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# MULTIVARIATE TIME SERIES WITH VARIOUS HIDDEN UNIT ROOTS, PART II

## *Estimation and Testing*

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This paper extends the statistical results obtained by Gregoir and Laroque (1994, *Journal of Econometrics* 63, 183–214). It develops statistical tools to analyze multivariate time series that can be represented under an autoregressive equation of finite order with various polynomial error correction terms at various frequencies with possibly a non-null deterministic part as introduced by Gregoir (1999, *Econometric Theory* 15, 435–468). We propose an estimation procedure that proceeds through repeated applications of principal component analysis and a specification test for the omission of a polynomial relation of cointegration at each frequency.

## 1. INTRODUCTION

The questionable use of seasonally adjusted data in empirical works by applied econometricians has been the subject of many works. For instance, Bell and Hillmer (1984) and Ghysels (1988) have pointed out that seasonal adjustment may lead to mistaken inference on relationships between economic time series. It may correspond to an important loss of information if seasonal fluctuations are a major source of variations in the time series under study. On the other hand, Ghysels and Perron (1993) and Maravall (1993) have, respectively, illustrated the effects of seasonal adjustment on test statistics for unit root and properties of multivariate representation. Seasonal adjustment may cause severe dynamic misspecifications in vectorial autoregressive regression (VAR). It may therefore be important to develop statistical tools that allow applied econometricians to work on multivariate nonseasonally adjusted times series. This means also for preliminary studies, tests for seasonal unit roots such as that proposed by Hasza and Fuller (1982), Dickey, Hasza, and Fuller (1984), or Hylleberg, Engle, Granger, and Yoo (1991).

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In Gregoir (1999), a general autoregressive representation theorem has been provided for a large class of real multivariate covariance stationary time series. This class is composed of multivariate processes with possibly various unit roots with various order of multiplicity in the determinant of the matrix series associated to their Wold representation. This kind of model must be considered when linear covariance stationary relationships exist between variables of different order of integration. This property is described as polynomial cointegration by Gregoir and Laroque (1993) but was previously introduced in some examples by Yoo (1986) and Granger and Lee (1989a, 1989b) as multicointegration. In this paper, we present an estimation procedure and a specification test that extend the results obtained by Gregoir and Laroque (1994) for a smaller class of processes in which only the unit root one was considered.

Cointegration at seasonal frequency was introduced by Engle, Granger, and Hallman (1989), and tests for seasonal cointegration have recently been designed. Engle, Granger, Hylleberg, and Lee (1993) proposed a generalization of the two-step Engle and Granger (1987) procedure to the case of monomial seasonal cointegration. Lee (1992) extended the maximum likelihood framework proposed by Johansen (1988, 1991) to the VAR of time series that are seasonally cointegrated. Both considered the representation of monomial cointegration under some restrictive assumptions. Recently Johansen and Schaumburg (1997) provided a complete treatment for the maximum likelihood estimation of the model without any restriction. Besides, some empirical studies on quarterly data have been performed. Engle et al. (1993) failed to find seasonal cointegration between Japanese household consumption and income, but Lee (1992) could not reject the null that there is cointegration at frequency $\pi$ between Canadian immigration and unemployment, and Kunst (1993) found some seasonal cointegration (at frequency $\pi/2$ and $\pi$) in a model of some macroeconomic time series for different European countries. The purpose of this paper is to present some results on tests for polynomial cointegration at various frequencies without any restrictions on the shape of (polynomial) error correction terms or on the (polynomial) loading factors.

Following Gregoir and Laroque (1993, 1994), we first develop an approach for covariance stationary processes without deterministic trend with the help of integral operators and do not consider the consequences of pretesting for the existence of various unit roots in the time series under study. For sake of simplicity, we consider a general framework of polynomial cointegration but assume that we know the frequencies of interest and test for the dimensions and degrees of the cointegration relationships for each frequency. We do not assume that a prior analysis allows us to define a subset of variables that appears in the relations of cointegration, as is done in the approach proposed by Phillips and Loretan (1991), Saikkonen (1991), Stock and Watson (1993), and others. The estimation procedure we propose cannot therefore provide the efficient estimators obtained by these authors. Similarly, a maximum likelihood approach in the spirit of that proposed by Johansen (1988) would be computationally difficult in the general case. In the case of monomial cointegration at frequency $\pi/2$, it has been analyzed by Johansen and Schaumburg (1997) and amounts to iterative procedures.

