Proceedings of the Royal Society of Edinburgh, **155**, 2345–2363, *2025*
DOI:10.1017/prm.2024.40

Unconditional convergence of eigenfunction expansions for abstract and elliptic operators

Vladimir Mikhailets
Institute of Mathematics of the Czech Academy of Sciences, Praha 11567, Czech Republic
Institute of Mathematics of the National Academy of Sciences of Ukraine, Kyiv 01024, Ukraine (vladimir.mikhailets@gmail.com; mikhailets@imath.kiev.ua)

Aleksandr Murach
Institute of Mathematics of the National Academy of Sciences of Ukraine, Kyiv 01024, Ukraine (murach@imath.kiev.ua)

(Received 25 December 2023; accepted 4 March 2024)

We study the most general class of eigenfunction expansions for abstract normal operators with pure point spectrum in a complex Hilbert space. We find sufficient conditions for such expansions to be unconditionally convergent in spaces with two norms and also estimate the degree of this convergence. Our result essentially generalizes and complements the known theorems of Krein and of Krasnosel'skiĭ and Pustyl'nik. We apply it to normal elliptic pseudodifferential operators on compact boundaryless C^∞-manifolds. We find generic conditions for eigenfunction expansions induced by such operators to converge unconditionally in the Sobolev spaces W_p^ℓ with $p > 2$ or in the spaces C^ℓ (specifically, for the p-th mean or uniform convergence on the manifold). These conditions are sufficient and necessary for the indicated convergence on Sobolev or Hörmander function classes and are given in terms of parameters characterizing these classes. We also find estimates for the degree of the convergence on such function classes. These results are new even for differential operators on the circle and for multiple Fourier series.

Keywords: Spectral expansion; eigenfunction expansion; unconditional convergence; space with two norms; normal operator; elliptic operator

2020 *Mathematics Subject Classification:* 47B90; 42B37

1. Introduction

Let $(X, \|\cdot\|)$ be a complex or real Banach space. A series $\sum_{n=1}^{\infty} x_n$ formed by elements $x_n \in X$ is called unconditionally convergent in X if the rearranged series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges in X for an arbitrary permutation $\sigma : \mathbb{N} \leftrightarrow \mathbb{N}$. In this case there exists a unique element $x \in X$ such that $\sum_{n=1}^{\infty} x_{\sigma(n)} = x$ for every permutation σ. If $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then the original series is called absolutely convergent in X. Every absolutely convergent series in X converges unconditionally therein. If the space is finite-dimensional, then the converse is true according to Riemann's

© The Author(s), 2024. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

theorem. However, if X is infinite-dimensional, then there exist unconditionally convergent series in X that do not converge absolutely (as the Dvoretzky–Rogers theorem states [**13**]).

In this paper, we study series in abstract and function spaces with two norms. The weaker norm among the two is induced by an inner product. Terms of series are pairwise orthogonal and are eigenfunctions of a certain (generally, unbounded) normal operator. Specifically, it may be a self-adjoint operator with discrete spectrum.

The simplest example of such type is given by the trigonometric Fourier series. From the operator-theoretic viewpoint, this is the spectral expansion of a function corresponding to the Laplace operator on the unit circle $\mathbb{T}$. The study of the convergence of such series has a very long history and huge literature (see e.g. monographs [**8**, **14**, **36**], surveys [**2**, **3**, **16**] and references therein).

It is known that the trigonometric Fourier series of every function $f(\tau)$ from the Lebesgue space $L_p(\mathbb{T})$, with $1 < p < \infty$, converges to f in this space. This is true for the Fourier series with respect to any rearranged trigonometric system

$$f(\tau) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \bigl[a_{\sigma(n)} \cos(\sigma(n)\tau) + b_{\sigma(n)} \sin(\sigma(n)\tau)\bigr] \tag{1.1}$$

if and only if $p = 2$. Moreover, there exists a certain permutation σ and a continuous function f on $\mathbb{T}$ such that series (1.1) does not converge to f in the space $L_p(\mathbb{T})$ whatever $p > 2$ [**35**, Theorem 8]. Let us indicate another example. The Carleson theorem [**12**] states that the Fourier trigonometric series of any function $f \in L_2(\mathbb{T})$ converges to f almost everywhere (a.e.) on $\mathbb{T}$. However, as was shown by Kolmogorov [**21**], there exists a function $f \in L_2(\mathbb{T})$ such that its certain rearranged Fourier series of form (1.1) diverges a.e. on $\mathbb{T}$. Thus, conditions for function series to be convergent or unconditionally convergent can be essentially different.

The study of solutions to initial-boundary-value problems for parabolic or hyperbolic differential equations in cylindrical domains by the Fourier method leads to the analysis of the convergence of expansions in eigenfunctions of self-adjoint elliptic differential operators in bounded Euclidean domains (see e.g. [**19**, **24**, **30**]). Let G be a bounded domain in $\mathbb{R}^n$ with smooth boundary, and suppose that an elliptic differential expression with smooth coefficients on $\overline{G}$ induces a self-adjoint operator L with discrete spectrum in the Hilbert space $L_2(G)$. Then this space possesses an orthonormal basis $\{e_j(\tau) : j \in \mathbb{N}\}$ formed by eigenfunctions of L. Choosing a function $f \in L_2(G)$ arbitrarily, we consider its spectral expansion

$$f(\tau) \sim \lim_{\lambda\to\infty} \sum_{j:|\lambda_j|\leqslant\lambda} c_j(f)\, e_j(\tau). \tag{1.2}$$

Here, λ_j is the eigenvalue of L associated with the eigenfunction e_j, and $c_j(f)$ is the Fourier coefficient of f with respect to e_j. This expansion converges in the $L_2(G)$ norm. Therefore, it is reasonable to ask under which supplementary conditions for f expansion (1.2) converges to f in the stronger $L_p(G)$ norm with $p > 2$ or uniformly on G. There are no comprehensive answers to this question yet even in the case of ordinary differential equations. Results obtained in the many-dimensional case show that it is necessary to narrow down the problem statement essentially and to consider the convergence of the Fourier series (1.2) not for single functions but on

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

the classes of differential functions with compact supports in the domain G. Let us formulate one complete result of this type, which became a starting point for us [**2**, Chapter II, § 3, Subsection 2].

Let $W^s_{p,0}(G)$ denote the class of all functions that belong to the L_p-Sobolev space of real order $s > 0$ over G and that their supports are compact subsets of the open set G. Spectral expansion (1.2) of an arbitrary function $f \in W^s_{p,0}(G)$ converges uniformly on every compact set $K \subset G$ if and only if the following three conditions are satisfied:

$$ps > n, \quad s \geqslant \frac{n-1}{2} \quad \text{and} \quad p \geqslant 1. \tag{1.3}$$

Note that the first inequality guarantees the continuity functions of class $W^s_{p,0}(G)$, whereas the second inequality provides the validity of the localization principle for spectral expansions on the class of elliptic operators under consideration. If $p \leqslant 2n/(n-1)$, then the second inequality in (1.3) follows from the first one. The claim for the compactness of the support of f is stipulated by the fact that then f satisfies certain homogeneous boundary conditions depending on the elliptic operator L. The uniform convergence of the spectral expansions only on compact subsets of G is connected with the presence of the boundary ∂G.

The main results of this paper are formulated and proved in § 4 and 5. There we study the unconditional convergence of expansions in eigenfunctions of an arbitrary normal elliptic pseudodifferential operator of positive order on a closed smooth manifold Γ. The convergence is considered with respect to the norm in the Sobolev space $W^\ell_p(\Gamma)$ or in the space $C^\ell(\Gamma)$, with $p > 2$ and $0 \leqslant \ell \in \mathbb{Z}$. We find necessary and sufficient conditions for this unconditional convergence to hold true on various classes of differentiable functions and estimate its degree. To this end, we use the Hörmander function classes $H^\alpha(\Gamma)$ whose differentiation order is a function $\alpha : [1, \infty) \to (0, \infty)$ that O-regularly varies at infinity in the sense of Avakumović. Such classes and their connection with elliptic pseudodifferential operators are discussed in § 3.

Sections 6 and 7 specify these results for differential operators on the circle and for multiple Fourier series, which allows us to obtain a number of new valuable results for these objects.

Our proofs of the aforementioned results are based on an abstract theorem about the unconditional convergence (see definition 2.1) of expansions in eigenvectors of a normal operator in a space with two norms. This theorem is stated and proved in § 2. It develops and refines known theorems of Krein [**23**, Theorem 4] and of Krasnosel'skiĭ and Pustyl'nik [**22**, Theorem 22.1] with respect to normal operators with pure point spectrum in a Hilbert space.

Final remarks and comments to the obtained results are given in § 8.

2. The basic abstract result

Throughout the paper, we consider complex linear spaces. Let H be an infinite-dimensional (not necessarily separable) Hilbert space, and let N be a normed space. Suppose that H and N are algebraically embedded in a certain linear space. As usual, $\|\cdot\|_H$ and $(\cdot,\cdot)_H$ respectively denote the norm and inner product in H, and

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

$\|\cdot\|_N$ stands for the norm in N. Thus, the linear space $H \cap N$ is endowed with two norms.

