/\ell,$$

where we note that

$$\sum_{j=1}^{k-1}(1 - \tau_j) \geqslant k - \frac{n}{2\ell}.$$

Then $T = \prod_{j=1}^{k} T_{k+1-j}$ is an integral operator whose kernel $K(x, y)$ is measurable in $\Omega' \times \Omega'$ and has the following properties:

(i) *$K(x^0, \cdot) \in L_2(\Omega')$ for every $x^0 \in \Omega'$;*

(ii) *$K(x, \cdot)$ is a continuous function of x in Ω' with values in $L_2(\Omega')$; and*

(iii) $$\left(\int_{\Omega'} |K(x,y)|^2 \, dy\right)^{1/2} \leqslant c \prod_{j=1}^{k} \|T_j\|^{\tau_j}_{X_{p_j}(\Omega') \to W^\ell_{p_j}(\Omega')} \|T_j\|^{1-\tau_j}_{X_{p_j}(\Omega') \to L_{p_j}(\Omega')}$$

for $x \in \Omega'$, where $\tau_k = n/\ell p_k$ and the constant c depends only upon ℓ, n, the p_j and Ω'.

Lastly, if $\Omega' = \Omega$, then T is a Hilbert–Schmidt operator.

Proof. If $k = 1$, then this proposition is an immediate consequence of proposition 3.2 (with $p = 2$), and hence we can suppose from now on that $k > 1$. Then, since it follows from our hypotheses and the text preceding (2.2) that the Sobolev embedding operator $S_j : W^\ell_{p_j}(\Omega') \to L_{p_{j+1}}(\Omega')$ is continuous for $j = 1, \dots, k-1$, we may view T as a continuous mapping of $L_2(\Omega')$ into $W^\ell_{p_k}(\Omega')$ given by the succession of continuous mappings:

$$L_2(\Omega') \xrightarrow{T_1} X_{p_2}(\Omega') \xrightarrow{T_2} X_{p_3}(\Omega') \to \cdots \to X_{p_{k-1}}(\Omega') \xrightarrow{T_{k-1}} X_{p_k}(\Omega') \xrightarrow{T_k} W^\ell_{p_k}(\Omega').$$

Thus, in view of (2.2) we now have

$$\|T\|_{L_2(\Omega') \to W^\ell_{p_k}(\Omega')} \leqslant c\|T_k\|_{X_{p_k}(\Omega') \to W^\ell_{p_k}(\Omega')} \prod_{j=1}^{k-1} \|T_j\|^{\tau_j}_{X_{p_j}(\Omega') \to W^\ell_{p_j}(\Omega')} \|T_j\|^{1-\tau_j}_{X_{p_j}(\Omega') \to L_{p_j}(\Omega')}$$

and

$$\|T\|_{L_2(\Omega') \to L_{p_k}(\Omega')} \leqslant c\|T_k\|_{X_{p_k}(\Omega') \to L_{p_k}(\Omega')} \prod_{j=1}^{k-1} \|T_j\|^{\tau_j}_{X_{p_j}(\Omega') \to W^\ell_{p_j}(\Omega')} \|T_j\|^{1-\tau_j}_{X_{p_j}(\Omega') \to L_{p_j}(\Omega')},$$

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where the constant c does not depend upon the T_j. In light of these results, all the assertions of the proposition now follow from proposition 3.2. □

Proof of theorem 3.1. Let us put $q = 2k$ and, concentrating on proposition 3.3, let us now take $\Omega' = \Omega$, $\ell = 2m$ and $T_j = R(\lambda)$ for $j = 1, \dots, k$ and for $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \geqslant \lambda_0$ (where λ_0 is chosen sufficiently large that (2.1) holds for $p = p_j$, $j = 1, \dots, k$). It then follows from proposition 3.3 and (2.1) that, for each fixed λ, $R(\lambda)^k$ is a Hilbert–Schmidt operator whose kernel $\tilde{K}(x, y, \lambda)$ satisfies the estimates

$$\left(\int_\Omega |\tilde{K}(x, y, \lambda)|^2 \, \mathrm{d}y \right)^{1/2} \leqslant c|\lambda|^{(n/4m)-k} \quad \text{for } x \in \Omega, \tag{3.3}$$

$$\left(\int_{\Omega \times \Omega} |\tilde{K}(x, y, \lambda)|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2} \leqslant c|\Omega|^{1/2}|\lambda|^{(n/4m)-k}, \tag{3.4}$$

where the constant c does not depend upon x and λ, and $|\Omega|$ denotes the measure of Ω. Hence, we conclude from [4, theorem 12.21, p. 205], the Schwarz inequality and Fubini's theorem that, for each fixed λ, $R(\lambda)^q$ is an integral operator of trace class (and hence of Hilbert–Schmidt type) whose kernel

$$K(x, y, \lambda) = \int_\Omega \tilde{K}(x, z, \lambda) \tilde{K}(z, y, \lambda) \, \mathrm{d}z$$

satisfies the following conditions:

$$K(x, x, \lambda) \text{ is integrable in } \Omega \text{ and } \operatorname{tr} R(\lambda)^q = \int_\Omega K(x, x, \lambda) \, \mathrm{d}x, \tag{3.5}$$

$$|K(x, x, \lambda)| = |\lambda|^{(n/2m)-q} \eta(x, \lambda) \text{ for almost every } x \text{ in } \Omega, \tag{3.6}$$

where $\eta(x, \lambda)$ is a real-valued, non-negative function in $L_2(\Omega)$ (and hence in $L_1(\Omega)$) satisfying $\int_\Omega \eta(x, \lambda)^2 \, \mathrm{d}x \leqslant c$, where the constant c does not depend upon λ.

We will now use (3.5), (3.6) and some localization arguments to complete the proof of the theorem. To this end we will proceed in several steps.

STEP 1. As the first step in completing the proof, we shall need some results concerning the local behaviour of $K(x, x, \lambda)$. Accordingly, we will henceforth suppose that $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \geqslant \lambda_0$ and let $x^0 \in \Omega$ such that $x^0 \notin \bigcup_{r=1}^N \Gamma_r$ if $\omega(x)$ satisfies condition 2 (see definition 2.2). We also let $\{\phi_j(x)\}_0^{k+1}$ be functions in $C^\infty(\mathbb{R}^n)$ such that for $j = 0, \dots, k+1$, $0 \leqslant \phi_j(x) \leqslant 1$, $\operatorname{supp} \phi_j(x) \subset \{x \in \mathbb{R}^n \mid |x| < 2\}$, where supp denotes support, while $\phi_0(x) = 1$ for $|x| < 1$ and $\phi_{j-1}(x)\phi_j(x) = \phi_{j-1}(x)$ for $j = 1, \dots, k+1$. Then, choosing δ so that $0 < 2\delta \leqslant \operatorname{dist}\{x^0, \Gamma\}$ if $\omega(x)$ satisfies condition 1 and

$$0 < 2\delta \leqslant \operatorname{dist}\left\{ x^0, \bigcup_{r=0}^N \Gamma_r \right\}$$

if $\omega(x)$ satisfies condition 2, where $\Gamma_0 = \Gamma$, let us now put $\phi_j^\delta(x) = \phi_j(\delta^{-1}(x - x^0))$ for $x \in \Omega$ and $j = 0, \dots, k+1$ and let $R_{j,\delta}(\lambda) = \phi_{j-1}^\delta R(\lambda) \phi_j^\delta$, $R_{j,\delta}^\dagger(\lambda) = \phi_j^\delta R(\lambda) \phi_{j-1}^\delta$ for $j = 1, \dots, k$, where here we use the notation ϕ_j^δ to denote the operator of multiplication induced by the function $\phi_j^\delta(x)$. Now let us observe that, for $1 \leqslant j \leqslant k$,

$$\phi_{j-1}^\delta R(\lambda) = \phi_{j-1}^\delta \phi_j^\delta R(\lambda) = R_{j,\delta}(\lambda) + \phi_{j-1}^\delta [\phi_j^\delta, R(\lambda)]$$