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Consequently, the cointegrating vectors we obtain in our approach by principal component analysis are not efficient estimators but consistent ones, relatively simple to compute. Nevertheless, as described by Harris (1997), they can be the starting point of the computation of efficient estimators once some nonparametric corrections have been computed in the spirit of the fully modified estimators proposed by Phillips and Hansen (1990). We are here mainly interested in the test for the ranks and degrees of polynomial relations of cointegration. Moreover, any efficient estimator of these relations that would satisfy an expansion of the kind we obtain can be used to test for the null hypotheses we consider.

The paper is organized as follows. In Section 2, the usual model of monomial seasonal cointegration considered by Lee (1992), Engle et al. (1993), and Johansen and Schaumburg (1997) is used to illustrate the estimation and test procedures that are presented, respectively, in Sections 3 and 4. The main purpose of Section 5 is to extend the results to a more general class of processes. This class is composed of processes in the class introduced previously but that have a deterministic trend that we restrict to be a linear combination of monomials $(t^{d_j}\cos\omega_j t, t^{d_j}\sin\omega_j t)_{(\omega_j\in[0,\pi],d_j\in\mathbb{N})}$. In the case of monomial cointegration at frequency 0, we know that inference on the dimension of the cointegration space based on maximum likelihood ratio test (Johansen, 1991; Johansen and Juselius, 1990) can be appreciably modified according to the presence and position of a constant term or linear trend. The framework introduced in the preceding sections allows us to provide a specification test for the omission of a relation of (polynomial) cointegration (at various frequencies) in presence of a deterministic term. A short conclusion is given in Section 6. The technical assumptions on the processes and proofs are gathered in Appendix A. The exact definitions of the integral operators denoted $G_\omega$ (which play roughly the role of the inverse of the first-difference operators $\Delta_\omega$), of $\mathcal{T}_\omega^l$ (the vector space of deterministic polynomial generated by $\{t^j\cos\omega t, t^j\sin\omega t,\ t\le l-1\}$), and the meaning of the notation $(A,B)^{*\omega}$ where $A$ and $B$ are two real matrices with the same dimensions are, respectively, introduced in Gregoir (1999) in Section 2 and Definition 3.2. We refer the reader to this paper for most of the results related to the representation and identification problem. In the sequel, the symbol $\Rightarrow$ signifies weak convergence.

## 2. INTRODUCTORY EXAMPLE

By way of illustration, we propose in this first section to describe through a simple example the main results we develop in the following sections in the general case of polynomial cointegration at different frequencies. In this example, we deal with the usual model of monomial cointegration at frequency 0, $\pi/2$, and $\pi$. We work with an $N$-dimensional process whose representation equation is of finite order $p$. We assume that we know the dimensions of the space of cointegration at each frequency: $r_0$, $r_{\pi/2}$, and $r_\pi$ and for sake of simplicity, there do not exist deterministic trends in the error correction terms at each frequency. Thus under the null, our process satisfies the following equation:

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$$y_t = \Theta_0 \alpha_0 G_0 y_{t-1} + \Theta_\pi \alpha_\pi G_\pi y_{t-1} + (\Theta^1_{\pi/2} + B\Theta^2_{\pi/2})$$

$$\times (\alpha^1_{\pi/2} G_{\pi/2} y_{t-1} + \alpha^2_{\pi/2} G_{\pi/2} y_{t-2}) + \sum_{j=1}^{p} \phi_j y_{t-j} + \epsilon_t, \quad \textbf{(1)}$$

where $\Theta_0$ and $\alpha_0'$ are $N \times r_0$, $\Theta_\pi$ and $\alpha_\pi'$ are $N \times r_\pi$, $\Theta^g_{\pi/2}$ and $\alpha^{g\prime}_{\pi/2}$ are $N \times r_{\pi/2}$, $\phi_j$ are $N \times N$, for all $j$, and at last $\epsilon_t$ is an $N$-dimensional martingale difference such that $E\epsilon_t \epsilon_t' = \Sigma$ and each component has a finite fourth moment. We first present an estimation procedure, then a statistical procedure to test for the specification of the model, i.e., to check that we do not omit a relation of cointegration at one frequency, precisely a test for $H_0^0 : r_0 = r$ against $H_a^0 : r_0 > r$, a test for $H_0^\pi : r_\pi = r$ against $H_a^\pi : r_\pi > r$, and a test for $H_0^{\pi/2} : r_{\pi/2} = r$ against $H_a^{\pi/2} : r_{\pi/2} > r$.