Let L be a normal (specifically, self-adjoint) linear operator in H. We allow the case where L is unbounded in H as well as the case where L is bounded on H. We suppose that L has pure point spectrum, i.e. the Hilbert space H has an orthonormal basis $\{e_j : j \in \Theta\}$ formed by certain eigenvectors e_j of L. Here, Θ is an index set whose cardinality coincides with the Hilbert dimension of H. Thus, every vector $f \in H$ expands in the series

$$f = \sum_{j\in\Theta} (f, e_j)_H \, e_j \tag{2.1}$$

in the topology of H. (Of course, this series contains only a countable number of non-zero terms, and its sum does not depend on their order.) Let λ_j denote the eigenvalue of L such that $Le_j = \lambda_j e_j$. As usual, $\sigma(L)$ stands for the spectrum of L, and $\sigma_p(L)$ denotes the point spectrum of L, i.e. $\sigma_p(L)$ is the set of all eigenvalues of L. Note that eigenvalues of L may be of infinite geometric multiplicity and that the set of all eigenvalues of L may have accumulation points.

Let $\mathcal{Q}$ denote the system of all finite subsets of Θ. We interpret $\mathcal{Q}$ as an upper directed set with respect to the relation $\subseteq$. Equality (2.1) means for $f \in H$ that the net

$$\left\{ \sum_{j\in\Upsilon} (f, e_j)_H \, e_j : \Upsilon \in \mathcal{Q} \right\} \tag{2.2}$$

converges to f in H.

Definition 2.1. *Let $f \in H \cap N$. We say that the expansions of f in eigenvectors of L converge unconditionally in the normed space N if net (2.2) lies in N and converges to f in N for every orthonormal basis $\{e_j : j \in \Theta\}$ in H formed by eigenvectors of L. If this condition is satisfied for every vector f from some class $M \subseteq H \cap N$, we also say that the expansions in eigenvectors of L converge unconditionally in N on the class M.*

Remark 2.2. Providing H is separable, eigenvector expansion (2.1) becomes

$$f = \sum_{j=1}^{\infty} (f, e_j)_H \, e_j. \tag{2.3}$$

In this case, the expansions of f in eigenvectors of L converge unconditionally in N if and only if series (2.3) converges to f in N for every orthonormal basis $\{e_j : j \in \mathbb{N}\}$ formed by eigenvectors of L (i.e. for any choices of orthonormal bases of eigenspaces of L associated with non-degenerate eigenvalues). It follows from this condition that series (2.3) converges under an arbitrary permutation of its terms.

Theorem 2.3. *Let $\omega, \eta : \sigma(L) \to \mathbb{C} \setminus \{0\}$ be Borel measurable bounded functions, and define the bounded operators $\omega(L)$ and $\eta(L)$ on H as functions of the normal operator L. Assume that the mapping $u \mapsto \omega(L)u$ sets a bounded operator from H to N, and denote this operator by R. Let f be an arbitrary vector from the image*

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

of the operator $(\omega\eta)(L)$. *Then the expansions of* f *in eigenvectors of* L *converge unconditionally in the space* N. *Moreover, the degree of this convergence admits the estimate*

$$\left\| f - \sum_{j\in\Upsilon} (f, e_j)_H \, e_j \right\|_N \leqslant \|R\|_{H\to N} \cdot \|g\|_H \cdot \sup_{j\in\Theta\setminus\Upsilon} |\eta(\lambda_j)| \cdot r_{g,e}(\Upsilon) \tag{2.4}$$

for each set $\Upsilon \in \mathcal{Q}$ *and every orthonormal basis* $e := \{e_j : j \in \Theta\}$ *indicated in definition 2.1 and with some net* $\{r_{g,e}(\Upsilon) : \Upsilon \in \mathcal{Q}\} \subseteq [0, 1]$ *that tends monotonically to zero and depends neither on* ω *nor on* η. *Here,* $g \in H$ *is the unique vector such that* $(\omega\eta)(L)g = f$.

Proof. Since $\omega(L)\eta(L)e_j = (\omega\eta)(\lambda_j)e_j$ whenever $j \in \Theta$ and since $(\omega\eta)(t) \neq 0$ whenever $t \in \sigma(L)$, we conclude that each $e_j \in N$ because the image of $\omega(L)$ lies in N. Thus, the left-hand side of (2.4) makes sense. Since the operator $(\omega\eta)(L)$ is algebraically reversible, the vector g is well defined for every above-mentioned f. Suppose that $f \neq 0$, otherwise the conclusion of this theorem is trivial. Given $\Upsilon \in \mathcal{Q}$, we let P_Υ denote the orthoprojector of H on the span of $\{e_j : j \in \Upsilon\}$ and obtain the following estimate:

$$\begin{aligned} \|f - P_\Upsilon f\|_N &= \|(\omega\eta)(L)g - P_\Upsilon(\omega\eta)(L)g\|_N \\ &= \|(\omega\eta)(L)(I - P_\Upsilon)g\|_N = \|\omega(L)\eta(L)(I - P_\Upsilon)^2 g\|_N \\ &\leqslant \|R\|_{H\to N} \cdot \|\eta(L)(I - P_\Upsilon)\|_{H\to H} \cdot \|(I - P_\Upsilon)g\|_H, \end{aligned} \tag{2.5}$$

with I being the identity operator on H. We put

$$r_{g,e}(\Upsilon) := \|(I - P_\Upsilon)g\|_H \cdot \|g\|_H^{-1} \tag{2.6}$$

and see that the net $\{r_{g,e}(\Upsilon) : \Upsilon \in \mathcal{Q}\}$ is required. Note also that

$$\|\eta(L)(I - P_\Upsilon)\|_{H\to H} \leqslant \sup_{j\in\Theta\setminus\Upsilon} |\eta(\lambda_j)|. \tag{2.7}$$

Indeed, since

$$\eta(L)(I - P_\Upsilon)h = \eta(L) \sum_{j\in\Theta\setminus\Upsilon} (h, e_j)_H \, e_j = \sum_{j\in\Theta\setminus\Upsilon} (h, e_j)_H \, \eta(\lambda_j)e_j$$

for every $h \in H$ (the convergence holds in H), we have the inequality

$$\|\eta(L)(I - P_\Upsilon)h\|_H^2 = \sum_{j\in\Theta\setminus\Upsilon} |(h, e_j)_H \, \eta(\lambda_j)|^2 \leqslant \|h\|_H^2 \cdot \sup_{j\in\Theta\setminus\Upsilon} |\eta(\lambda_j)|^2,$$

which gives (2.7). Now (2.5)–(2.7) yield estimate (2.4), which implies the required unconditional convergence. □

Remark 2.4. Assume that the norms in H and N are compatible on the linear space $H \cap N$. (By definition [**15**, Chapter I, Section 2.2], this compatibility means the following: if a sequence lies in $H \cap N$ and converges to zero in one of these

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

norms and is a Cauchy sequence with respect to the second norm, then it converges to zero in the second norm.) Let R be a bounded linear operator on H (specifically, $R : u \mapsto \omega(L)u$ where $u \in H$, as in theorem 2.3). Then the inclusion $R(H) \subseteq N$ implies that R acts continuously from H to N. Indeed, suppose that $R(H) \subseteq N$. Then, owing to the closed graph theorem, the operator $R : H \to \widetilde{N}$ is bounded if and only if it is closed; here, $\widetilde{N}$ is the completion of the normed space N. We therefore have to prove that this operator is closable. Assume that a sequence $(u_k)_{k=1}^{\infty} \subset H$ satisfies the following two conditions: $u_k \to 0$ in H and

$$Ru_k \to v \text{ in } \widetilde{N} \text{ for certain } v \in \widetilde{N}, \tag{2.8}$$

as $k \to \infty$. Then $Ru_k \to 0$ in H because R is bounded on H. Besides, $(Ru_k)_{k=1}^{\infty}$ is a Cauchy sequence in N. Hence, $Ru_k \to 0$ in N because the norms in N and H are compatible on $H \cap N$. Thus, $v = 0$ in view of (2.8), which means that the operator $R : H \to \widetilde{N}$ is closable.

Theorem 2.3 contains Krein's result [**23**, Theorem 4] according to which series (2.3) converges in N for every $f \in L(H)$ if L is a compact self-adjoint operator in H and if L acts continuously from H to a Banach space N. The latter theorem generalizes (to abstract operators) the Hilbert–Schmidt theorem about the uniform decomposability of sourcewise representable functions in eigenfunctions of a symmetric integral operator.