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and that $R(\lambda)\phi_{j-1}^{\delta} = R_{j,\delta}^{\dagger} + [R(\lambda), \phi_j^{\delta}]\phi_{j-1}^{\delta}$, where $[\phi_j^{\delta}, R(\lambda)] = -[R(\lambda), \phi_j^{\delta}]$ denotes the commutator $\phi_j^{\delta}R(\lambda) - R(\lambda)\phi_j^{\delta}$. Hence, if we apply this observation to the expression

$$\phi_0^{\delta}R(\lambda)^q\phi_0^{\delta} = (\phi_0^{\delta}R(\lambda)^k)(R(\lambda)^k\phi_0^{\delta}), \tag{3.7}$$

and if, in the first bracket on the right-hand side of (3.7), we pass through the jth term in the product $R(\lambda)^k$, proceeding from left to right, twice by ϕ_j^{δ} for $j = 1, \ldots, k-1$, while if, in the second bracket, we pass through the jth term in the product $R(\lambda)^k$, proceeding from right to left, twice by ϕ_j^{δ} for $j = 1, \ldots, k-1$, then it is not difficult to verify that

$$\phi_0^{\delta}R(\lambda)^q\phi_0^{\delta} = \left(\prod_{j=1}^{k} R_{j,\delta}(\lambda)\right)\left(\prod_{j=1}^{k} R_{k+1-j,\delta}^{\dagger}(\lambda)\right) + P_{\delta}(\lambda), \tag{3.8}$$

where

$$P_{\delta}(\lambda) = \sum_{r=1}^{k} P_{\delta}^{(r)}(\lambda)(R(\lambda)^k\phi_0^{\delta}) + \sum_{r=1}^{k}\left(\prod_{j=1}^{k} R_{j,\delta}(\lambda)\right)\tilde{P}_{\delta}^{(r)}(\lambda), \tag{3.9}$$

$$P_{\delta}^{(r)}(\lambda) = \left(\prod_{j=1}^{r-1} R_{j,\delta}(\lambda)\right)(\phi_{r-1}^{\delta}[\phi_r^{\delta}, R(\lambda)]\psi_r^{\delta})R(\lambda)^{k-r}, \tag{3.10}$$

$$\tilde{P}_{\delta}^{(r)}(\lambda) = R(\lambda)^{k-r}(\psi_r^{\delta}[R(\lambda), \phi_r^{\delta}]\phi_{r-1}^{\delta})\prod_{j=1}^{r-1} R_{r-j,\delta}^{\dagger}(\lambda), \tag{3.11}$$

and the products $\prod_{j=1}^{r-1} R_{j,\delta}(\lambda)$ and $\prod_{j=1}^{r-1} R_{j,\delta}^{\dagger}(\lambda)$ are defined to be equal to 1 when $r = 1$, while $\psi_r^{\delta} = 1 + \phi_r^{\delta}$ for $r < k$ and is equal to 1 otherwise.

We will now use proposition 3.3 with $\Omega' = \Omega$ and $\ell = 2m$ to derive information about the operator $P_{\delta}(\lambda)$. Accordingly, let us suppose for the remainder of this proof that $|\lambda| \geqslant \max\{\lambda_0, \delta^{-2m}\}$ and concentrate on the term $P_{\delta}^{(r)}(\lambda)$ in the first summation on the right-hand side of (3.9) for some r satisfying $1 \leqslant r \leqslant k$. Then, bearing in mind (3.10), we may make use of (2.1) and (2.3) to deduce that, for $r > 1$ and $1 \leqslant j \leqslant r-1$,

$$\|R_{j,\delta}(\lambda)\|_{L_{p_{k+1-j}}(\Omega)\to W_{p_{k+1-j}}^{2m}(\Omega)} \leqslant C, \quad \|R_{j,\delta}(\lambda)\|_{L_{p_{k+1-j}}(\Omega)\to L_{p_{k+1-j}}(\Omega)} \leqslant C/|\lambda|, \tag{3.12}$$

while, for $r < k$ and $1 \leqslant j \leqslant k-r$,

$$\|R(\lambda)\|_{L_{p_j}(\Omega)\to W_{p_j}^{2m}(\Omega)} \leqslant C, \qquad \|R(\lambda)\|_{L_{p_j}(\Omega)\to L_{p_j}(\Omega)} \leqslant C/|\lambda|. \tag{3.13}$$

Here and in the remainder of this proof, C denotes a generic constant which may vary from inequality to inequality, but which in all cases does not depend upon λ, δ and the indices j and r indicated above, or on any space variables and functions which may appear in the inequality under consideration. Also, if we make use of the fact that $[\phi_r^{\delta}, R(\lambda)] = R(\lambda)[V^{-1}A, \phi_r^{\delta}]R(\lambda)$, then we can likewise show, using the Leibniz rule and observing that $[V^{-1}A, \phi_r^{\delta}]$ is a differential operator of order

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less than $2m$, that

$$\|\phi_{r-1}^{\delta}[\phi_r^{\delta}, R(\lambda)]\psi_r^{\delta}\|_{L_{p_{k+1-r}}(\Omega)\to W^{2m}_{p_{k+1-r}}(\Omega)} \leqslant C/\delta|\lambda|^{1/2m},$$
$$\|\phi_{r-1}^{\delta}[\phi_r^{\delta}, R(\lambda)]\psi_r^{\delta}\|_{L_{p_{k+1-j}}(\Omega)\to L_{p_{k+1-j}}(\Omega)} \leqslant C/\delta|\lambda|^{1+1/2m}.$$

Hence, it follows from (3.10) and proposition 3.3 that $P_{\delta}^{(r)}(\lambda)$ is an integral operator of Hilbert–Schmidt type whose kernel $K_{\delta}^{(r)}(x, y, \lambda)$ satisfies

$$\left(\int_{\Omega} |K_{\delta}^{(r)}(x, y, \lambda)|^2 \, dy\right)^{1/2} \leqslant C\delta^{-1}|\lambda|^{(n/4m)-k-(1/2m)} \quad \text{for } x \in \Omega. \tag{3.14}$$

Likewise, if we concentrate on the term $\tilde{P}_{\delta}^{(r)}(\lambda)$ in the second summation on the right-hand side of (3.9) for some r satisfying $1 \leqslant r \leqslant k$, then, bearing in mind (3.11), we can argue as in the previous paragraph to show that

(i) if $r < k$ and $1 \leqslant j \leqslant k - r$, then the estimates (3.12) hold with $R_{j,\delta}(\lambda)$ replaced by $R(\lambda)$,

(ii) if $r > 1$ and $1 \leqslant j \leqslant r - 1$, then the estimates (3.13) hold with $R(\lambda)$ replaced by $R_{j,\delta}^{\dagger}(\lambda)$, and

(iii)
$$\|\psi_r^{\delta}[R(\lambda), \phi_r^{\delta}]\phi_{r-1}^{\delta}\|_{L_{p_r}(\Omega)\to W^{2m}_{p_r}(\Omega)} \leqslant \frac{C}{\delta|\lambda|^{1/2m}},$$
$$\|\psi_r^{\delta}[R(\lambda), \phi_r^{\delta}]\phi_{r-1}^{\delta}\|_{L_{p_r}(\Omega)\to L_{p_r}(\Omega)} \leqslant \frac{C}{\delta|\lambda|^{1+1/2m}}.$$

Hence, it follows from (3.11) and proposition 3.3 that $\tilde{P}_{\delta}^{(r)}(\lambda)$ is an integral operator of Hilbert–Schmidt type whose kernel $\tilde{K}_{\delta}^{(r)}(x, y, \lambda)$ satisfies (3.14) with $K_{\delta}^{(r)}(x, y, \lambda)$ replaced by $\tilde{K}_{\delta}^{(r)}(x, y, \lambda)$. Furthermore, given the above estimates for $R(\lambda)$ and the $R_{j,\delta}(\lambda)$, it also follows that $R(\lambda)^k\phi_0^{\delta}$ (respectively, $\prod_{j=1}^{k} R_{j,\delta}(\lambda)$) is an integral operator of Hilbert–Schmidt type whose kernel $K_{\delta}^{\dagger}(x, y, \lambda)$ (respectively, $K_{\delta}^{\#}(x, y, \lambda)$) satisfies (3.3) with $\tilde{K}(x, y, \lambda)$ replaced by $K_{\delta}^{\dagger}(x, y, \lambda)$ (respectively, $K_{\delta}^{\#}(x, y, \lambda)$) and with c replaced by C.