## 2.1. Estimation Procedure

A possible way of estimating the cointegration terms at frequency 0 (resp. $\pi$) is to compute the $r_0$ (resp. $r_\pi$) principal components associated to the $r_0$ (resp. $r_\pi$) smallest eigenvalues of the cross-product of $G_0 y_t$ (resp. $G_\pi y_t$). We denote $\hat{\alpha}_0$ (resp. $\hat{\alpha}_\pi$) the $r_0$ (resp. $r_\pi$) estimates of the basis of the cointegration space at frequency 0 (resp. $\pi$) and $\hat{\gamma}_0$ (resp. $\hat{\gamma}_\pi$) the $N - r_0$ (resp. $N - r_\pi$) vectors associated with the remaining principal components that are $I_0(1)$ (resp. $I_\pi(1)$). The way these estimators are built satisfies the identification rule presented in Theorem 3.2 in Gregoir (1999).

At frequency $\pi/2$, the estimation procedure is more complex, because according to Remark 3.1 in Gregoir (1999), we know that there exist $2r_{\pi/2}$ cointegrating vectors that are related two by two by linear constraints: if $(\alpha^1_{\pi/2} \quad \alpha^2_{\pi/2})$ is a set of cointegrating vectors then $(-\alpha^2_{\pi/2} \quad \alpha^1_{\pi/2})$ is also a set of cointegrating vectors. Various procedures can be used to estimate the cointegrating vectors according to whether or not we take into account the cross-restrictions that exist between the $2r_{\pi/2}$ vectors. We propose to follow a procedure that relies on the identification criterion proposed in Theorem 3.3 in Gregoir (1999). First we compute the smallest principal component of the cross-product of $(G_{\pi/2} y_t' \quad G_{\pi/2} y_{t-1}')'$, which we denote $(\hat{\alpha}^1_{\pi/2,(1)} \quad \hat{\alpha}^2_{\pi/2,(1)})$, and build $c^{\pi/2}_{(1)}$, the $2(N-1) \times 2N$ matrix whose orthogonal rows span the vector space orthogonal to $(\hat{\alpha}^1_{\pi/2,(1)} \quad \hat{\alpha}^2_{\pi/2,(1)})$ and $(-\hat{\alpha}^2_{\pi/2,(1)} \quad \hat{\alpha}^1_{\pi/2,(1)})$, which are linearly independent. We then compute the smallest principal component of the cross-product of $c^{\pi/2}_{(1)}(G_{\pi/2} y_t' \quad G_{\pi/2} y_{t-1}')'$ and get the coordinates of $(\hat{\alpha}^1_{\pi/2,(2)} \quad \hat{\alpha}^2_{\pi/2,(2)})$ in the basis $c^{\pi/2}_{(1)}$. We build $c^{\pi/2}_{(2)}$, the $2(N-2) \times 2N$ matrix whose orthonormal rows span the vector space orthogonal to $(\hat{\alpha}^1_{\pi/2,(1)} \quad \hat{\alpha}^2_{\pi/2,(1)})$, $(-\hat{\alpha}^2_{\pi/2,(1)} \quad \hat{\alpha}^1_{\pi/2,(1)})$, $(\hat{\alpha}^1_{\pi/2,(2)} \quad \hat{\alpha}^2_{\pi/2,(2)})$, and $(-\hat{\alpha}^2_{\pi/2,(2)} \quad \hat{\alpha}^1_{\pi/2,(2)})$, and compute the smallest principal component of the cross-product $c^{\pi/2}_{(2)}(G_{\pi/2} y_t' \quad G_{\pi/2} y_{t-1}')'$, and so on. At the end of the procedure, we obtain $2r_{\pi/2}$ vectors of cointegration that can be divided into two sets of $r_{\pi/2}$ vectors satisfying the preceding linear constraints. Finally, we denote $(\hat{\gamma}^1_{\pi/2} \quad \hat{\gamma}^2_{\pi/2})$, the $(N - r_{\pi/2}) \times 2N$ matrix whose rows are orthonormal to $(\hat{\alpha}^1_{\pi/2} \quad \hat{\alpha}^2_{\pi/2})$ and $(-\hat{\alpha}^2_{\pi/2} \quad \hat{\alpha}^1_{\pi/2})$,

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which have been built sequentially using the same procedure. We have the following result.

THEOREM 2.1. *The principal component estimates* $(\hat{\alpha}_0, \hat{\alpha}_\pi, (\hat{\alpha}^1_{\pi/2} \quad \hat{\alpha}^2_{\pi/2}))$ *are consistent estimators of* $(\alpha_0, \alpha_\pi, (\alpha^1_{\pi/2} \quad \alpha^2_{\pi/2}))$.