If L is a positive definite self-adjoint operator with discrete spectrum and if $\omega(t) \equiv t^{-\delta}$ and $\eta(t) \equiv t^{-\tau}$ for certain numbers $\delta, \tau \geqslant 0$ and if $R := L^{-\delta}$ acts continuously from H to a Banach space N, Krasnosel'skiĭ and Pustyl'nik [**22**, Theorem 22.1] proved that

$$\left\| f - \sum_{j=1}^{k} (f, e_j)_H \, e_j \right\|_N = o(\lambda_k^{-\tau}) \text{ as } k \to \infty$$

for any f from the domain of $L^{\delta+\tau}$ (the Hilbert space H is assumed to be separable). Here, the eigenvalues $\lambda_k > 0$ of L are numbered so that $\lambda_k \leqslant \lambda_{k+1}$. Formula (2.4) implies a more general result for normal operators with discrete spectrum.

3. Hörmander spaces and elliptic operators

We will use them to estimate the degree of the convergence of expansions in eigenfunctions of some elliptic operators given on a closed manifold. Such spaces were introduced over $\mathbb{R}^n$ and investigated by Hörmander [**17**, Section 2.2]. Their smoothness index (or differentiation order) is a general enough function of the frequency variables, which allows us to obtain finer estimates than those [**1**, Subsection 6.1 a] received with the help of classical function spaces such as Sobolev spaces, whose smoothness index is a number. We use a broad class of isotropic inner product Hörmander spaces that admit a reasonable definition on smooth closed manifolds. The class was introduced and investigated in [**27**], [**28**, Section 2.4.2] and [**29**] (note that these works develop an approach proposed in [**25**, **26**]). It has various applications to elliptic operators [**28**, Section 2.4.3] and elliptic boundary-value problems [**4**, **5**].

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

This class is formed by certain function spaces H^α whose smoothness index α ranges over a set denoted by OR. By definition, OR consists of all Borel functions $\alpha : [1, \infty) \to (0, \infty)$ such that

$$c^{-1} \leqslant \frac{\alpha(\lambda t)}{\alpha(t)} \leqslant c \text{ whenever } t \geqslant 1 \text{ and } 1 \leqslant \lambda \leqslant b,$$

with the numbers $b > 1$ and $c \geqslant 1$ being independent of t and λ (but may depend on α). Such functions were introduced by Avakumović [**7**], are called OR-varying (or O-regularly varying) at infinity, and are well investigated [**9**, **10**].

The class OR admits the following description [**9**, Theorem 2.2.7]: $\alpha \in \mathrm{OR}$ if and only if

$$\alpha(t) = \exp\left(\beta(t) + \int_1^t \frac{\gamma(\tau)}{\tau}\, \mathrm{d}\tau\right) \text{ whenever } t \geqslant 1 \tag{3.1}$$

for some bounded Borel functions $\beta, \gamma : [1, \infty) \to \mathbb{R}$.

For every function $\alpha \in \mathrm{OR}$ there exist numbers $s_0, s_1 \in \mathbb{R}$, with $s_0 \leqslant s_1$, and numbers $c_0, c_1 > 0$ such that

$$c_0 \lambda^{s_0} \leqslant \frac{\alpha(\lambda t)}{\alpha(t)} \leqslant c_1 \lambda^{s_1} \quad \text{whenever} \quad t \geqslant 1 \text{ and } \lambda \geqslant 1. \tag{3.2}$$

Given $\alpha \in \mathrm{OR}$, we let $s_*(\alpha)$ denote the supremum of the set of all numbers s_0 that the left-hand inequality in (3.2) holds true (for certain c_0 depending on s_0), and we let $s^*(\alpha)$ denote the infimum of the set of all numbers s_1 that the right-hand inequality in (3.2) holds true (for certain c_1 depending on s_1). Of course, $-\infty < s_*(\alpha) \leqslant s^*(\alpha) < \infty$. The numbers $s_*(\alpha)$ and $s^*(\alpha)$ are equal to the lower and upper Orlicz–Matuszewska indices of α, resp. These facts follow from [**9**, Theorem 2.1.7 and Proposition 2.2.1].

For instance, every continuous function $\alpha : [1, \infty) \to (0, \infty)$ such that

$$\alpha(t) = t^s (\log t)^{s_1} (\log\log t)^{s_2} \dots (\underbrace{\log \dots \log}_{k \text{ times}} t)^{s_k} \quad \text{whenever} \quad t \gg 1$$

belongs to OR; here, $1 \leqslant k \in \mathbb{Z}$ and $s, s_1, \dots, s_k \in \mathbb{R}$. It is well known that $s_*(\alpha) = s^*(\alpha) = s$ for this function.

An example of a function $\alpha \in \mathrm{OR}$ with the different Orlicz–Matuszewska indices is given by formula (3.1) in which

$$\gamma(\tau) := \begin{cases} r & \text{if } \tau \in [\theta_1 \cdots \theta_{2j-1}, \theta_1 \cdots \theta_{2j-1}\theta_{2j}] \text{ for certain integer } j \geqslant 1, \\ s & \text{otherwise} \end{cases} \tag{3.3}$$

provided that $r < s$, $1 = \theta_1 < \theta_2 < \theta_3 < \dots$, and $\theta_k \to \infty$ as $k \to \infty$. Then $s_*(\alpha) = r$ and $s^*(\alpha) = s$, as is noticed, e.g., in [**4**, Section 3].

Let $\alpha \in \mathrm{OR}$ and $1 \leqslant n \in \mathbb{Z}$. By definition, the linear space $H^\alpha(\mathbb{R}^n)$ consists of all tempered distributions w on $\mathbb{R}^n$ that their Fourier transform $\widehat{w} := \mathcal{F}w$ is locally

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

Lebesgue integrable over $\mathbb{R}^n$ and satisfies the condition

$$\int\limits_{\mathbb{R}^n} \alpha^2(\langle\xi\rangle)\,|\widehat{w}(\xi)|^2\,\mathrm{d}\xi < \infty.$$

As usual, $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ is the smoothed absolute value of $\xi \in \mathbb{R}^n$. The space $H^\alpha(\mathbb{R}^n)$ is endowed with the inner product

$$(w_1, w_2)_{H^\alpha(\mathbb{R}^n)} := \int\limits_{\mathbb{R}^n} \alpha^2(\langle\xi\rangle)\,\widehat{w_1}(\xi)\,\overline{\widehat{w_2}(\xi)}\,d\xi$$

and the corresponding norm. This space is an isotropic Hilbert case of the Hörmander spaces $\mathcal{B}_{p,k}$ considered in [**17**, Section 2.2] and [**18**, Section 10.1]. Namely, $H^\alpha(\mathbb{R}^n) = \mathcal{B}_{2,k}$ provided that $k(\xi) = \alpha(\langle\xi\rangle)$ whenever $\xi \in \mathbb{R}^n$.

Consider a version of the space $H^\alpha(\mathbb{R}^n)$ for smooth closed manifolds. In the sequel, Γ is a closed (i.e. compact and boundaryless) manifold of dimension $n \geqslant 1$ and class C^∞. Suppose that a positive C^∞-density $\mathrm{d}x$ is given on Γ. We arbitrarily choose a finite atlas from the C^∞-structure on Γ; let this atlas be formed by $\varkappa$ local charts $\pi_j : \mathbb{R}^n \leftrightarrow \Gamma_j$, with $j = 1, \ldots, \varkappa$. Here, the open sets $\Gamma_1, \ldots, \Gamma_\varkappa$ form a covering of Γ. We also arbitrarily choose functions $\chi_j \in C^\infty(\Gamma)$, with $j = 1, \ldots, \varkappa$, such that $\chi_1(x) + \cdots + \chi_\varkappa(x) = 1$ whenever $x \in \Gamma$ and that $\operatorname{supp}\chi_j \subset \Gamma_j$ whenever $1 \leqslant j \leqslant \varkappa$. Thus, these functions form a partition of unity on Γ subordinate to the above covering.

By definition, the linear space $H^\alpha(\Gamma)$ consists of all distributions f on Γ such that $(\chi_j f)\circ\pi_j \in H^\alpha(\mathbb{R}^n)$ for each $j \in \{1, \ldots, \varkappa\}$. Here, $(\chi_j f)\circ\pi_j$ is the representation of the distribution $\chi_j f$ in the local chart π_j. The space $H^\alpha(\Gamma)$ is endowed with the inner product

$$(f_1, f_2)_{H,\alpha} := \sum_{j=1}^{\varkappa} ((\chi_j f_1)\circ\pi_j, (\chi_j f_2)\circ\pi_j)_{H^\alpha(\mathbb{R}^n)}$$

and the corresponding norm $\|f\|_{H,\alpha} := (f,\, f)_{H,\alpha}^{1/2}$. This space is complete (i.e. Hilbert) and separable and does not depend up to equivalence of norms on the choice of the atlas and partition of unity on Γ; the set $C^\infty(\Gamma)$ is dense in $H^\alpha(\Gamma)$. These properties are due to [**28**, Theorem 2.21].

If $\alpha(t) \equiv t^s$ for some $s \in \mathbb{R}$, then $H^\alpha(G)$, where $G = \mathbb{R}^n$ or $G = \Gamma$, becomes the inner product Sobolev space $W_2^s(G)$ of order s.