In the light of (3.9) and the foregoing results, we can now argue as at the beginning of this proof (see the text preceding this step) to deduce that, for each fixed λ, $P_{\delta}(\lambda)$ is an integral operator in $L_2(\Omega)$ of trace class whose kernel $K_{\delta}(x, y, \lambda)$ satisfies the following conditions:

(i) $K_{\delta}(x, x, \lambda)$ is integrable in Ω and

$$\operatorname{tr} P_{\delta}(\lambda) = \int_{\Omega} K_{\delta}(x, x, \lambda) \, dx;$$

(ii)
$$|K_{\delta}(x, x, \lambda)| \leqslant \delta^{-1}|\lambda|^{(n/2m)-q-(1/2m)}\eta_{\delta}(x, \lambda) \quad \text{for almost every } x \in \Omega, \tag{3.15}$$

where $\eta_{\delta}(x, \lambda)$ is a real-valued, non-negative function in $L_2(\Omega)$ satisfying

$$\int_{\Omega} \eta_{\delta}(x, \lambda)^2 \, dx \leqslant C.$$

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STEP 2. In order to obtain some further results concerning the local behaviour of $K(x,x,\lambda)$, we shall need to derive some results concerning constant coefficient operators. Accordingly, with x^0 as defined in step 1, let $A_0(D) = \sum_{|\alpha|=2m} a_\alpha(x^0)D^\alpha$ and let $\mathcal{A}_p$ denote the operator in $L_p(\mathbb{R}^n)$ that acts as $A_0(D)$ and has domain $W_p^{2m}(\mathbb{R}^n)$. Also let $\mathcal{V}_p$ denote the operator of multiplication induced on $L_p(\mathbb{R}^n)$ by $\omega(x^0)$. Then it is well known (cf. [6, proposition 2.5]) that, for those values of λ under consideration here, λ belongs to the resolvent set of $\mathcal{V}_p^{-1}\mathcal{A}_p$ and the estimate

$$|||S_p(\lambda)f|||_{2m,p,\mathbb{R}^n} \leqslant c_p\|f\|_{0,p,\mathbb{R}^n} \tag{3.16}$$

holds for every $f \in L_p(\mathbb{R}^n)$, where $S_p(\lambda)$ denotes the resolvent of $\mathcal{V}_p^{-1}\mathcal{A}_p$ and the constant c_p does not depend upon x^0, f and λ. We shall henceforth write $\mathcal{A}$ for $\mathcal{A}_2$, $\mathcal{V}$ for $\mathcal{V}_2$ and $S(\lambda)$ for $S_2(\lambda)$, where we note for later use that $S_p(\lambda)f = S(\lambda)f$ for $f \in L_p(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$. Note also that $S(\lambda)^q$ is an integral operator in $L_2(\mathbb{R}^n)$ with kernel

$$G_0(x,y,\lambda) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i(x-y)\cdot\xi}}{[\omega(x^0)^{-1}a_0(x^0,\xi)-\lambda]^q}\,d\xi, \tag{3.17}$$

which is continuous in $\mathbb{R}^n \times \mathbb{R}^n$ for each fixed λ and here satisfies the estimate

$$|G_0(x,y,\lambda)| \leqslant C|\lambda|^{(n/2m)-q}, \tag{3.18}$$

where the centred dot denotes the inner product in $\mathbb{R}^n$ and C denotes the generic constant introduced in step 1.

Next let $A_0^*(D)$ denote the formal adjoint of $A_0(D)$ and let $\mathcal{A}_p^*$ denote the operator in $L_p(\mathbb{R}^n)$ that acts as $A_0^*(D)$ and has the domain $W_p^{2m}(\mathbb{R}^n)$. Also let $S_p^*(\lambda)$ denote the resolvent of $\mathcal{V}_p^{-1}\mathcal{A}_p^*$, where we note that, for the values of λ under consideration here, $S_p^*(\bar\lambda)$ has properties analogous to those of $S_p(\lambda)$ cited above and, in particular, (3.16) holds with $S_p(\lambda)$ replaced by $S_p^*(\bar\lambda)$. Note also that $\mathcal{A}_{p'}^*$ and $S_{p'}^*(\bar\lambda)$ are the adjoints of $\mathcal{A}_p$ and $S_p(\lambda)$, respectively, in the sense of Banach space operators, where $p' = p/(p-1)$, and we will henceforth use '$*$' to denote adjoint in this sense. Then, recalling the definitions of δ and the ϕ_j^δ introduced in step 1, but assuming for the remainder of step 2 that the functions $\phi_j^\delta(x)$ have been extended to all of $\mathbb{R}^n$ by 0 and that the ϕ_j^δ are now multiplication operators acting on function spaces over $\mathbb{R}^n$, let us next set $S_{j,\delta}(\lambda) = \phi_{j-1}^\delta S(\lambda)\phi_j^\delta$, $S_{j,\delta}^{\#}(\bar\lambda) = \phi_{j-1}^\delta S^*(\bar\lambda)\phi_j^\delta$ and $S_{j,\delta}^{\dagger}(\lambda) = \phi_j^\delta S(\lambda)\phi_{j-1}^\delta$ for $j = 1,\dots,k$, where $S^*(\bar\lambda) = S_2^*(\bar\lambda)$. Then, by arguing as in step 1 we can readily show that

$$\begin{aligned}\phi_0^\delta S(\lambda)^q\phi_0^\delta &= (\phi_0^\delta S(\lambda)^k)(\phi_0^\delta S^*(\bar\lambda)^k)^*\phi_{k+1}^\delta \\ &= \Big(\prod_{j=1}^k S_{j,\delta}(\lambda)\Big)\Big(\prod_{j=1}^k S_{j,\delta}^{\#}(\bar\lambda)\Big)^*\phi_{k+1}^\delta + Q_\delta(\lambda)\phi_{k+1}^\delta,\end{aligned} \tag{3.19}$$

where

$$Q_\delta(\lambda) = \sum_{r=1}^k Q_\delta^{(r)}(\lambda)(\phi_0^\delta S^*(\bar\lambda)^k)^* + \sum_{r=1}^k\Big(\prod_{j=1}^k S_{j,\delta}(\lambda)\Big)(\tilde Q_\delta^{(r)}(\lambda))^*, \tag{3.20}$$

$$Q_\delta^{(r)}(\lambda) = \Big(\prod_{j=1}^{r-1} S_{j,\delta}(\lambda)\Big)(\phi_{r-1}^\delta[\phi_r^\delta, S(\lambda)]\psi_r^\delta)S(\lambda)^{k-r}, \tag{3.21}$$

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$$\tilde{Q}_\delta^{(r)}(\lambda) = \left(\prod_{j=1}^{r-1} S_{j,\delta}^{\#}(\bar{\lambda})\right)(\phi_{r-1}^\delta[\phi_r^\delta, S^*(\bar{\lambda})]\psi_r^\delta)S^*(\bar{\lambda})^{k-r}. \tag{3.22}$$

We will now use proposition 3.3, with $\Omega' = \mathbb{R}^n$ and $\ell = 2m$, to derive information concerning the operator $Q_\delta(\lambda)$. Accordingly, let us fix our attention upon the term $Q_\delta^{(r)}(\lambda)$ in the first summation on the right-hand side of (3.20) for some r satisfying $1 \leqslant r \leqslant k$. Then, bearing in mind (3.21), we may make use of (2.3) and (3.16) to deduce that, for $r > 1$ and $1 \leqslant j \leqslant r-1$,

$$\left.\begin{aligned} \|S_{j,\delta}(\lambda)\|_{L_{p_{k+1-j}}(\mathbb{R}^n)\to W^{2m}_{p_{k+1-j}}(\mathbb{R}^n)} &\leqslant C, \\ \|S_{j,\delta}(\lambda)\|_{L_{p_{k+1-j}}(\mathbb{R}^n)\to L_{p_{k+1-j}}(\mathbb{R}^n)} &\leqslant \frac{C}{|\lambda|}, \end{aligned}\right\} \tag{3.23}$$

while, for $r < k$ and $1 \leqslant j \leqslant k-r$,

$$\|S(\lambda)\|_{L_{p_j}(\mathbb{R}^n)\to W^{2m}_{p_j}(\mathbb{R}^n)} \leqslant C, \qquad \|S(\lambda)\|_{L_{p_j}(\mathbb{R}^n)\to L_{p_j}(\mathbb{R}^n)} \leqslant \frac{C}{|\lambda|},$$

and, lastly,

$$\begin{aligned} \|\phi_{r-1}^\delta[\phi_r^\delta, S(\lambda)]\psi_r^\delta\|_{L_{p_{k+1-r}}(\mathbb{R}^n)\to W^{2m}_{p_{k+1-r}}(\mathbb{R}^n)} &\leqslant C/\delta|\lambda|^{1/2m}, \\ \|\phi_{r-1}^\delta[\phi_r^\delta, S(\lambda)]\psi_r^\delta\|_{L_{p_{k+1-r}}(\mathbb{R}^n)\to L_{p_{k+1-r}}(\mathbb{R}^n)} &\leqslant C/\delta|\lambda|^{1+1/2m}. \end{aligned}$$