Once the cointegration relationships have been consistently estimated, the parameters $\Theta_0$, $\Theta_\pi$, $\Theta^1_{\pi/2}$, $\Theta^2_{\pi/2}$, and $\phi_j$, $j = 1, \ldots, p$ can be estimated by running ordinary least squares regressions of $y_t$ on the various cointegration terms and the lagged dependent variables. Nonetheless, this analysis is conditional on the null hypothesis on $r_0$, $r_\pi$, and $r_{\pi/2}$. It seems of interest to design a procedure to test for these hypotheses against possible alternatives.

### 2.2. A Specification Test

The test procedure we propose allows us to test for the omission of a relation of cointegration at a given frequency. It relies on the same idea as the test suggested by Dickey and Fuller (1979). If we introduce the terms $\hat{\gamma}_0 G_0 y_t$, $\hat{\gamma}_\pi G_\pi y_t$, and $\hat{\gamma}^1_{\pi/2} G_{\pi/2} y_t + \hat{\gamma}^2_{\pi/2} G_{\pi/2} y_{t-1}$, which are nonstationary under the null, in the right-hand side of the representation equation, the estimators of the coefficients of these variables superconverge toward zero, whereas under the alternative at least one of the estimators of these coefficients has a nonzero limit. We compute a statistic as the ratio of the variance explained by the integrated regressors for each frequency to the variance of the residuals, whose nonstandard asymptotic distribution is free of nuisance parameters. A similar test has been proposed by Gregoir and Laroque (1994) in case of cointegration at frequency 0. The test at frequency $\pi$ is very close to this one. In case of frequency $\pi/2$, the construction of a test statistic free of nuisance parameter requires an additional regression because of the involvement of lagged error correction terms in the evolution of the $N$-dimensional variable under study.

Let $y_t$ be a process whose data generating process is given by (1); let $(\hat{\Psi}_0, \hat{\Psi}_\pi, \hat{\Psi}^1_{\pi/2}, \hat{\Theta}_0, \hat{\Theta}_\pi, \hat{\Theta}^1_{\pi/2} \hat{\Theta}^2_{\pi/2}, \hat{\phi}_j, j = 1, \ldots, p')$ and $\hat{\eta}$ be, respectively, the ordinary least squares estimators of the $N \times N - r_0$ coefficient matrix $\Psi_0$, of the $N \times N - r_\pi$ coefficient matrix $\Psi_\pi$, of the $N \times 2(N - r_{\pi/2})$ coefficient matrix $\Psi^1_{\pi/2}$, of $\Theta_0, \Theta_\pi, \Theta^1_{\pi/2}, \Theta^2_{\pi/2}, \phi_j, j = 1, \ldots, p'$ and of the $N \times T$ matrix of residuals in the $N$-regression equations

$$
\begin{aligned}
y_t = {} & \Psi_0 \hat{\gamma}_0 G_0 y_{t-1} + \Psi_\pi \hat{\gamma}_\pi G_\pi y_{t-1} + \Psi^1_{\pi/2} \begin{pmatrix} \hat{\gamma}^1_{\pi/2} G_{\pi/2} y_{t-2} + \hat{\gamma}^2_{\pi/2} G_{\pi/2} y_{t-3} \\ \hat{\gamma}^1_{\pi/2} G_{\pi/2} y_{t-3} + \hat{\gamma}^2_{\pi/2} G_{\pi/2} y_{t-4} \end{pmatrix} \\
& + \Theta_0 \hat{\alpha}_0 G_0 y_{t-1} + \Theta_\pi \hat{\alpha}_\pi G_\pi y_{t-1} \\
& + (\Theta^1_{\pi/2} + B\Theta^2_{\pi/2})(\hat{\alpha}^1_{\pi/2} G_{\pi/2} y_{t-1} + \hat{\alpha}^2_{\pi/2} G_{\pi/2} y_{t-2}) + \sum_{j=1}^{p'} \phi_j y_{t-j} + \eta_t,
\end{aligned}
$$

where $p' \geq p$.

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Let $(\tilde{\Psi}_0, \tilde{\Psi}_\pi, \hat{\Psi}^2_{\pi/2}, \tilde{\Theta}_0, \tilde{\Theta}_\pi, \tilde{\Theta}^1_{\pi/2}, \tilde{\Theta}^2_{\pi/2}, \tilde{\phi}_j, j = 1, \ldots, p')$ and $\tilde{\eta}$ be respectively the ordinary least squares estimators of the $N \times N - r_0$ coefficient matrix $\Psi_0$, of the $N \times N - r_\pi$ coefficient matrix $\Psi_\pi$, of the $N \times 2(N - r_{\pi/2})$ coefficient matrix $\Psi^2_{\pi/2}$, of $\Theta_0, \Theta_\pi, \Theta^1_{\pi/2}, \Theta^2_{\pi/2}, \phi_j, j = 1, \ldots, p'$ and of the $N \times T$ matrix of residuals in the $N$-regression equations