We also need an equivalent definition of the space $H^\alpha(\Gamma)$, with $\alpha \in \mathrm{OR}$, in terms of a function of some elliptic pseudodifferential operators (PsDOs). Following [**1**, Section 2.1], we let $\Psi^m_{\mathrm{ph}}(\Gamma)$ denote the class of all classical (i.e. polyhomogeneous) PsDOs on Γ of order $m \in \mathbb{R}$. Suppose that $m > 0$ and that a PsDO $A \in \Psi^m_{\mathrm{ph}}(\Gamma)$ is elliptic on Γ, i.e. $a_0(x,\,\xi) \neq 0$ whenever $x \in \Gamma$ and $0 \neq \xi \in T^*_x\Gamma$, with $a_0(x,\,\xi)$ denoting the principal symbol of A and with $T^*_x\Gamma$ standing (as usual) for the cotangent space to Γ at x. We may and will consider A as a closed unbounded operator in the Hilbert Lebesgue space $L_2(\Gamma) := L_2(\Gamma,\,\mathrm{d}x)$ and with the domain $\operatorname{Dom} A = W_2^m(\Gamma)$ (see [**1**, Theorem 2.3.5]). Suppose that A is a self-adjoint operator in $L_2(\Gamma)$ and

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

that $\sigma(A) \subseteq [1, \infty)$. Then the operator $\alpha(A^{1/m})$ is well defined in the Hilbert space $L_2(\Gamma, \mathrm{d}x)$ via Spectral Theorem as the Borel function $\alpha(t^{1/m})$ of A.

PROPOSITION 3.1. *Let $\alpha \in \mathrm{OR}$. Then the norm in the space $H^\alpha(\Gamma)$ is equivalent to the norm*

$$f \mapsto \|\alpha(A^{1/m})f\|_{L_2(\Gamma)} \tag{3.4}$$

on the dense set $C^\infty(\Gamma)$. Thus, $H^\alpha(\Gamma)$ coincides with the completion of $C^\infty(\Gamma)$ with respect to norm (3.4). Hence, if the function $1/\alpha$ is bounded on $[1, \infty)$, then $H^\alpha(\Gamma)$ is the domain of the operator $\alpha(A^{1/m})$ and this operator sets an isomorphism between $H^\alpha(\Gamma)$ and $L_2(\Gamma)$.

This proposition is due to [**28**, Theorem 2.23]. We may put $A := 1 - \Delta_\Gamma$ and $m = 2$, where Δ_Γ is the Laplace–Beltrami operator on Γ, with Γ being endowed with the Riemannian metric inducing $\mathrm{d}x$.

4. Mean convergence

Given $s \in \mathbb{R}$ and $p \in (1, \infty)$, we let $W_p^s(\Gamma)$ denote the Sobolev space over Γ with the smoothness index s and the integral-exponent p and let $\|\cdot\|_{W,s,p}$ stand for the norm in this space. The space $W_p^s(\Gamma)$ is defined on the base of $W_p^s(\mathbb{R}^n)$ with the help of the above atlas and partition of unity like the definition of $H^\alpha(\Gamma)$ on the base of $H^\alpha(\mathbb{R}^n)$. Recall that the Sobolev space $W_p^s(\mathbb{R}^n)$ consists of all tempered distributions w on $\mathbb{R}^n$ such that the distribution $v := \mathcal{F}^{-1}[\langle\xi\rangle^s\, \widehat{w}(\xi)]$ belongs to the Lebesgue space $L_p(\mathbb{R}^n)$, with the norm of w in $W_p^s(\mathbb{R}^n)$ being equal to the norm of v in $L_p(\mathbb{R}^n)$. Of course, $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. Note that $W_p^0(\Gamma)$ coincides with the Lebesgue space $L_p(\Gamma) := L_p(\Gamma, \mathrm{d}x)$, and assume that this holds with equality of norms.

We use the term 'mean convergence' to refer to the convergence in the normed Sobolev space $W_p^\ell(\Gamma)$ subject to $0 \leqslant \ell \in \mathbb{Z}$ and $1 < p < \infty$, specifically in the Lebesgue space $L_p(\Gamma)$.

Let real $m > 0$, and let $A \in \Psi^m_{\mathrm{ph}}(\Gamma)$. Suppose that the PsDO A is elliptic on Γ. If A is a normal operator in $L_2(\Gamma)$, the separable Hilbert space $L_2(\Gamma)$ has an orthonormal basis formed by some eigenfunctions of A (see e.g. [**32**, Section 15.3]), every eigenfunction of A pertaining to $C^\infty(\Gamma)$ and each eigenvalue of A being of finite multiplicity. Then every vector $f \in L_2(\Gamma)$ admits the spectral decomposition

$$f = \lim_{r\to\infty} \sum_{\substack{\lambda\in\sigma_p(A)\\ |\lambda|\leqslant r}} P(\lambda)f, \tag{4.1}$$

the series converging in $L_2(\Gamma)$. As usual, $P(\lambda)$ denotes the orthoprojector on the eigenspace of A associated with the eigenvalue λ. Note that the sum in (4.1) contains only a finite number of terms for every $r > 0$. The convergence in $L_2(\Gamma)$ implies one in each normed space $L_p(\Gamma)$ with $1 \leqslant p < 2$; hence, we are interested in conditions for the convergence of (4.1) in the mean with index $p > 2$.

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

Given real $m > 0$, we let $\mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$ denote the set of all PsDOs of class $\Psi^m_{\mathrm{ph}}(\Gamma)$ that are elliptic on Γ and normal in $L_2(\Gamma)$. Specifically, every elliptic constant-coefficient partial differential operator on Γ of order m belongs to $\mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$. Recall that $n = \dim\Gamma \geqslant 1$.

THEOREM 4.1. *Suppose that* $0 < m \in \mathbb{R}$, $0 \leqslant \ell \in \mathbb{Z}$, s, p, $q \in \mathbb{R}$ *and* $1 < q \leqslant 2 < p$. *Then the following three conditions are equivalent:*

(i) $s \geqslant \ell + n/q - n/p$;

(ii) *there exists a PsDO* $A \in \mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$ *that spectral decomposition* (4.1) *induced by* A *converges in* $W^\ell_p(\Gamma)$ *on the class* $W^s_q(\Gamma)$;

(iii) *the expansions in eigenfunctions of an arbitrary PsDO from* $\mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$ *converge unconditionally in* $W^\ell_p(\Gamma)$ *on the class* $W^s_q(\Gamma)$.

Proof. We will first prove that (i) $\Rightarrow$ (iii). Assuming (i) to hold true, we arbitrarily choose a PsDO $A \in \mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$ and put $L := (I + A^*A)$, with I denoting the identity operator on $L_2(\Gamma)$. Let $\sigma := \ell + n/2 - n/p > 0$. Since $L \in \mathrm{NE}\Psi^{2m}_{\mathrm{ph}}(\Gamma)$ and $\sigma(L) \subseteq [1, \infty)$, the operator $L^{\sigma/(2m)}$ is well defined in $L_2(\Gamma)$ as a power function of L and sets an isomorphism between $W^\sigma_2(\Gamma)$ and $L_2(\Gamma)$; see [**1**, Corollary 5.3.2]. The inverse of L, i.e. the operator $L^{-\sigma/(2m)}$, acts continuously from $H = L_2(\Gamma)$ to $N := W^\ell_p(\Gamma)$ because of the continuous embedding $W^\sigma_2(\Gamma) \hookrightarrow W^\ell_p(\Gamma)$. This embedding follows directly from its well-known analogue for Sobolev spaces over $\mathbb{R}^n$ (see e.g. [**33**, Theorem 2.8.1(b)]). Using theorem 2.3, we put $\omega(t) := t^{-\sigma/(2m)}$ and $\eta(t) := 1$ whenever $t \geqslant 1$ and remark that $W^\sigma_2(\Gamma)$ is the image of the operator $(\omega\eta)(L)$. We conclude by this theorem that the expansions in eigenfunctions of L converge unconditionally in $W^\ell_p(\Gamma)$ on the class $W^\sigma_2(\Gamma)$. Observe that $W^s_q(\Gamma) \hookrightarrow W^\sigma_2(\Gamma)$ because $s \geqslant \sigma + n/q - n/2$ by (i). This yields (iii) since all eigenfunctions of A are eigenfunctions of L as well. We have proved that (i) $\Rightarrow$ (iii).

The implication (iii) $\Rightarrow$ (ii) is obvious.