Hence, it follows from (3.21) and proposition 3.3 that $Q_\delta^{(r)}(\lambda)$ is an integral operator whose kernel $G_\delta^{(r)}(x,y,\lambda)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$ and has the following properties: for each fixed λ, $G_\delta^{(r)}(x,\cdot,\lambda)$ is a continuous function of x for $x \in \mathbb{R}^n$ with values in $L_2(\mathbb{R}^n)$ and here satisfies the estimate

$$\left(\int_{\mathbb{R}^n} |G_\delta^{(r)}(x,y,\lambda)|^2\,dy\right)^{1/2} \leqslant C\delta^{-1}|\lambda|^{(n/4m)-k-(1/2m)}. \tag{3.24}$$

If we now concentrate on the term $\tilde{Q}_\delta^{(r)}(\lambda)$ in the second summation on the right-hand side of (3.20) for some r satisfying $1 \leqslant r \leqslant k$ and bear in mind what was said above about $S^*(\bar{\lambda})$, then we can make use of (3.22) in the same way as we did with (3.21), above, to show that $\tilde{Q}_\delta^{(r)}(\lambda)$ is an integral operator whose kernel $\tilde{G}_\delta^{(r)}(x,y,\lambda)$ has properties completely analogous to those of $G_\delta^{(r)}(x,y,\lambda)$ cited above and, in particular, (3.24) holds with $G_\delta^{(r)}(x,y,\lambda)$ replaced by $\tilde{G}_\delta^{(r)}(x,y,\lambda)$. Likewise, we can show that $\phi_0^\delta S^*(\bar{\lambda})^k$ and $\prod_{j=1}^k S_{j,\delta}(\lambda)$ are integral operators whose kernels $G_\delta^\dagger(x,y,\lambda)$ and $G_\delta^{\#}(x,y,\lambda)$, respectively, have properties completely analogous to those of $G_\delta^{(r)}(x,y,\lambda)$, except that now we are to replace (3.24) by the estimate

$$\left(\int_{\mathbb{R}^n} |G_\delta^\dagger(x,y,\lambda)|^2\,dy\right)^{1/2} + \left(\int_{\mathbb{R}^n} |G_\delta^{\#}(x,y,\lambda)|^2\,dy\right)^{1/2} \leqslant C|\lambda|^{(n/4m)-k}.$$

It follows immediately from the foregoing results and a standard argument (cf. [3], [4, theorem 12.21], [6, lemma 5.3]) that, for each fixed λ, $Q_\delta(\lambda)\phi_{k+1}^\delta$ is an integral operator whose kernel $G_\delta(x,y,\lambda)$ is continuous in $\mathbb{R}^n \times \mathbb{R}^n$ and in this set the estimate

$$|G_\delta(x,y,\lambda)| \leqslant C\delta^{-1}|\lambda|^{(n/2m)-q-(1/2m)} \tag{3.25}$$

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holds. Observe also from (3.19)—(3.22) that if $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^n)$ such that, for at least one j, $\operatorname{supp}\psi_j \cap \Omega = \emptyset$, then

$$0 = (Q_\delta(\lambda)\phi_{k+1}^\delta \psi_1, \psi_2) = \int_{\mathbb{R}^n \times \mathbb{R}^n} G_\delta(x, y, \lambda)\psi_1(y)\bar{\psi}_2(x)\,dx\,dy,$$

where $(\cdot\,,\cdot)$ denotes the inner product in $L_2(\mathbb{R}^n)$ and an overline denotes complex conjugation. Thus, $Q_\delta(\lambda)\phi_{k+1}^\delta$ is supported in Ω, $Q_\delta(\lambda)\phi_{k+1}^\delta f|_{\mathbb{R}^n \setminus \Omega} = 0$ for every $f \in L_2(\mathbb{R}^n)$, and $G_\delta(x, y, \lambda) = 0$ when x or y lie outside of Ω. Therefore, we shall from now on suppose, as before, that the ϕ_j^δ operate on function spaces over Ω and consider $Q_\delta(\lambda)\phi_{k+1}^\delta$ as an integral operator on $L_2(\Omega)$ for each fixed λ (thus, for $f \in L_2(\Omega)$, we are to interpret $Q_\delta(\lambda)\phi_{k+1}^\delta f$ as $r_\Omega Q_\delta(\lambda) i_\Omega \phi_{k+1}^\delta f$, where i_Ω denotes the extension operator mapping function spaces over Ω into function spaces over $\mathbb{R}^n$, where the extension is by 0 outside of Ω, while r_Ω denotes the restriction operator mapping function spaces over $\mathbb{R}^n$ into function spaces over Ω).

Step 3. We will now use the results of steps 1 and 2 to derive our main results concerning the local behaviour of $K(x, x, \lambda)$. Accordingly, we observe from (3.8) and (3.19) that

$$\phi_0^\delta R(\lambda)^q \phi_0^\delta = \phi_0^\delta S(\lambda)^q \phi_0^\delta + P_\delta(\lambda) - Q_\delta(\lambda)\phi_{k+1}^\delta - F_\delta(\lambda), \tag{3.26}$$

where

$$F_\delta(\lambda) = \Big(\prod_{j=1}^k S_{j,\delta}(\lambda)\Big)\Big(\prod_{j=1}^k S_{k+1-j,\delta}^\dagger(\lambda)\Big) - \Big(\prod_{j=1}^k R_{j,\delta}(\lambda)\Big)\Big(\prod_{j=1}^k R_{k+1-j,\delta}^\dagger(\lambda)\Big).$$

Note that, unlike the supposition in step 2, we now consider the operators $S_{j,\delta}(\lambda) = \phi_{j-1}^\delta S(\lambda)\phi_j^\delta$ and $S_{j,\delta}^\dagger(\lambda) = \phi_j^\delta S(\lambda)\phi_{j-1}^\delta$ as operators acting in $L_p(\Omega)$ (thus, we may here interpret the mappings ϕ_j^δ either as

$$i_\Omega \phi_j^\delta : L_p(\Omega) \to L_p(\mathbb{R}^n)$$

or as

$$\phi_j^\delta r_\Omega : W_p^{2m}(\mathbb{R}^n) \to W_p^{2m}(\Omega),$$

where i_Ω and r_Ω are defined in the last paragraph of step 2). Likewise, bearing in mind the final paragraph of step 2, we now consider $Q_\delta(\lambda)\phi_{k+1}^\delta$ as an integral operator acting in $L_2(\Omega)$.

For later use we require some further terminology. Accordingly, let h denote an arbitrary positive number and

(i) if $\omega(x)$ satisfies condition 1, then let $\Phi_\alpha(h)$ denote the supremum of

$$|\omega(x)^{-1}a_\alpha(x) - \omega(y)^{-1}a_\alpha(y)|$$

over the set $\{x, y \in \Omega \mid |x-y| \leqslant h\}$ for each multi-index α satisfying $|\alpha| = 2m$, and let $\Phi(h) = \max_{|\alpha|=2m} \Phi_\alpha(h)$,

(ii) if $\omega(x)$ satisfies condition 2, then let $\Phi_{j,\alpha}(h)$ denote the supremum of

$$|\omega(x)^{-1}a_\alpha(x) - \omega(y)^{-1}a_\alpha(y)|$$

over the set $\{x, y \in \Omega_j \mid |x - y| \leqslant h\}$ for each j and multi-index α satisfying $0 \leqslant j \leqslant N$, $|\alpha| = 2m$, and let $\Phi(h) = \max_{0\leqslant j\leqslant N} \max_{|\alpha|=2m} \Phi_{j,\alpha}(h)$.