$$y_t = \Psi_0 \hat{\gamma}_0 G_0 y_{t-1} + \Psi_\pi \hat{\gamma}_\pi G_\pi y_{t-1} + \Psi^2_{\pi/2} \begin{pmatrix} \hat{\gamma}^1_{\pi/2} G_{\pi/2} y_{t-1} + \hat{\gamma}^2_{\pi/2} G_{\pi/2} y_{t-2} \\ \hat{\gamma}^1_{\pi/2} G_{\pi/2} y_{t-2} + \hat{\gamma}^2_{\pi/2} G_{\pi/2} y_{t-3} \end{pmatrix}$$
$$+ \Theta_0 \hat{\alpha}_0 G_0 y_{t-1} + \Theta_\pi \hat{\alpha}_\pi G_\pi y_{t-1}$$
$$+ (\Theta^1_{\pi/2} + B\Theta^2_{\pi/2})(\hat{\alpha}^1_{\pi/2} G_{\pi/2} y_{t-1} + \hat{\alpha}^2_{\pi/2} G_{\pi/2} y_{t-2}) + \sum_{j=1}^{p'} \phi_j y_{t-j} + \eta_t,$$

where $p' \geq p$.

Then when the sample size $T$ goes to infinity, $\hat{\Theta}_0$, $\hat{\Theta}_\pi$, $\hat{\Theta}^1_{\pi/2}$, $\hat{\Theta}^2_{\pi/2}$, and $\hat{\phi}_j$, $j = 1, \ldots, p$ are consistent asymptotically normal estimators of $\Theta_0$, $\Theta_\pi$, $\Theta^1_{\pi/2}$, $\Theta^2_{\pi/2}$, and $\phi_j$, $j = 1, \ldots, p$.

Moreover, we have the following result.

THEOREM 2.2. *The following convergence in distribution holds when T goes to infinity*:

$$\text{Trace}\{T\Sigma(\hat{\Theta}_{\omega,\perp})^{-1}\hat{\Theta}'_{\omega,\perp}\hat{\Psi}_\omega[\hat{\gamma}_\omega G_\omega y(\hat{\gamma}_\omega G_\omega y)']\hat{\Psi}'_\omega \hat{\Theta}_{\omega,\perp}\}$$
$$\Rightarrow \text{Trace}\left\{\left(\int_0^1 dW'_\omega W_\omega\right)\left(\int_0^1 W_\omega(\tau)' W_\omega(\tau)\, d\tau\right)^{-1}\left(\int_0^1 W'_\omega\, dW_\omega\right)\right\},$$

*where*

(i) *when* $\omega \in \{0, \pi\}$, $\hat{\Theta}_{\omega,\perp}$ *is an* $N \times N - r_\omega$ *matrix of full rank such that* $\hat{\Theta}'_{\omega,\perp}\hat{\Theta}_\omega = 0, \Sigma(\hat{\Theta}_{\omega,\perp}) = \hat{\Theta}'_{\omega,\perp}\hat{\eta}'\hat{\eta}\hat{\Theta}_{\omega,\perp}$, $\hat{\gamma}_\omega G_\omega y$ *stands for the* $T \times N - r_\omega$ *matrix of generic element* $\hat{\gamma}_\omega G_\omega y_{t-1}$, $t = 1, \ldots, T$, *and* $W_\omega$ *is a standard Wiener process of dimension* $(1 \times N - r_\omega)$;

(ii) *when* $\omega = \pi/2$, $\hat{\Theta}_{\pi/2,\perp}$ *is a* $2N \times 2N - 2r_{\pi/2}$ *matrix of full rank such that*

$$\hat{\Theta}'_{\pi/2,\perp}\begin{pmatrix} \hat{\Theta}^1_{\pi/2} & \hat{\Theta}^2_{\pi/2} \\ -\hat{\Theta}^2_{\pi/2} & \hat{\Theta}^1_{\pi/2} \end{pmatrix} = 0,$$