It remains to prove that (ii) $\Rightarrow$ (i). Assuming (ii) to hold true, we obtain the embedding of $W^s_q(\Gamma)$ in $W^\ell_p(\Gamma)$. The embedding implies (i), which is considered to be a known fact. Specifically, this fact follows from [**34**, p. 60, property (ii)]. Namely, the above embedding implies that $W^s_q(G) \hookrightarrow W^\ell_p(G)$ for any open ball G in $\mathbb{R}^n$. (Recall that $W^s_q(G)$, e.g. consists of the restrictions of all distributions $w \in W^s_q(\mathbb{R}^n)$ to G.) If (i) was not true, i.e. $s + \varepsilon = \ell + n/q - n/p$ for some $\varepsilon > 0$, then the Besov space $B^{s+\varepsilon}_{q,\theta}(G)$ (as a part of $W^s_q(G)$) would be embedded in $W^\ell_p(G)$ for each $\theta \geqslant 1$. However, by [**34**, p. 60, property (ii)], the last embedding is equivalent to $\theta \leqslant p$. This contradiction shows that (ii) $\Rightarrow$ (i). □

Let $m > 0$, and let $A \in \mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$. Consider an orthonormal basis $e := (e_j)_{j=1}^\infty$ in $L_2(\Gamma)$ formed by eigenfunctions $e_j \in C^\infty(\Gamma)$ of A. Let λ_j denote the eigenvalue of A such that $Ae_j = \lambda_j e_j$. We enumerate the eigenfunctions so that $|\lambda_j| \leqslant |\lambda_{j+1}|$ whenever $j \geqslant 1$. Then

$$|\lambda_j| \sim \widetilde{c}\, j^{m/n} \quad \text{as} \quad j \to \infty, \tag{4.2}$$

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

where $\widetilde{c}$ is a certain positive number that does not depend on j (see e.g. [**32**, Section 15.3]). According to theorem 4.1, the series (2.3), where $H = L_2(\Gamma)$, converges unconditionally in $W_p^\ell(\Gamma)$ on the class $W_2^{\ell+n/2-n/p}(\Gamma)$. This class is the broadest one among the spaces $W_q^s(\Gamma)$ indicated by theorem 4.1 as classes of convergence of the above series. Using the Hörmander spaces $H^\alpha(\Gamma) \subset W_2^{\ell+n/2-n/p}(\Gamma)$ as classes of the convergence, we can estimate its degree.

THEOREM 4.2. *Let $0 \leqslant \ell \in \mathbb{Z}$, $2 < p \in \mathbb{R}$ and $\alpha \in \mathrm{OR}$. Suppose that*

$$h(t) := t^{\ell+n/2-n/p}\,(\alpha(t))^{-1} \to 0 \quad \textit{as} \quad t \to \infty$$

and that the function h decreases on $[1, \infty)$. Then

$$\Big\| f - \sum_{j:|\lambda_j|\leqslant\lambda} (f, e_j)_H\, e_j \Big\|_{W,\ell,p} \leqslant c \cdot \|f\|_{H,\alpha} \cdot h(\lambda^{1/m}) \tag{4.3}$$

and

$$\Big\| f - \sum_{j=1}^{k} (f, e_j)_H\, e_j \Big\|_{W,\ell,p} \leqslant c \cdot \|f\|_{H,\alpha} \cdot h(k^{1/n}) \tag{4.4}$$

for all $f \in H^\alpha(\Gamma)$, $\lambda \geqslant 1$ and integer-valued $k \geqslant 1$. Here, c is a certain positive number that does not depend on f, λ and k.

Proof. Put $\sigma := l + n/2 - n/p$, $L := (I + A^*A)$ and $\omega(t) := t^{-\sigma/(2m)}$ and $\eta(t) := h(t^{1/(2m)})$ whenever $t \geqslant 1$. Since $(\omega\eta)(t) = (\alpha(t^{1/(2m)}))^{-1}$ whenever $t \geqslant 1$, the operator $(\omega\eta)(L)$ sets an isomorphism between $L_2(\Gamma)$ and $H^\alpha(\Gamma)$ due to proposition 3.1. Hence, according to theorem 2.3 and since h decreases, we have the estimates

$$\Big\| f - \sum_{j:\lambda'_j\leqslant 1+\lambda^2} (f, e_j)_H\, e_j \Big\|_{W,\ell,p} \leqslant c_0 \cdot \|g\|_H \cdot \eta(1+\lambda^2) \tag{4.5}$$

and

$$\Big\| f - \sum_{j=1}^{k} (f, e_j)_H\, e_j \Big\|_{W,\ell,p} \leqslant c_0 \cdot \|g\|_H \cdot \eta(\lambda'_k), \tag{4.6}$$

where c_0 is the norm of $\omega(L)$ considered as a bounded operator from $H = L_2(\Gamma)$ to $N = W_p^\ell(\Gamma)$ (see the proof of theorem 4.1), $g := ((\omega\eta)(L))^{-1} f$ and the number λ'_j satisfies $Le_j = \lambda'_j e_j$, i.e. $\lambda'_j = 1 + |\lambda_j|^2$. Hence, $\|g\|_H \asymp \|f\|_{H,\alpha}$, $\eta(1+\lambda^2) \asymp h(\lambda^{1/m})$ as $\lambda \geqslant 1$, and

$$\eta(\lambda'_k) = h((1 + |\lambda_k|^2)^{1/(2m)}) \asymp h(k^{1/n}) \quad \text{as} \quad k \geqslant 1,$$

due to $h \in \mathrm{OR}$ and (4.2). (As usual, the symbol $\asymp$ means the weak equivalence of positive values.) Thus, (4.5) and (4.6) imply the required estimates (4.3) and (4.4), resp. □

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

REMARK 4.3. If A is a positive definite operator in $L_2(\Gamma)$, we have the equivalence of norms

$$\|f\|_{H,\alpha} \asymp \left(\sum_{j=1}^{\infty} \alpha^2(j^{1/n})\,|(f,e_j)_H|^2\right)^{1/2} \asymp \left(\sum_{j=1}^{\infty} \alpha^2((1+\lambda_j)^{1/m})\,|(f,e_j)_H|^2\right)^{1/2},$$

which follows from proposition 3.1 (cf. [**28**, Theorem 2.7]).

Let us consider three examples of a function α that satisfy hypotheses of theorem 4.2. As above, $0 \leqslant \ell \in \mathbb{Z}$ and $2 < p \in \mathbb{R}$. We arbitrarily choose a number $\varepsilon > 0$.

Putting $\alpha(t) := t^{l+\varepsilon+n/2-n/p}$ whenever $t \geqslant 1$, we obtain the power estimates (4.3) and (4.4) with $h(\lambda^{1/m}) = \lambda^{-\varepsilon/m}$ and $h(k^{1/n}) = k^{-\varepsilon/n}$, resp. In this case, $H^{\alpha}(\Gamma)$ is a Sobolev space.

Taking $s := l + n/2 - n/p$ and $r := s + \varepsilon$ in (3.3) and defining α by formula (3.1) with $\beta(t) \equiv 1$, we receive the Hörmander space $H^{\alpha}(\Gamma)$ that is not a part of the union of all Sobolev spaces $W_2^s(\Gamma)$ such that $s > \ell + n/2 - n/p$. In this case, $h(t) \searrow 0$ as $t \to \infty$.

The third example is given by any function $\alpha \in \mathrm{OR}$ such that

$$\alpha(t) := t^{l+n/2-n/p}\,(\underbrace{\log \ldots \log}_{k \text{ times}} t)^{\varepsilon} \quad \text{whenever } t \gg 1.$$

In this case, estimate (4.4) is of a logarithmic kind, and the space $H^{\alpha}(\Gamma)$ is broader than the above union of Sobolev spaces.

REMARK 4.4. Theorems 4.1 and 4.2 remain valid for every fractional $\ell > 0$, which follows from their proofs.

REMARK 4.5. Let $0 \leqslant \ell \in \mathbb{R}$, $1 \leqslant q < 2 < p$ and $s = \ell + n/q - n/p$. Assertions (ii) and (iii) of theorem 4.1 hold true if we replace the normed space $W_p^{\ell}(\Gamma)$ with the Triebel–Lizorkin space $F_{p,\theta}^{\ell}(\Gamma)$, where $1 \leqslant \theta \leqslant \infty$, or with the Besov space $B_{p,\theta}^{\ell}(\Gamma)$, where $2 \leqslant \theta \leqslant \infty$, and replace the convergence class $W_q^s(\Gamma)$ with $F_{q,\eta}^s(\Gamma)$, where $1 \leqslant \eta \leqslant \infty$, or with $B_{q,\eta}^s(\Gamma)$, where $1 \leqslant \eta \leqslant 2$. This follows from the continuous embeddings

$$F_{q,\eta}^s(\Gamma) \hookrightarrow W_2^{\ell+n/2-n/p}(\Gamma) \hookrightarrow F_{p,\theta}^{\ell}(\Gamma)$$

and

$$B_{q,\eta}^s(\Gamma) \hookrightarrow W_2^{\ell+n/2-n/p}(\Gamma) \hookrightarrow B_{p,\theta}^{\ell}(\Gamma)$$

[**34**, p. 60, properties (ii) and (iii)], as we can conclude analysing the proof of this theorem. Of course, it is natural to restrict ourselves to the marginal cases where $\eta = \infty$ and $\theta = 1$ for F-spaces and where $\eta = \theta = 2$ for B-spaces. In these cases we obtain narrower normed spaces than $W_p^{\ell}(\Gamma)$ and get broader convergence classes than $W_q^s(\Gamma)$. Of course, theorem 4.2 is also valid under this replacement of $W_p^{\ell}(\Gamma)$.