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Next let us observe that

$$F_\delta(\lambda) = F_{\delta,1}(\lambda) + F_{\delta,2}(\lambda), \tag{3.27}$$

where

$$F_{\delta,1}(\lambda) = \sum_{r=1}^{k} F_{\delta,1}^{(r)}(\lambda)\Big(\prod_{j=1}^{k} S_{j,\delta}^{\#}(\bar\lambda)\Big)^*, \qquad F_{\delta,2}(\lambda) = \sum_{r=1}^{k}\Big(\prod_{j=1}^{k} R_{j,\delta}(\lambda)\Big)F_{\delta,2}^{(r)}(\lambda), \tag{3.28}$$

$$F_{\delta,1}^{(r)}(\lambda) = \Big(\prod_{j=1}^{r-1} R_{j,\delta}(\lambda)\Big)(S_{r,\delta}(\lambda) - R_{r,\delta}(\lambda))\Big(\prod_{j=r+1}^{k} S_{j,\delta}(\lambda)\Big), \tag{3.29}$$

$$F_{\delta,2}^{(r)}(\lambda) = \Big(\prod_{j=1}^{r-1} R_{k+1-j,\delta}^{\dagger}(\lambda)\Big)(S_{k+1-r,\delta}^{\dagger}(\lambda) - R_{k+1-r,\delta}^{\dagger}(\lambda))\Big(\prod_{j=r+1}^{k} S_{k+1-j,\delta}^{\dagger}(\lambda)\Big). \tag{3.30}$$

Let us now use proposition 3.3 with $\Omega' = \Omega$ and $\ell = 2m$ to derive some information concerning the operators $F_{\delta,1}(\lambda)$ and $F_{\delta,2}(\lambda)$. To this end let us firstly fix our attention upon the term $F_{\delta,1}^{(r)}(\lambda)$ in the first summation in (3.28) for some r satisfying $1 \leqslant r \leqslant k$. Then, bearing in mind (3.29), we can argue as we did in step 2 to show that, for $r < k$ and $r+1 \leqslant j \leqslant k$, (3.23) holds with p_{k+1-j} and $\mathbb{R}^n$ replaced by p_j and Ω, respectively, while if $r > 1$ and $1 \leqslant j \leqslant r-1$, then (3.12) holds. Additionally, if we make use of the fact that $S_{r,\delta}(\lambda) - R_{r,\delta}(\lambda) = \phi_{r-1}^{\delta} R(\lambda)((V^{-1}A - \mathcal{V}^{-1}\mathcal{A})\phi_{k+1}^{\delta} + [\mathcal{V}^{-1}\mathcal{A}, \phi_{k+1}^{\delta}])S(\lambda)\phi_r^{\delta}$, then we can make use of (2.1), (2.3) and (3.16) to show that

$$\|S_{r,\delta}(\lambda) - R_{r,\delta}(\lambda)\|_{L_{p_{k+1-r}}(\Omega)\to W^{2m}_{p_{k+1-r}}(\Omega)} \leqslant C(\Phi(2\delta) + \delta^{-1}|\lambda|^{-1/2m} + |\lambda|^{-1/2m})$$

and

$$\|S_{r,\delta}(\lambda) - R_{r,\delta}(\lambda)\|_{L_{p_{k+1-r}}(\Omega)\to L_{p_{k+1-r}}(\Omega)} \leqslant \frac{C(\Phi(2\delta) + \delta^{-1}|\lambda|^{-1/2m} + |\lambda|^{-1/2m})}{|\lambda|}.$$

Hence, it follows from (3.29) and proposition 3.3 that $F_{\delta,1}^{(r)}(\lambda)$ is an integral operator of Hilbert–Schmidt type whose kernel $H_{\delta,1}^{(r)}(x,y,\lambda)$ has the following properties: $H_{\delta,1}^{(r)}(x,\cdot,\lambda) \in L_2(\Omega)$ for fixed (x,λ) while, for fixed λ, $H_{\delta,1}^{(r)}(x,\cdot,\lambda)$ is a continuous function of x in Ω with values in $L_2(\Omega)$ and satisfies here the estimate

$$\Big(\int_\Omega |H_{\delta,1}^{(r)}(x,y,\lambda)|^2\,dy\Big)^{1/2} \leqslant C(\Phi(2\delta) + \delta^{-1}|\lambda|^{-1/2m} + |\lambda|^{-1/2m})|\lambda|^{(n/4m)-k}. \tag{3.31}$$

Likewise, we can make use of $S_{j,\delta}^{\#}(\bar\lambda)$ as we did before when dealing with equation (3.22) (but recalling that there the $S_{j,\delta}^{\#}(\bar\lambda)$ were considered as operators acting on function spaces over $\mathbb{R}^n$) to show that

$$\prod_{j=1}^{\ell} S_{j,\delta}^{\#}(\bar\lambda)$$

is an integral operator of Hilbert–Schmidt type whose kernel $H_\delta^{\#}(x,y,\lambda)$ has properties analogous to those of $H_{\delta,1}^{(r)}(x,y,\lambda)$, except that now we are to replace (3.31)

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by (3.3), with $\tilde{K}(x,y,\lambda)$ there replaced by $H_\delta^{\#}(x,y,\lambda)$. In light of the foregoing results, (3.28) and a standard argument (see the final paragraph of step 2), we conclude that, for each fixed λ, $F_{\delta,1}(\lambda)$ is an integral operator of trace class in $L_2(\Omega)$ whose kernel $H_{\delta,1}(x,y,\lambda)$ is continuous in $\Omega \times \Omega$ and in this set the estimate

$$|H_{\delta,1}(x,y,\lambda)| \leqslant C(\Phi(2\delta) + \delta^{-1}|\lambda|^{-1/2m} + |\lambda|^{-1/2m})|\lambda|^{(n/2m)-q} \tag{3.32}$$

holds.

Turning next to the term $F_{\delta,2}^{(r)}(\lambda)$ in the second summation in (3.28) for some r satisfying $1 \leqslant r \leqslant k$ and, bearing in mind (3.30), we can argue with $F_{\delta,2}^{(r)}(\lambda)$ as we argued with $F_{\delta,1}^{(r)}(\lambda)$ above to show that $F_{\delta,2}^{(r)}(\lambda)$ is an integral operator of Hilbert–Schmidt type whose kernel $H_{\delta,2}^{(r)}(x,y,\lambda)$ has properties analogous to those of $H_{\delta,1}^{(r)}(x,y,\lambda)$ that were cited above and, in particular, (3.31) holds with $H_{\delta,1}^{(r)}(x,y,\lambda)$ replaced by $H_{\delta,2}^{(r)}(x,y,\lambda)$. Furthermore, by arguing as we did before with $\prod_{j=1}^{k} R_{j,\delta}(\lambda)$ (see (3.9)), we can readily show that $\prod_{j=1}^{k} R_{j,\delta}(\lambda)$ is an integral operator of Hilbert–Schmidt type whose kernel $H_\delta^{\dagger}(x,y,\lambda)$ has properties completely analogous to those of $H_\delta^{\#}(x,y,\lambda)$ that were cited above. Hence, it follows from these results, (3.28) and the arguments used in the text preceding step 1 that, for fixed λ,

$$\left(\prod_{j=1}^{k} R_{j,\delta}(\lambda)\right) F_{\delta,2}^{(r)}(\lambda)$$

is an integral operator in $L_2(\Omega)$ of trace class whose kernel

$$\tilde{H}_\delta^{(r)}(x,y,\lambda) = \int_\Omega H_\delta^{\dagger}(x,z,\lambda) H_{\delta,2}^{(r)}(z,y,\lambda)\,\mathrm{d}z$$

has the property that $\tilde{H}_\delta^{(r)}(x,x,\lambda)$ is integrable in Ω.