$\Sigma(\hat{\Theta}_{\pi/2,\perp}) = (\hat{\Theta}'_{\pi/2,\perp} I_2 \otimes \hat{\eta}\hat{\eta}'\hat{\Theta}_{\pi/2,\perp})$, $\hat{\gamma}_{\pi/2} G_{\pi/2} y$ *stands for the* $T \times 2(N - r_{\pi/2})$ *matrix of generic element*

$$\begin{pmatrix} \hat{\gamma}^1_{\pi/2} G_{\pi/2} y_{t-1} + \hat{\gamma}^2_{\pi/2} G_{\pi/2} y_{t-2} \\ \hat{\gamma}^1_{\pi/2} G_{\pi/2} y_{t-2} + \hat{\gamma}^2_{\pi/2} G_{\pi/2} y_{t-3} \end{pmatrix},$$

$t = 1, \ldots, T$

$$\hat{\Psi}_{\pi/2} = \begin{pmatrix} \hat{\Psi}^1_{\pi/2} \\ \hat{\Psi}^2_{\pi/2} \end{pmatrix},$$

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*and*

$$W_{\pi/2} = \begin{pmatrix} W_{c,\pi/2} & W_{s,\pi/2} \\ -W_{s,\pi/2} & W_{c,\pi/2} \end{pmatrix}$$

*where* $W_{c,\pi/2}$ *and* $W_{s,\pi/2}$ *are two independent standard Wiener processes of dimension* $(1 \times N - r_{\pi/2})$.

The trace statistics of the preceding theorem provide consistent tests against the alternative hypotheses where $r_\omega$ is larger than under the null. Under these alternatives, some linear combinations of variables introduced for test purposes in the right-hand side of the equation are covariance stationary. The estimates of the nonzero coordinates of $\psi_\omega$ associated with these combinations are consistent so that the test statistic is at least $O_p(T)$.

Remark 2.1. In the general data generating process, there may exist deterministic terms in each error correction term. We have

$$\begin{aligned} y_t = {} & \Theta_0\{\alpha_0 G_0 y_{t-1} + \beta_0\} + \Theta_\pi\{\alpha_\pi G_\pi y_{t-1} + (-1)^{t-1}\beta_\pi\} \\ & + (\Theta^1_{\pi/2} + B\Theta^2_{\pi/2}) \\ & \times \left(\alpha^1_{\pi/2} G_{\pi/2} y_{t-1} + \alpha^2_{\pi/2} G_{\pi/2} y_{t-2} + \cos\frac{\pi}{2}(t-1)\beta^1_{\pi/2} + \sin\frac{\pi}{2}(t-1)\beta^2_{\pi/2}\right) \\ & + \sum_{j=1}^{p} \phi_j y_{t-j} + \epsilon_t. \end{aligned}$$

The estimation procedure can be easily amended to take into account the presence of these deterministic trends. We have to regress each $I_\omega(1)$ process on the corresponding deterministic trend(s) and apply the same estimation procedure to the residuals of these regressions. For the test procedure, we have to introduce the deterministic trends in the right-hand side of the equation in addition to the $I_\omega(1)$ linear relationships. Therefore the coefficient matrices $\psi_0$ and $\psi_\pi$ are, respectively, $N \times N - r_0 + 1$ and $N \times N - r_\pi + 1$ matrices and $\psi^g_{\pi/2}$, $g = 1,2$, are $N \times 2(N - r_{\pi/2} + 1)$ matrices. We then compute the same kind of ratio but with the corresponding $N - r_0 + 1 \times N - r_0 + 1$ (resp. $N - r_\pi + 1 \times N - r_\pi + 1$) or $2(N - r_{\pi/2} + 1) \times 2(N - r_{\pi/2} + 1)$ cross-product matrices built up with the extra terms introduced in the right-hand side of the equation. The asymptotic distribution of the test statistics is then modified. For 0 and $\pi$, we have to replace $W_0$ and $W_\pi$ by, respectively, $(W_0 \quad 1)$ and $(W_\pi \quad 1)$. For $\pi/2$, we replace $W_{c,\pi/2}$ and $W_{s,\pi/2}$, respectively, by $(W_{c,\pi/2} \quad 0)$ and $W_{s,\pi/2} \quad 1)$. We refer to Theorems 4.3 and 4.4 for more details.

## 3. GENERAL ESTIMATION PROCEDURE

As already mentioned in the introduction, the estimation of the model in this section, and in the following one, is made under the null hypothesis that we know the degrees and ranks of the cointegrating polynomials at each frequency.