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

5. Uniform convergence

We use this term to refer to the convergence in the normed space $C^\ell(\Gamma)$ of ℓ times continuously differentiable functions on the manifold Γ, with $0 \leqslant \ell \in \mathbb{Z}$. Specifically, if $\ell = 0$, we get the convergence in the space $C(\Gamma)$ of continuous functions on Γ, i.e. the uniform convergence on Γ. Let $\|\cdot\|_{C,\ell}$ denote the norm in $C^\ell(\Gamma)$. If $\ell \geqslant 1$, we use the norm

$$\|f\|_{C,\ell} := \sum_{j=1}^{\varkappa} \max_{|\varrho|\leqslant\ell} \sup_{y\in\mathbb{R}^n} |\partial^\varrho((\chi_j f)\circ\pi_j)(y)|,$$

with π_j and χ_j being taking from the definition of Hörmander spaces over Γ. Here, of course, $\varrho = (\varrho_1, \ldots, \varrho_n)$ is a multi-index, $|\varrho| = \varrho_1 + \cdots + \varrho_n$, and ∂^ϱ is the partial derivative corresponding to ϱ.

THEOREM 5.1. *Suppose that* $0 < m \in \mathbb{R}$, $0 \leqslant \ell \in \mathbb{Z}$ *and* $\alpha \in \mathrm{OR}$. *Then the following three conditions are equivalent:*

(i)
$$\int_1^\infty \frac{t^{2\ell+n-1}}{\alpha^2(t)}\,\mathrm{d}t < \infty;$$

(ii) *there exists a PsDO* $A \in \mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$ *that spectral decomposition* (4.1) *induced by* A *converges in* $C^\ell(\Gamma)$ *on the class* $H^\alpha(\Gamma)$*;*

(iii) *the expansions in eigenfunctions of an arbitrary PsDO from* $\mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$ *converge unconditionally in* $C^\ell(\Gamma)$ *on the class* $H^\alpha(\Gamma)$.

Proof. It follows from Hörmander's embedding theorem [**17**, Theorem 2.2.7] that

$$\text{condition (i)} \iff H^\alpha(\Gamma) \subset C^\ell(\Gamma), \tag{5.1}$$

as is seen from [**28**, Proposition 2.6 (vi)].

Let us now prove that (i) $\Rightarrow$ (iii). We assume (i) to hold true and arbitrarily choose a PsDO $A \in \mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$. Owing to (5.1), the space $H^\alpha(\Gamma)$ lies in $L_2(\Gamma)$. Hence, the function $1/\alpha$ is bounded on $[1, \infty)$ in view of [**17**, Theorem 2.2.2]. We put $\omega(t) := 1/\alpha(t^{1/(2m)})$ whenever $t \geqslant 1$ and consider the function $\omega(L)$ of the operator $L = (I + A^*A) \in \mathrm{NE}\Psi^{2m}_{\mathrm{ph}}(\Gamma)$ acting in $L_2(\Gamma)$. According to proposition 3.1, the operator $\omega(L)$ sets an isomorphism between $L_2(\Gamma)$ and $H^\alpha(\Gamma)$. Hence, owing to (5.1), this operator acts continuously from $H = L_2(\Gamma)$ to $N := C^\ell(\Gamma)$, as noticed in remark 2.4. We therefore conclude by theorem 2.3, where $\eta(t) \equiv 1$, that the expansions in eigenfunctions of L converge unconditionally in $C^\ell(\Gamma)$ on the class $H^\alpha(\Gamma)$. We have proved that (i) $\Rightarrow$ (iii).

The implication (iii) $\Rightarrow$ (ii) is obvious.

The implication (ii) $\Rightarrow$ (i) is true because assertion (ii) entails the inclusion $H^\alpha(\Gamma) \subset C^\ell(\Gamma)$ and hence implies (i) by (5.1). □

Let $m > 0$ and $A \in \mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$. As in the previous section, $e := (e_j)_{j=1}^\infty$ is an orthonormal basis in $H = L_2(\Gamma)$ formed by eigenfunctions e_j of A, with $Ae_j = \lambda_j e_j$

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

and $|\lambda_j| \leqslant |\lambda_{j+1}|$ whenever $j \geqslant 1$. We complement theorem 5.1 by estimating the degree of the uniform convergence of series (2.3) on Hörmander classes.

THEOREM 5.2. *Let $0 \leqslant \ell \in \mathbb{Z}$, and suppose that certain functions h, $\beta \in \mathrm{OR}$ satisfy the following conditions: $h(t) \to 0$ as $t \to \infty$, h decreases on $[1, \infty)$, and*

$$\int_1^\infty \frac{t^{2\ell+n-1}}{\beta^2(t)}\,\mathrm{d}t < \infty.$$

Put $\alpha := \beta/h$, and note that the function α belongs to OR *and satisfies hypothesis (i) of theorem 5.1. Then*

$$\Big\|f - \sum_{j:|\lambda_j|\leqslant\lambda} (f, e_j)_H\, e_j\Big\|_{C,\ell} \leqslant c \cdot \|f\|_{H,\alpha} \cdot h(\lambda^{1/m}) \tag{5.2}$$

and

$$\Big\|f - \sum_{j=1}^{k} (f, e_j)_H\, e_j\Big\|_{C,\ell} \leqslant c \cdot \|f\|_{H,\alpha} \cdot h(k^{1/n}) \tag{5.3}$$

for all $f \in H^\alpha(\Gamma)$, $\lambda \geqslant 1$, and integer-valued $k \geqslant 1$. Here, c is a certain positive number that does not depend on f, λ and k.

Proof. Let, as above, $L := (I + A^*A)$, and consider the bounded functions $\omega(t) := (\beta(t^{1/(2m)}))^{-1}$ and $\eta(t) := h(t^{1/(2m)})$ of $t \geqslant 1$. (As to ω note that every function of class $\mathrm{OR} \cap L_1[1, \infty)$ is bounded on $[1, \infty)$). Since $(\omega\eta)(t) \equiv (\alpha(t^{1/(2m)}))^{-1}$, the operator $(\omega\eta)(L)$ sets an isomorphism between $H = L_2(\Gamma)$ and $H^\alpha(\Gamma)$ due to [**28**, Theorem 2.23]. Hence, according to theorem 2.3 and since the function h decreases, we obtain the estimates of form (4.5) and (4.6), with the norm $\|\cdot\|_{C,\ell}$ being instead of $\|\cdot\|_{W,\ell,p}$. These estimates imply (5.2) and (5.3) with the help of the same reasoning as that given in the proof of theorem 4.2. □

Let us indicate important examples of the function α satisfying hypotheses of theorem 5.2. We let $0 \leqslant \ell \in \mathbb{Z}$ and arbitrarily choose numbers ε and δ such that $0 < \delta < \varepsilon$.

Putting $\alpha(t) := t^{l+n/2+\varepsilon+\delta}$ whenever $t \geqslant 1$, we obtain power estimates (5.2) and (5.3) with $h(t) \equiv t^{-\delta}$. Such power estimates hold true for functions

$$f \in \bigcup_{s>l+n/2} W_2^s(\Gamma) \subsetneqq W_2^{l+n/2}(\Gamma).$$

To achieve the limiting value $s = l + n/2$, we put $\alpha(t) := t^{l+n/2}\log^{1/2+\varepsilon+\delta}(t+1)$ whenever $t \geqslant 1$ and receive the estimates of a logarithmic kind with $h(t) \equiv \log^{-\delta}(t+1)$.

6. Applications to ordinary differential operators

Let us discuss results of § 4 and 5 in the case where Γ is a circle $\mathbb{T}$ of the unit radius and when A is an (ordinary) differential operator on $\mathbb{T}$ of order m with

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

infinitely smooth complex-valued coefficients. Let τ, with $0 \leqslant \tau \leqslant 2\pi$, set a natural parametrization of $\mathbb{T}$. We suppose that the leading coefficient of A does not equal zero for any τ, which is equivalent to the ellipticity of A on $\mathbb{T}$. We also assume that A is a normal operator in the Hilbert space $H := L_2(\mathbb{T})$. If all coefficients of A are constant, this assumption holds true. In the case of variables coefficients, we recall the following necessary and sufficient condition for A to be self-adjoint in $L_2(\mathbb{T})$ [**31**, Chapter I, Section 1, Subsection 5]: A is a sum of differential operators of the form

$$A_{2k}u := (\mu_k u^{(k)})^{(k)} \quad \text{and} \quad A_{2k-1}u := \frac{i}{2}\bigl((\nu_k u^{(k)})^{(k-1)} + (\nu_k u^{(k-1)})^{(k)}\bigr)$$

provided that μ_k and ν_k are real-valued functions of class $C^\infty(\mathbb{T})$. Specifically, if all coefficients of A are real-valued, the above condition means that m is even and that A is the sum of A_{2k} with $k = 0, 1, \ldots, m/2$.