Next let $B_{2\delta}(x^0) = \{x \in \mathbb{R}^n \mid |x - x^0| < 2\delta\}$, let $\psi_1, \psi_2 \in C_0^\infty(\Omega)$ such that, for at least one j, $\operatorname{supp}\psi_j \cap B_{2\delta}(x^0) = \emptyset$, and let $(\cdot\,,\cdot)$ denote the inner product in $L_2(\Omega)$. It then follows from the definitions that, for fixed λ,

$$0 = (F_{\delta,2}^{(r)}(\lambda)\psi_1, \psi_2) = \int_{\Omega\times\Omega} H_{\delta,2}^{(r)}(x,y,\lambda)\psi_1(y)\bar{\psi}_2(x)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$0 = \left(\left(\prod_{j=1}^{k} R_{j,\delta}(\lambda)\right)\psi_1, \psi_2\right) = \int_{\Omega\times\Omega} H_\delta^{\dagger}(x,y,\lambda)\psi_1(y)\bar{\psi}_2(x)\,\mathrm{d}x\,\mathrm{d}y.$$

Hence, we conclude that $H_{\delta,2}^{(r)}(x,y,\lambda)$ and $H_\delta^{\dagger}(x,y,\lambda)$ both vanish when x or y lies outside of $B_{2\delta}(x^0)$. Consequently, in light of the foregoing results we see that, for fixed λ and for any compact set X of Ω, we have

$$\begin{aligned}
&\int_X |\tilde{H}_\delta^{(r)}(x,x,\lambda)|\,\mathrm{d}x \\
&\quad\leqslant \int_{B_{2\delta}(x^0)} \left(\int_{B_{2\delta}(x^0)} |H_\delta^{\dagger}(x,z,\lambda)|^2\,\mathrm{d}z\right)^{1/2} \left(\int_{B_{2\delta}(x^0)} |H_{\delta,2}^{(r)}(z,x,\lambda)|^2\,\mathrm{d}z\right)^{1/2} \mathrm{d}x \\
&\quad\leqslant C\delta^n(\Phi(2\delta) + \delta^{-1}|\lambda|^{-1/2m} + |\lambda|^{-1/2m})|\lambda|^{(n/2m)-q},
\end{aligned} \tag{3.33}$$

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where we have used the Schwarz inequality. Thus, it follows from these results and (3.28) that $F_{\delta,2}(\lambda)$ is an integral operator in $L_2(\Omega)$ of trace class whose kernel $H_{\delta,2}(x,y,\lambda)$ has the property that $H_{\delta,2}(x,x,\lambda)$ is integrable in Ω and with X as above we see that

$$\int_X |H_{\delta,2}(x,x,\lambda)|\,\mathrm{d}x$$

is majorized by the expression on the right-hand side of (3.33).

As a consequence of the foregoing results we are now in a position to finally state our main result concerning the local behaviour of $K(x,x,\lambda)$. Indeed, bearing in mind (3.17) and the definition of $c_q(x)$ given in the statement of the theorem, it follows from (3.26), (3.27) and the foregoing results that

$$\begin{aligned} K(x,x,\lambda) = (-\lambda)^{(n/2m)-q}c_q(x) + (-\lambda)^{(n/2m)-q}(c_q(x^0) - c_q(x)) \\ + K_\delta(x,x,\lambda) - G_\delta(x,x,\lambda) - H_{\delta,1}(x,x,\lambda) - H_{\delta,2}(x,x,\lambda) \end{aligned} \tag{3.34}$$

almost everywhere in the set $\{x \in \Omega \mid |x - x^0| \leqslant \delta\}$.

STEP 4. We will now use (3.34) to complete the proof of the theorem. Accordingly, let ε be an arbitrary positive number. Then, bearing in mind (3.6) and (3.18), let us choose the compact set X, where $X \subset \Omega$ if $\omega(x)$ satisfies condition 1 and $X \subset \bigcup_{r=0}^N \Omega_r$ if $\omega(x)$ satisfies condition 2, such that

$$\int_{\Omega\setminus X} (|K(x,x,\lambda)| + |G(x,x,\lambda)|)\,\mathrm{d}x < \varepsilon|\lambda|^{(n/2m)-q}, \tag{3.35}$$

where $G(x,x,\lambda)$ denotes the expression on the right-hand side of (3.17) with $y = x$ and with x^0 replaced by x (so that $G(x,x,\lambda) = (-\lambda)^{(n/2m)-q}c_q(x)$). Let δ denote an arbitrary positive number satisfying $0 < 3\delta \leqslant \operatorname{dist}\{X, \Gamma\}$ if $\omega(x)$ satisfies condition 1 and $0 < 3\delta \leqslant \operatorname{dist}\{X, \bigcup_{r=0}^N \Gamma_r\}$ if $\omega(x)$ satisfies condition 2, where $\Gamma_0 = \Gamma$. Then we can cover X by the sequence of open cubes $\{U_\nu\}_1^{n_0}$ in $\mathbb{R}^n$ with sides parallel to the coordinate axes and of length $2\delta/n^{1/2}$ such that each point of X is contained in at most 2^n cubes. We shall denote by $x^{(\nu)}$ the centre of U_ν for $\nu = 1, \ldots, n_0$, and where we note that $n_0 \leqslant C\delta^{-n}$. Lastly, let $\{\psi_\nu\}_1^{n_0}$ be a partition of unity subordinate to the covering $\{U_\nu\}_1^{n_0}$ of X.

Bearing in mind (3.34), it now follows from (3.15), (3.25), (3.32), (3.33) (with $\int_X |\tilde{H}_\delta^{(r)}(x,x,\lambda)|\,\mathrm{d}x$ replaced by $\int_X |H_{\delta,2}(x,x,\lambda)|\,\mathrm{d}x$) and (3.35) that

$$\begin{aligned} |\lambda|^{q-(n/2m)} &\left| \int_\Omega K(x,x,\lambda)\,\mathrm{d}x - (-\lambda)^{(n/2m)-q}c_q \right| \\ &\leqslant \varepsilon + \sum_{\nu=1}^{n_0} \int_{U_\nu\cap X} \psi_\nu(x)|c_q(x) - c_q(x^{(\nu)})|\,\mathrm{d}x \\ &\quad + \delta^{-1}|\lambda|^{-1/2m} \sum_{\nu=1}^{n_0} \int_{U_\nu\cap X} \psi_\nu(x)\eta_\delta^{(\nu)}(x,\lambda)\,\mathrm{d}x \\ &\quad + C(\Phi(2\delta) + \delta^{-1}|\lambda|^{-1/2m} + |\lambda|^{-1/2m})(|\Omega| + \delta^n n_0), \end{aligned} \tag{3.36}$$

where the $\eta_\delta^{(\nu)}(x,\lambda)$ have the same properties as $\eta_\delta(x,\lambda)$ of (3.15). Now observe that, from the uniform continuity of $c_q(x)$, we can always choose δ to be suffi-

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ciently small that $|c_q(x) - c_q(x^{(\nu)})| < \varepsilon/|\Omega|$ for $x \in U_\nu$, $\nu = 1, \dots, n_0$. Furthermore, since $\delta^n n_0 \leqslant C$, we can choose δ to be even smaller if necessary, so that $C\Phi(2\delta)(|\Omega| + \delta^n n_0) < \varepsilon$. Then, with δ thus chosen, and observing that

$$\sum_{\nu=1}^{n_0} \int_{U_\nu \cap X} \psi_\nu(x) \eta_\delta^{(\nu)}(x, \lambda)\, \mathrm{d}x \leqslant C\delta^{n/2} n_0,$$

we can now choose λ_0' to be sufficiently large that, for $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \geqslant \lambda_0'$, the expression on the right-hand side of (3.36) is dominated by 5ε. Thus, because ε was chosen arbitrarily and due to (3.5), the proof of the theorem is complete. □

4. Eigenvalue asymptotics

The object of this section is to use theorem 3.1 to derive results concerning the asymptotic behaviour of the eigenvalues of $V^{-1}A$. However, since theorem 3.1 has also been derived in [6], but under much stronger assumptions than are supposed here, and since all of the theorems established in [6] pertaining to eigenvalue asymptotics depend only upon the validity of theorem 3.1 and not upon the smoothness assumptions required for its validity, all the results of [6] pertaining to eigenvalue asymptotics carry over in full force to the problem under consideration here. Thus, we shall content ourselves here with merely stating the results and briefly outlining the proofs.