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In case of monomial cointegration, Johansen (1988, 1991) has developed a full information maximum likelihood approach that has been recently extended by Johansen and Schaumburg (1997) to the case of seasonal monomial cointegration. In the general case we consider, it looks difficult to carry out in practice. We therefore intend to use the technique proposed by Gregoir and Laroque (1994) based on principal component analysis, which yields consistent estimates of the cointegrating vectors. In this framework, we work with a covariance stationary process. The idea of the procedure consists of considering the vectors of integrated series $(G_\omega y, G_\omega^2 y, \ldots, G_\omega^{m_\omega} y)$ and estimating sequentially the error correction terms starting with the terms of highest degree. The basic intuition is that knowing the number $r_{\omega,h}$ of terms of degree $h$ that are stationary, we estimate them with the polynomials of degree $h$ in the corresponding integral operator with the smallest variances, under some appropriate constraints.

We derive an asymptotic expansion of the estimates of the principal components, and we give explicitly the order of magnitudes of their weights on the integrated directions, which are required to derive the asymptotic tests of the next section. The procedure and the proofs of the theorems when $\omega \in \{0, \pi\}$ are close to those in Gregoir and Laroque (1994). Nevertheless for convenience, we present the procedure and restate the theorems.

### 3.1. A Principal Components Estimator When $\omega \in \{0,\pi\}$

Under the null, we know that there are exactly $r_{\omega,m_\omega}$ terms of degree $m_\omega$ in $G_\omega y$ and propose a $m_\omega$-step procedure in which we take into account the fact that the highest degree coefficients in a cointegrating vector are orthogonal to those associated with cointegrating vector of larger order of integration under the identification restrictions of Theorem 3.2 in Gregoir (1999).

Let us define the matrix $c_1 = I_N$ and the integer $s_1 = N$. The principal components estimator can be iteratively obtained through the replication of the following generic step, for $l = 1$ to $m_\omega$.

(Step $l$):

(a) Regress $c_l G_\omega^{m_\omega - l+1} y$ on $G_\omega y, \ldots, G_\omega^{m_\omega - l} y$ and a basis of $\mathcal{T}_\omega^{m_\omega - l+1}$ and take the principal components of the residuals. The coordinates of the $r_{\omega,m_\omega - l+1}$ smallest components denoted $\hat{a}^1_{\omega,m_\omega - l+1}$ give $\hat{\alpha}^{1,m_\omega - l+1}_{\omega,m_\omega - l+1} = \hat{\alpha}^1_{\omega,m_\omega - l+1} c_l$, whereas the associated regression coefficients (i.e., premultiplied by $\hat{a}^1_{\omega,m_\omega - l+1}$ with the opposite sign) give $\hat{\alpha}^{1,m_\omega - l+1}_{\omega,h}$, for $h \le m_\omega - l$ and $\hat{\beta}^{m_\omega - l+1}_{\omega,h}$ for $0 \le h \le m_\omega - l$.

(b) Denote $s_{l+1} = s_l - r_{m_\omega - l+1}$, $\hat{a}^{1,m_\omega - l+1}_\omega$ the $s_{l+1} \times s_l$ matrix of the largest principal components and $c_{l+1} = \hat{a}^{1,m_\omega - l+1}_\omega c_l$ and go to the step $l+1$.

We have the following result.

THEOREM 3.1. *The principal components estimates* $(\hat{\alpha}^{1,h}_\omega(G_\omega), \hat{\beta}^h_\omega)$, $h = 1, \ldots, m_\omega$ *are consistent estimators of* $(\alpha^{1,h}_\omega(G_\omega), \beta^h_\omega)$, $h = 1, \ldots, m_\omega$.

We need a more precise characterization of the asymptotic behavior of the estimators of the principal components to derive the statistics of the next section.

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In particular, it is useful to know the order of magnitude in $T$ of the estimation error. To describe the speed of convergence of the estimators, we take a particular basis, say, $\bar{\alpha}_\omega(G_\omega)$, of the $Nm_\omega$ space of polynomial $N$-vectors in $G_\omega$, of degree inferior or equal to $m_\omega$, without constant terms. First, we take the $k_\omega = \sum_{h=1}^{m_\omega} hr_{\omega,h}$ terms described in Remark 3.2 in Gregoir (1999), which are such that $\bar{\alpha}_\omega(G_\omega)y$ is stationary with possibly a deterministic trend. Then, we take the $N - r_{m_\omega}$ terms

$$G_\omega(\alpha_\omega^{0\prime} \quad \alpha_\omega^{1,1}(G_\omega)' \ldots \alpha_\omega^{1,m_\omega-1}(G_\omega)')'$$

that, applied to $y$, all yield $I_\omega(1)$ series. Iterating the procedure, at stage $h$, we take the $N - r_{m_\omega} - \cdots - r_{m_\omega-h+1}$ terms

$$G_\omega^h(\alpha_\omega^{0\prime} \quad \alpha_\omega^{1,1}(G_\omega)' \ldots \alpha^{1,m_\omega-h}(G_\omega)')'$$

that, applied to $y$, all yield $I_\omega(h)$ series, and so on, until $\alpha_\omega^0 G_\omega^{m_\omega}$. It is easy to see that by construction the resulting $\bar{\alpha}_\omega(G_\omega)$ is a full rank $(Nm_\omega \times N)$ matrix that forms a basis of the $N$-vector polynomials in $G_\omega$.