Let $e := (e_j)_{j=1}^\infty$ be an orthonormal basis in $H = L_2(\mathbb{T})$ formed by eigenfunctions e_j of A, with $Ae_j = \lambda_j e_j$ and $|\lambda_j| \leqslant |\lambda_{j+1}|$ whenever $j \geqslant 1$. Given a function $f \in L_2(\mathbb{T})$, we consider its expansion

$$f(\tau) = \sum_{j=1}^{\infty} (f, e_j)_H \, e_j(\tau), \quad \text{with} \quad 0 \leqslant \tau \leqslant 2\pi, \tag{6.1}$$

in these eigenfunctions, the expansion converging in $L_2(\mathbb{T})$. Let $0 \leqslant \ell \in \mathbb{Z}$ and $2 < p \in \mathbb{R}$. If $1 < q \leqslant 2$ and $s \geqslant \ell + 1/q - 1/p$, then theorem 4.1 implies that, for every $f \in W_q^s(\mathbb{T})$, this expansion converges in the p-th mean unconditionally and preserves this convergence after the termwise differentiation up to ℓ times. If a function parameter $\alpha \in \mathrm{OR}$ satisfies

$$\int_1^\infty \frac{t^{2\ell}}{\alpha^2(t)}\,\mathrm{d}t < \infty,$$

then theorem 5.1 implies that, for every $f \in H^\alpha(\mathbb{T})$, the expansion converges uniformly and unconditionally and preserve this convergence after the same differentiation. Note that the derivatives of $e_j(\tau)$ may not form an orthogonal basis in $L_2(\mathbb{T})$. Assuming α to satisfy the hypotheses of theorem 4.2 or 5.2, we see that estimates (4.3) and (5.2) become

$$\sum_{k=0}^{\ell}\biggl(\int_0^{2\pi}\Bigl|f^{(k)}(\tau) - \sum_{j:|\lambda_j|\leqslant\lambda}(f,e_j)_H\, e_j^{(k)}(\tau)\Bigr|^p \mathrm{d}\tau\biggr)^{1/p} \leqslant c\cdot\|f\|_{H,\alpha}\cdot h(\lambda^{1/m})$$

and

$$\sum_{k=0}^{\ell}\sup_{0\leqslant\tau\leqslant2\pi}\Bigl|f^{(k)}(\tau) - \sum_{j:|\lambda_j|\leqslant\lambda}(f,e_j)_H\, e_j^{(k)}(\tau)\Bigr| \leqslant c\cdot\|f\|_{H,\alpha}\cdot h(\lambda^{1/m})$$

whenever $f \in H^\alpha(\mathbb{T})$ and $\lambda \geqslant 1$; here, $h(\lambda^{1/m}) \to 0$ as $\lambda \to \infty$. Formulas (4.4) and (5.3) are rewritten quite similarly.

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

Considering remark 4.3 in the $A := 1 - d^2/d^2\tau$ case, we conclude that the norm $\|f\|_{H,\alpha}$ in the space $H^\alpha(\mathbb{T})$ is equivalent to the norm

$$\left(|c_0(f)|^2 + \sum_{j=1}^{\infty} \alpha^2(|j|)\left(|c_j(f)|^2 + |c_{-j}(f)|^2\right)\right)^{1/2},$$

with $c_j(f)$ being the Fourier coefficient of f with respect to the eigenfunction $e_j(\tau) := (2\pi)^{-1}e^{ij\tau}$ of the differential operator $1 - d^2/d^2\tau$.

In this case or if $A = d/d\tau$, expansion (6.1) becomes the Fourier series with respect to the complex trigonometric system on $[0,\, 2\pi]$. The above results on the mean convergence are new even for this series.

Ending this section, we remark that theorems 4.1, 4.2, 5.1 and 5.2 remain valid in the case where the coefficients of the ordinary differential operator A have finite smoothness depending, of course, on the parameters involved in these theorems. We will not specify the smoothness.

7. Applications to multiple Fourier series

Let $2 \leqslant n \in \mathbb{Z}$, and consider the n-dimensional torus $\mathbb{T}^n$; as above, $\mathbb{T}$ is a circle of the unit radius. Let A denote the Laplace–Beltrami operator on $\mathbb{T}^n$. It belongs to $\mathrm{NE}\Psi^2_{\mathrm{ph}}(\Gamma)$, is a self-adjoint operator in the Hilbert space $H := L_2(\mathbb{T}^n)$ and takes the form $A = \partial^2/\partial^2\tau_1 + \cdots + \partial^2/\partial^2\tau_n$, where τ_k, with $0 \leqslant \tau_k \leqslant 2\pi$, sets a natural parametrization of the k-th specimen of $\mathbb{T}$. The eigenfunctions $e_j(\tau) := (2\pi)^{-n}e^{i(j_1\tau_1+\cdots+j_n\tau_n)}$ of A, where $\tau = (\tau_1,\, \ldots,\, \tau_n) \in [0,\, 2\pi]^n$ and $j = (j_1,\, \ldots,\, j_n) \in \mathbb{Z}^n$, form an orthonormal basis in the Hilbert space $L_2(\mathbb{T}^n)$. Given a function $f \in L_2(\mathbb{T}^n)$, we consider its expansion in the multiple Fourier series

$$f(\tau) = \sum_{j\in\mathbb{Z}^n} c_j(f)\, e_j(\tau), \quad \text{with} \quad \tau \in [0, 2\pi]^n, \tag{7.1}$$

the expansion converging in $L_2(\mathbb{T}^n)$. Here, $c_j(f)$ denotes the Fourier coefficient of f with respect to e_j,

Since the termwise differentiation of series (7.1) gives the multiple Fourier series again, we restrict ourselves to the $\ell = 0$ case applying results of § 4 and 5 to expansion (7.1). Let $2 < p < \infty$. If $1 < q \leqslant 2$ and $s \geqslant n/q - n/p$, it follows from theorem 4.1 that, whatever $f \in W^s_q(\mathbb{T}^n)$, expansion (7.1) is unconditionally convergent in the p-th mean. If a function parameter $\alpha \in \mathrm{OR}$ satisfies

$$\int_1^{\infty} \frac{t^{n-1}}{\alpha^2(t)}\,\mathrm{d}t < \infty, \tag{7.2}$$

it follows from theorem 5.1 that, for every $f \in H^\alpha(\mathbb{T}^n)$, this expansion converges unconditionally uniformly (on $\mathbb{T}^n$). Considering the last result in the case of power functions $\alpha(t) \equiv t^s$, we conclude that expansion (7.1) converges unconditionally uniformly on each class $W^s_2(\mathbb{T}^n)$ where $s > n/2$. If n is odd, such a conclusion is substantiated in [**20**, Theorem A.1] concerning the uniform convergence over the torus.

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

Assuming α to satisfy hypotheses of theorem 4.2 or 5.2, we rewrite estimates (4.3) and (5.2) as follows:

$$\left(\int\limits_{[0,2\pi]^n} \left| f(\tau) - \sum_{j\in\mathbb{Z}^n:\|j\|\leqslant\lambda} c_j(f)\, e_j(\tau) \right|^p \mathrm{d}\tau \right)^{1/p} \leqslant c\cdot\|f\|_{H,\alpha}\cdot h(\lambda)$$

and

$$\sup_{\tau\in[0,2\pi]^n} \left| f(\tau) - \sum_{j\in\mathbb{Z}^n:\|j\|\leqslant\lambda} c_j(f)\, e_j(\tau) \right| \leqslant c\cdot\|f\|_{H,\alpha}\cdot h(\lambda),$$

whenever $f\in H^\alpha(\mathbb{T})$ and $\lambda\geqslant 1$. Here, $h(\lambda)\to 0$ as $\lambda\to\infty$, and we take into account that $Ae_j=-\|j\|^2 e_j$, with $\|j\|:=(j_1^2+\cdots+j_n^2)^{1/2}$. According to remark 4.3, the norm $\|f\|_{H,\alpha}$ in $H^\alpha(\mathbb{T}^n)$ is equivalent to the norm

$$\left(|c_0(f)|^2 + \sum_{j\in\mathbb{Z}^n, j\neq 0} \alpha^2(\|j\|)\, |c_j(f)|^2 \right)^{1/2}.$$

If α belongs to OR and satisfies (7.2), then

$$\sum_{j\in\mathbb{Z}^n} |c_j(f)| < \infty \tag{7.3}$$

for every $f\in H^\alpha(\mathbb{T}^n)$, which means the absolute convergence of the Fourier series (7.1) of f and follows from the unconditional uniform convergence of this series. As is known [**2**, Section 6, Subsection 3], property (7.3) is fulfilled provided that f belongs to the Hölder space $\mathcal{C}^\ell(\mathbb{T}^n)$ for some $\ell>n/2$ and is not fulfilled for certain functions $f\in\mathcal{C}^{n/2}(\mathbb{T}^n)$. Our sufficient condition for (7.3) to hold true is weaker than that indicated in terms of Hölder spaces. This is demonstrated by the example when $\alpha(t)\equiv t^{n/2}\log^{1/2+\varepsilon}(t+1)$ for some $\varepsilon>0$, which gives the broader space $H^\alpha(\mathbb{T}^n)$ (satisfying (7.2)) than the union of all $\mathcal{C}^\ell(\mathbb{T}^n)$ with $\ell>n/2$.