Accordingly, throughout this section it will be supposed that the boundary problem (1.1), (1.2) is elliptic with parameter in $\mathcal{L}$. Then we know from § 2 that $V^{-1}A$ has a discrete spectrum, while it follows from theorem 2.4 that at most a finite number of eigenvalues of $V^{-1}A$ lie in $\mathcal{L}$. Consequently, by introducing a shift in the spectral parameter if necessary, we see that there is no loss of generality in supposing henceforth that 0 is in the resolvent set of $V^{-1}A$. Also, assuming that the spectrum of $V^{-1}A$ is not empty, let $\{\lambda_j\}_{j \geqslant 1}$ denote the sequence of eigenvalues of $V^{-1}A$ arranged so that $|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$, where each eigenvalue is repeated according to its multiplicity, and let us introduce the counting function $N(t) = \max\{j \mid |\lambda_j| \leqslant t\}$. Now let us observe that if the boundary problem (1.1), (1.2) is elliptic with parameter in $\mathcal{L}$ and the constant $\rho(|\rho| = 1)$ is suitably chosen, then the boundary problem (1.1)′, (1.2) is elliptic with parameter in $\mathcal{L}_\theta$, where (1.1)′ is obtained from (1.1) by replacing $A(x, D)$ by $\rho A(x, D)$ and replacing λ by $\rho\lambda$. Furthermore, $N(t)$ is invariant under this rotation. Thus, it follows that, when obtaining estimates for $N(t)$, there is no loss of generality in supposing that the sector $\mathcal{L}$ of definition 2.3 coincides with the sector $\mathcal{L}_\theta$ of theorem 3.1 and that the sector $\mathcal{L}_\theta$ is contained in the resolvent set of $V^{-1}A$. Hence, if we bear this in mind and, referring to (3.2), if we define d_q by $c_q = b_{n/2m,q} d_q$, where $b_{t,q} = tB(t, q - t)$ and $B(t, s)$ is the Beta function, then we have the following result.

THEOREM 4.1. *Suppose that the boundary problem (1.1), (1.2) is elliptic with parameter in $\mathcal{L}_\theta$. Then $\overline{\lim}_{t\to\infty} N(t)t^{-n/2m} < \infty$, while, if $d_q \neq 0$, then*

$$\underline{\lim_{t\to\infty}}\, N(t)t^{-n/2m} > 0.$$

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Proof. We have shown in the proof of theorem 3.1 that, for $\lambda \in \mathcal{L}_\theta$ with $|\lambda|$ sufficiently large, $R(\lambda)^k$ is a Hilbert–Schmidt operator whose kernel $\tilde{K}(x, y, \lambda)$ satisfies (3.4). Hence, it follows that

$$S_q(\mu) = \int_0^\infty (t+\mu)^{-q}\, \mathrm{d}N(t) \leqslant \sum_{j \geqslant 1} \frac{1}{|\lambda_j + \mu|^q} \leqslant c\mu^{(n/2m)-q}$$

for all μ sufficiently large, where the constant c does not depend upon μ, and hence we conclude that

$$\overline{\lim_{\mu \to \infty}}\, S_q(\mu)\mu^{q-(n/2m)} \leqslant c. \tag{4.1}$$

The first assertion of the theorem is an immediate consequence of (4.1) and the results of [5]. Furthermore, it is shown in [5] that if $d_q \neq 0$, then (4.1) implies that

$$\underline{\lim_{\mu \to \infty}}\, S_q(\mu)\mu^{q-(n/2m)} \geqslant c \tag{4.2}$$

and, as pointed out in [5], (4.1) and (4.2) imply the second assertion of the theorem. □

REMARK 4.2. More precise information concerning $\overline{\lim}_{t\to\infty} N(t)t^{-n/2m}$ has been established in [6].

As a consequence of theorem 4.1, we have the following result.

THEOREM 4.3. *If $d_q \neq 0$, then $N(t) \asymp t^{n/2m}$ as $t \to \infty$.*

This means that the ratio $N(t)/t^{n/2m}$ lies between positive numbers for large t.

If $\omega(x)$ satisfies condition 1 or if $\omega(x)$ satisfies condition 2 and is of one sign in $\bigcup_{k=0}^N \Omega_k$, then for brevity we will henceforth say that $\omega(x)$ is of one sign almost everywhere in Ω and write $\omega(x) > 0$ almost everywhere in Ω or $\omega(x) < 0$ almost everywhere in Ω to indicate whether the sign of $\omega(x)$ is positive or negative. Then theorem 3.1 can also be used to sharpen the result of theorem 4.3 for the case where $a_0(x, \xi) > 0$ in $\bar{\Omega} \times \mathbb{R}^n \setminus \{0\}$ and $\omega(x)$ is of one sign almost everywhere in Ω. Indeed, let us suppose that this case holds, and in particular that $\omega(x) > 0$ almost everywhere in Ω. Then it follows from theorem 3.1 and the definitions of the terms involved that $d_q > 0$, while it is clear that condition (i) of definition 2.3 is satisfied for every ray $\mathcal{L} = \mathcal{L}(\theta) = \overline{\{\lambda \in \mathbb{C} \mid \arg \lambda = \theta\}}$ for which $\theta \neq 2k\pi$, $k \in \mathbb{Z}$, but is not satisfied for $\theta = 2k\pi$. Bearing these facts in mind, we now have the following result.

THEOREM 4.4. *Suppose that $a_0(x, \xi) > 0$ in $\bar{\Omega} \times \mathbb{R}^n \setminus \{0\}$ and that $\omega(x)$ is of one sign almost everywhere in Ω. Suppose also that the boundary problem (1.1), (1.2) is either elliptic with parameter in $\mathcal{L}(\theta)$ for every $\theta \neq 2k\pi$, $k \in \mathbb{Z}$, if $\omega(x) > 0$ almost everywhere in Ω, or elliptic with parameter in $\mathcal{L}(\theta)$ for every $\theta \neq (2k+1)\pi$ if $\omega(x) < 0$ almost everywhere in Ω. Then*

$$N(t) = \kappa t^{n/2m} + o(t^{n/2m}) \quad \text{as } t \to \infty, \tag{4.3}$$

where

$$\kappa = \frac{1}{(2\pi)^n} \int_\Omega |\omega(x)|^{n/2m}\, \mathrm{d}x \int_{a_0(x,\xi)<1} \mathrm{d}\xi.$$

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Proof. We shall prove only the theorem for the case $\omega(x) > 0$ almost everywhere in Ω, since the proof in the case of negative $\omega(x)$ can be obtained in the same way by replacing $\omega(x)$ and λ in (1.1) by $-\omega(x)$ and $-\lambda$, respectively. It then follows from hypothesis, from theorem 4.3 and from what was stated above that $V^{-1}A$ has infinitely many eigenvalues, while we know from [6, theorem 3.4] that, for any ε satisfying $0 < \varepsilon < \frac{1}{2}\pi$, there are infinitely many eigenvalues lying in the sector $|\arg\lambda| < \varepsilon$ and at most a finite number lying in the complement of this sector in the λ-plane. Hence, since it follows from theorem 3.1 that

$$\sum_{j=1}^{\infty} \frac{1}{(\lambda_j + t)^q} = b_{n/2m,q}\kappa t^{(n/2m)-q} + o(t^{(n/2m)-q}) \quad \text{as } t \to \infty, \tag{4.4}$$

we can argue in a manner somewhat similar to that in the proof of theorem 6.3 in [6] to show that (4.4) also holds with λ_j replaced by $|\lambda_j|$. The assertion (4.3) follows immediately from the latter result and from the Hardy–Littlewood Tauberian theorem (see [5]). □

Our goal now is to sharpen the result of theorem 4.3 for the case where $\omega(x)$ satisfies condition 2, but is not of one sign in $\bigcup_{k=0}^{N} \Omega_k$. To this end we firstly need to generalize theorem 3.1. Accordingly, by extending the definition of $R(\lambda)^q$ from q an even integer exceeding $n/2m$ to any real $q > n/2m$ as in [6, § 6], we can readily establish the following result (we refer the reader to [6, theorem 6.4] for the proof).

