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PROBLEM SECTION

The Minimum Independence Number of a Hasse Diagram

JIŘÍ MATOUŠEK AND ALEŠ PŘÍVĚTIVÝ

Department of Applied Mathematics and Institute of Theoretical Computer Science (ITI)
Charles University
Malostranské nám. 25, 118 00 Praha 1
Czech Republic
matousek@kam.mff.cuni.cz, privetivy@kam.mff.cuni.cz

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Let $P = (X, \leqslant)$ be a finite partially ordered set. That is, X is a finite ground set and $\leqslant$ is a partial ordering on X (a reflexive, transitive, and weakly antisymmetric relation). An $x \in X$ is an *immediate predecessor* of a $y \in X$ if $x < y$ and there is no $z \in X$ with $x < z < y$ (where $x < y$ means that $x \leqslant y$ and $x \neq y$). The *Hasse diagram* $H(P)$ is the undirected graph with vertex set X and with $\{x, y\}$ forming an edge if x is an immediate predecessor of y or if y is an immediate predecessor of x. We denote by $\alpha(H(P))$ the *independence number* of the Hasse diagram, that is, the maximum possible size of a subset $I \subseteq X$ such that no element of I is an immediate predecessor (in P) of another element of I. This quantity should not be confused with the maximum size of an antichain in P, which is sometimes denoted by $\alpha(P)$.

Problem 1.1. *Estimate $f(n)$, defined as the minimum of $\alpha(H(P))$ over all n-element partially ordered sets P.*

The best known upper bound is apparently $O(n \log\log\log n / \log n)$, provided by a probabilistic construction of Brightwell and Nešetřil [3]. As for lower bounds, it is not difficult to see that $f(n) = \Omega(\sqrt{n})$. This can be improved by a logarithmic factor: $H(P)$ is triangle-free and every triangle-free graph on n vertices has an independent set of size $\Omega(\sqrt{n \log n})$ (see [1, 2]). This is the best lower bound we are aware of.

Thus, there is a considerable gap between the known lower and upper bounds, roughly $\sqrt{n}$ versus roughly linear. It is not clear to us which of the two bounds should be

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improved. Improving the upper bound might bring along new interesting examples of partially ordered sets.

Problem 1.1 is motivated by a geometric-looking question originating in computer science [4]: Given a set X of n points in the plane, we want to find a coloring $c: X \to \{1, 2, \ldots, k\}$ of X by k colors, with $k = k(X)$ as small as possible, such that for every axis-parallel rectangle R with $R \cap X \neq \emptyset$, there exists a color that appears exactly once in $R \cap X$. Such a coloring is called *conflict-free*. (The conflict-free coloring problem has also been studied, in [4] and other papers, with many other families of geometric shapes in place of axis-parallel rectangles.)

Let $k(n)$ be the maximum of $k(X)$ over all n-point sets X in the plane. The best known bounds are $k(n) = O(\sqrt{n/\log n})$ from above and $k(n) = \Omega(\log n)$ from below [4].

To see the connection to Problem 1.1, let us define the *generalized Delaunay graph* $\mathrm{GD}(X)$ of a given planar point set X. The vertex set is X, and $\{x, y\}$ is an edge if there exists an axis-parallel rectangle R with $R \cap X = \{x, y\}$. Clearly, any conflict-free coloring of X is a proper coloring of $\mathrm{GD}(X)$, and hence, $k(X) \geqslant \chi(\mathrm{GD}(X)) \geqslant n/\alpha(\mathrm{GD}(X))$. On the other hand, if we let X_1 be an independent set in $\mathrm{GD}(X)$, X_2 an independent set in $\mathrm{GD}(X \setminus X_1)$, etc., it is easily verified that the coloring assigning the points of X_i color i is conflict-free [4]. Thus, the minimum number of colors in a conflict-free coloring and $\alpha(\mathrm{GD}(X))$ are closely related.

Now $\mathrm{GD}(X)$ is the union of two Hasse diagrams $H(X, \preceq_1)$ and $H(X, \preceq_2)$, where $p \preceq_1 q$ for two points p, q in the plane if $x(p) \leqslant x(q)$ and $y(p) \leqslant y(q)$, while $p \preceq_2 q$ means $x(p) \leqslant x(q)$ and $y(p) \geqslant y(q)$.

The partial orderings $\preceq_1$ and $\preceq_2$ have (Dushnik–Miller) dimension 2, meaning that each of them is an intersection of two linear orderings. We thus arrive at the following variant of Problem 1.1:

Problem 1.2. *Estimate $f_2(n)$, defined as the minimum of $\alpha(H(P))$ over all n-element partially ordered sets P* of dimension 2.

The considerations above yield the following connections between f_2 and conflict-free colorings: $k(n) \geqslant n/f_2(n)$, and if $f_2(n) = \Omega(n^{1-\varepsilon})$ for some fixed $\varepsilon > 0$, then $k(n) = O(n^{2\varepsilon - \varepsilon^2})$.

The best known lower bound for $f_2(n)$ is the same as that for $f(n)$, while no nontrivial upper bound is known. The only result in this direction is an ingenious construction of Kříž and Nešetřil [5] of finite 2-dimensional partially ordered sets P with arbitrarily large chromatic number of $H(P)$. However, the independence number in this construction is a constant fraction of n.

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