Note that of considerable interest is also the study of conditions for multiple Fourier series to converge almost everywhere. Relevant results are expounded in [**2**, **3**, **6**, **11**, **16**] and references therein.

8. Concluding remarks

Remark 8.1. Our results are also applied to the case where the operator $A\in \mathrm{NE}\Psi^m_{\mathrm{ph}}(\Gamma)$ is of negative order m because the resolvent of A belongs to $\mathrm{NE}\Psi^{-m}_{\mathrm{ph}}(\Gamma)$, is of positive order and has the same eigenfunctions as A does.

Remark 8.2. According to theorem 2.3, the left-hand side of (2.4) (i.e. the norm of a remainder of series (2.1)) depends monotonically on the corresponding set of eigenvectors of L. Hence, the estimates in theorems 4.2 and 5.2 remain true if the written sums additionally contain some other terms of the eigenfunction expansion.

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

Specifically, estimates (4.3) and (5.2) are valid if we replace the sum

$$\sum_{j:|\lambda_j|\leqslant\lambda} (f,e_j)_H\, e_j$$

with any partial sum of the series in the arbitrarily numbered eigenfunctions e_j provided that the later sum contains all e_j subject to $|\lambda_j|\leqslant\lambda$.

Remark 8.3. If the closed manifold Γ is a sphere (of any dimension), the results of § 4 and 5 admit further detailing and are also new.

Remark 8.4. Our approach allows other applications, specifically, to elliptic matrix pseudodifferential operators and elliptic differential operators in a bounded Euclidean domain.

Acknowledgements
This work was funded by the Czech Academy of Sciences within grant RVO:67985840 and by the National Academy of Sciences of Ukraine. The authors were supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 873071 (SOMPATY: Spectral Optimization: From Mathematics to Physics and Advanced Technology). The second named author was also supported by a grant from the Simons Foundation (1030291, A.A.M.).

References

1 M. S. Agranovich. Elliptic operators on closed manifolds, In: Partial Differential Equations, VI, Elliptic and Parabolic Operators, Encyclopaedia Math. Sci., Vol. 63 (Springer, Berlin, 1994), pp. 1–130.
2 Sh. A. Alimov, V. A. Il'in and E. M. Nikishin. Convergence problems of multiple trigonometric series and spectral decompositions. I. *Russian Math. Surveys* **31** (1976), 29–86.
3 Sh. A. Alimov, V. A. Il'in and E. M. Nikishin. Problems of convergence of multiple trigonometric series and spectral decompositions. II. *Russian Math. Surveys* **32** (1977), 115–140.
4 A. Anop, I. Chepurukhina and A. Murach. Elliptic problems with additional unknowns in boundary conditions and generalized Sobolev spaces. *Axioms* **10** (2021), 292.
5 A. Anop, R. Denk and A. Murach. Elliptic problems with rough boundary data in generalized Sobolev spaces. *Commun. Pure Appl. Anal.* **20** (2021), 697–735.
6 R. R. Ashurov. Generalized localization and summability almost everywhere of multiple Fourier series and integrals. *J. Math. Sci. (N.Y.)* **278** (2024), 408–425.
7 V. G. Avakumović. O jednom O-inverznom stavu. *Rad Jugoslovenske Akad. Znatn. Umjetnosti* **254** (1936), 167–186.
8 N. K. Bary (Bari). *A Treatise on Trigonometric Series*, Vols. I and II (The Macmillan Company, New York, 1964).
9 N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation, Encyclopedia of Mathematics and its Applications*, Vol. 27 (CUP, Cambridge, 1989).
10 V. V. Buldygin, K.-H. Indlekofer, O. I. Klesov and J. G. Steinebach. Pseudo-Regularly Varying Functions and Generalized Renewal Processes, Probability Theory and Stochastic Modelling, Vol. 91 (Springer, Cham, 2018).
11 A. Carbery and F. Soria. Almost-everywhere convergence of Fourier integrals for functions in Sobolev spaces and an L^2-localisation principle. *Rev. Mat. Iberoam.* **4** (1988), 319–337.

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press

12 L. Carleson. On convergence and growth of partial sums of Fourier series. *Acta Math.* **116** (1966), 135–157.

13 A. Dvoretzky and C. A. Rogers. Absolute and unconditional convergence in normed linear spaces. *Proc. Natl. Acad. Sci. USA* **36** (1950), 192–197.

14 R. E. Edwards. Fourier Series: A Modern Introduction, Vols. 1 and 2 [2nd edn], Graduate Text in Mathematics, Vols. 64 and 65 (Springer-Verlag, New York/Berlin, 1979, 1982).

15 I. M. Gel'fand and G. E. Shilov. Generalized Functions, Vol. 2, Spaces of Fundaqmental and Generalized Functions [Reprint of the 1968 English translation], (AMS Chelsea Publishing, USA, 2016).

16 B. I. Golubov. Multiple Fourier series and integrals. *J. Math. Sci* **24** (1984), 639–673.

17 L. Hörmander, Linear Partial Differential Operators, Die Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 116 (Springer, Berlin, 1963).

18 L. Hörmander, The Analysis of Linear Partial Differential Operators, Vol. II, Differential Operators with Constant Coefficients, Die Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 257 (Springer, Berlin, 1983).

19 V. A. Il'in. The solvability of mixed problems for hyperbolic and parabolic equations. *Russian Math. Surveys* **15** (1960), 85–142.

20 V. A. Ilin. *Spectral Theory of Differential Operators, Self-Adjoint Differential Operators* (Consultants Bureau, New York, 1995).

21 A. Kolmogoroff (Kolmogorov) and D. Menchoff (Men'shov). Sur la convergence des séries de fonetions orthogonales [French]. *Math. Z.* **26** (1927), 432–441.

22 M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik and P. E. Sobolevskiĭ. *Integral Operators in Spaces of Summable Functions* (Noordhoff International Publishing, Leiden, 1976).

23 M. G. Krein. On completely continuous operators in function spaces with two norms. *Works Inst. Math. Acad. Sci. Ukraine* **9** (1947), 104–129. [Russian] [English translation: M. G. Krein. Compact linear operators on functional spaces with two norms. *Integral Equations Operator Theory* **30** (1998), 140–162.

24 O. A. Ladyženskaya (Ladyzhenskaya), The Mixed Problem for a Hyperbolic Equation [Russian] (Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953).

25 V. A. Mikhailets and A. A. Murach. Refined scales of spaces and elliptic boundary-value problems. I. *Ukrainian Math. J.* **58** (2006), 244–262.

26 V. A. Mikhailets and A. A. Murach. Refined scales of spaces and elliptic boundary-value problems II. *Ukrainian Math. J.* **58** (2006), 398–417.

27 V. A. Mikhailets and A. A. Murach. Extended Sobolev scale and elliptic operators. *Ukrainian Math. J.* **65** (2013), 435–447.

28 V. A. Mikhailets and A. A. Murach. Hörmander Spaces, Interpolation, and Elliptic Problems, De Gruyter Studies in Mathematics, Vol. 60 (De Gruyter, Berlin, 2014).

29 V. A. Mikhailets and A. A. Murach. Interpolation Hilbert spaces between Sobolev spaces. *Results Math.* **67** (2015), 135–152.

30 K. Maurin. Methods of Hilbert Spaces [2nd edn], Monografie Matematyczne [Mathematical Monographs], Vol 45 (PWN—Polish Scientific Publishers, Warsaw, 1972).

31 M. A. Naimark. *Linear Differential Operators, Part I: Elementary Theory of Linear Differential Operators* (Frederick Ungar Publishing Co, New York, 1967).

32 M. A. Shubin. *Pseudodifferential Operators and Spectral Theory [2-nd edn]* (Springer, Berlin, 2001).

33 H. Triebel. *Interpolation Theory Function Spaces, Differential Operators [2nd edn]* (Johann Ambrosius Barth, Heidelberg, 1995).

34 H. Triebel. Theory of Function Spaces, III, Monographs in Mathematics, Vol. 100 (Birkhäuser, Basel, 2006).

35 P. L. Ul'janov (Ul'yanov). On series in a permuted trigonometric system [Russian]. *Izv. Akad. Nauk SSSR Ser. Mat.* **22** (1958), 515–542.

36 A. Zygmund. *Trigonometric Series, Vols. I and II [3rd edition]* (CUP, Cambridge, 2002).

https://doi.org/10.1017/prm.2024.40 Published online by Cambridge University Press