THEOREM 4.5. *Let the boundary problem (1.1), (1.2) be elliptic with parameter in $\mathcal{L}_\theta$. Then the asymptotic formula (3.1) holds for all real $q > n/2m$ with c_q as indicated in (3.2).*

Assuming for the remainder of this section that $\omega(x)$ satisfies condition 2 and is not of one sign in $\bigcup_{k=0}^{N} \Omega_k$, let us suppose now that the boundary problem (1.1), (1.2) is elliptic with parameter in the two closed sectors $\mathcal{L} = \mathcal{L}_1$ and $\mathcal{L} = \mathcal{L}_2$ that intersect only at the origin. Let Λ_1 and Λ_2 denote the two open sectors complementary to $\mathcal{L}_1 \cup \mathcal{L}_2$, and let us suppose that $\bar{R}_+$ is the bisector of Λ_1. Bearing in mind what was said in the text preceding theorem 4.1, it is also clear that we may suppose in addition that all the eigenvalues of $V^{-1}A$ are contained in $\Lambda_1 \cup \Lambda_2$. Let $\Omega^{(1)}$ denote the union of those Ω_k, $0 \leqslant k \leqslant N$, for which $\omega(x)^{-1}a_0(x,\xi) \in \Lambda_1$ for $x \in \Omega_k$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ (note that $\Omega^{(1)} \neq \emptyset$ and $\bigcup_{k=0}^{N} \Omega_k \setminus \Omega^{(1)} \neq \emptyset$), and let $\operatorname{tr}^{(1)} R(\lambda)^q$ denote that part of the trace of $R(\lambda)^q$ which corresponds to the eigenvalues of $V^{-1}A$ lying in Λ_1. Finally, let

$$d^{(1)} = \frac{1}{(2\pi)^n n} \int_{\Omega^{(1)}} \mathrm{d}x \int_{|\xi|=1} [\omega(x)^{-1}a_0(x,\xi)]^{-n/2m} \, \mathrm{d}S_\xi,$$

where the power is defined by using a cut along $\bar{\mathbb{R}}_-$.

THEOREM 4.6. *Let $n/2m < q < 1 + n/2m$. Then, under the foregoing assumptions, we have*

$$\operatorname{tr}^{(1)} R(\lambda)^q = b_{n/2m,q} d^{(1)} (-\lambda)^{(n/2m)-q} + o(|\lambda|^{(n/2m)-q}) \quad \text{as } |\lambda| \to \infty \tag{4.5}$$

uniformly in any closed sector that has no common points with $\bar{\Lambda}_1$ except 0, and where $(-\lambda)^{\cdots}$ is defined by using a cut along $\bar{\mathbb{R}}_+$.

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Proof. Although theorem 4.5 was established under the hypothesis that the boundary problem (1.1), (1.2) is elliptic with parameter in $\mathcal{L}_\theta$, it is clear from what was said in the text preceding theorem 4.1 that under the hypothesis of theorem 4.6 we can still deduce from theorem 4.5 that

$$\operatorname{tr} R(\lambda)^q = \frac{1}{(2\pi)^n}\int_\Omega \mathrm{d}x \int_{\mathbb{R}^n} \frac{\mathrm{d}\xi}{[\omega(x)^{-1}a_0(x,\xi)-\lambda]^q} + o(|\lambda|^{(n/2m)-q}) \quad \text{as } |\lambda| \to \infty \tag{4.6}$$

uniformly in $\mathcal{L}_1 \cup \mathcal{L}_2$. Consequently, in light of (4.6), the proof of the theorem can be completed by arguing as in the proofs of [6, theorem 6.5] and [5, theorem 1.17]. □

As a consequence of theorem 4.6 we have the following sharpening of theorem 4.3. Here we let $N^{(1)}(t) = \operatorname{card}\{j \mid |\lambda_j| \leqslant t,\ \lambda_j \in \Lambda_1\}$.

THEOREM 4.7. *Suppose that the hypotheses of theorem 4.6 hold. Then* $N^{(1)}(t) \asymp t^{n/2m}$ *as* $t \to \infty$ *if* $d^{(1)} \neq 0$.

Proof. The proof follows from the same kind of arguments that were used in establishing theorem 4.3, except that now we use (4.5) in place of (3.1). □

We are now going to use theorem 4.6 to extend the results of theorem 4.4 to the case under consideration here (i.e. when $\omega(x)$ satisfies condition 2, but is not of one sign in $\bigcup_{k=0}^N \Omega_k$. Accordingly, let us suppose henceforth that this case holds and that $a_0(x,\xi) > 0$ for $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. We note then that condition (i) of definition 2.3 is satisfied for every ray $\mathcal{L} = \mathcal{L}(\theta)$ for which $\theta \neq k\pi$, $k \in \mathbb{Z}$, but is not satisfied for $\theta = k\pi$. Bearing this in mind, let us also suppose that the boundary problem (1.1), (1.2) is elliptic with parameter in every ray $\mathcal{L}(\theta)$ for which $\theta \neq k\pi$. It then follows from [6, theorem 8.6] that, for any ε satisfying $0 < \varepsilon < \frac{1}{2}\pi$, there are infinitely many eigenvalues of $V^{-1}A$ lying in each of the sectors $|\arg\lambda| < \varepsilon$, $|\pi - \arg\lambda| < \varepsilon$, while there are at most a finite number of eigenvalues lying in the complement, in the λ-plane, of their union. Let $N^+(t) = \operatorname{card}\{j \mid |\lambda_j| \leqslant t,\ \operatorname{Re}\lambda_j \geqslant 0\}$, $N^-(t) = \operatorname{card}\{j \mid |\lambda_j| \leqslant t,\ \operatorname{Re}\lambda_j < 0\}$, so that $N(t) = N^+(t) + N^-(t)$. Lastly, let $\omega^+(x) = \max\{\omega(x), 0\}$, $\omega^-(x) = \max\{-\omega(x), 0\}$.

THEOREM 4.8. *Suppose that* $a_0(x,\xi) > 0$ *for* $x \in \bar{\Omega}$ *and* $\xi \in \mathbb{R}^n \setminus \{0\}$ *and that* $\omega(x)$ *satisfies condition 2, but is not of one sign in* $\bigcup_{k=0}^N \Omega_k$. *Suppose also that the boundary problem (1.1), (1.2) is elliptic with parameter in every ray* $\mathcal{L} = \mathcal{L}(\theta)$ *for which* $\theta \neq k\pi$, $k \in \mathbb{Z}$. *Then*

$$N^\pm(t) = \kappa^\pm t^{n/2m} + o(t^{n/2m}) \quad \text{as } t \to \infty,$$
$$N(t) = \kappa t^{n/2m} + o(t^{n/2m}) \quad \text{as } t \to \infty,$$

where

$$\kappa^\pm = \frac{1}{(2\pi)^n}\int_\Omega (\omega^\pm(x))^{n/2m}\,\mathrm{d}x \int_{a_0(x,\xi)<1} \mathrm{d}\xi,$$
$$\kappa = \frac{1}{(2\pi)^n}\int_\Omega |(\omega(x)|^{n/2m}\,\mathrm{d}x \int_{a_0(x,\xi)<1} \mathrm{d}\xi.$$

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Proof. It follows from theorem 4.6 that

$$\sum_{\operatorname{Re}\lambda_j \geqslant 0} \frac{1}{(\lambda_j + t)^q} = b_{n/2m,q}\kappa^+ t^{(n/2m)-q} + o(t^{(n/2m)-q}) \quad \text{as } t \to \infty, \tag{4.7}$$

and hence we find, from an argument somewhat similar to that used in the proof of theorem 6.3 in [6], that (4.7) holds with λ_j replaced by $|\lambda_j|$. The assertion for $N^+(t)$ now follows from the Hardy–Littlewood Tauberian theorem. The assertion for $N^-(t)$ can be obtained in the same way by replacing $\omega(x)$ and λ in (1.1) by $-\omega(x)$ and $-\lambda$, respectively. The assertion for $N(t)$ now follows from these results. □

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