We define similarly $\bar{\beta}_\omega(t)$ as the corresponding $Nm_\omega \times 1$ deterministic vector obtained by the same operations.

Now let $\Delta_\omega(T)$ be the $(Nm_\omega \times Nm_\omega)$ diagonal matrix that describes the speeds of convergence of the $\bar{\alpha}_\omega(G_\omega)y_t + \bar{\beta}_\omega(t)$ series. Its first $k_\omega$ terms are taken equal to $T^{-1/2}$, and the others are $T^{-h}$ where $h$ is the degree of integration of the corresponding term in $\bar{\alpha}_\omega(G_\omega)y_t + \bar{\beta}_\omega(t)$. Similarly, let $\Delta_h^\omega(T)$ be the $h \times h$ diagonal matrix whose term $(j,j)$ is equal to $T^{-j}$, associated with the $h$ elements of the canonical basis of $\mathcal{T}_\omega^h$ whose generic element is $t^j e^{-i\omega t}$. In the following statement we use the same symbol for the vector space and its canonical basis. We have the following result.

THEOREM 3.2. *When the sample size T goes to infinity, the principal component estimates have the following asymptotic behavior*:

$$\hat{\alpha}_\omega^{1,h}(G_\omega)y_t + \hat{\beta}_\omega^h(t) - \alpha_\omega^{1,h}(G_\omega)y_t - \beta_\omega^h(t)$$
$$= O_{p,1}^{\omega,h}(1)\Delta_\omega(T)[\bar{\alpha}_\omega(G_\omega)y_t + \bar{\beta}_\omega(t)] + T^{-1/2}O_{p,2}^{\omega,h}(1)\Delta_h^\omega(T)\mathcal{T}_\omega^h(t),$$

*where* $O_{p,1}^{\omega,h}(1)$ *and* $O_{p,2}^{\omega,h}(1)$ *are, respectively,* $(r_{\omega,h} \times Nm_\omega)$ *and* $(r_{\omega,h} \times h)$ *matrices that both converge weakly.*

### 3.2. A Principal Component Estimator When $\omega \in ]0,\pi[$

Using the same approach as in the previous section, we want to estimate sequentially the error correction terms when $\omega \in ]0,\pi[$. We know that in this case, each set of $2r_{\omega,h}$ independent polynomial cointegrating vectors of degree $h$ can be partitioned into $r_{\omega,h}$ couples whose elements are related by linear constraints. We therefore are led to build up iteratively one vector in each couple and to derive the second one by the operation described in Remark 3.1 in Gregoir (1999).

We start with the terms of highest degree. Under the null, we know that there are exactly $2r_{\omega,m_\omega}$ terms of degree $m_\omega$. Among the polynomials of degree $m_\omega$ in

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$G_\omega y_t$ and $G_\omega y_{t-1}$ with a deterministic polynomial in $\mathcal{T}_\omega^{m_\omega}$, they are the ones with the smallest variance. Some cross-restrictions allow us to derive $r_{\omega,m_\omega}$ cointegrating vectors from the $r_{\omega,m_\omega}$ others. We take into account these cross-restrictions and compute iteratively only $r_{\omega,m_\omega}$ principal components. To find the error correction terms of lower degree, we have to take into account the constraint that their coefficients of highest degree are orthogonal to those of the error correction terms of larger degree under the identification restrictions of Theorem 3.3 in Gregoir (1999). We now describe this iterative procedure. Let us define the matrix $c_1^{m_\omega} = I_{2N}$ and the integer $s_{1,1} = 1$. The principal components estimator can then be iteratively obtained through the replication of the following generic step $l$ (for $l = 1$ to $m_\omega$) composed of generic substep $j$ (for $j = 1$ to $r_{m_\omega - l + 1}$) in which we denote $(\alpha_{\omega,h,(j)}^{1,m_\omega-l+1} \quad \alpha_{\omega,h,(j)}^{2,m_\omega-l+1})$ the $j$th row of the $r_\omega \times 2N$ matrix coefficient associated to the terms in $G_\omega^h y_t$ and $G_\omega^h y_{t-1}$ in the cointegration relationship of degree $m_\omega - l + 1$:

(Step $l$, substep $j$):

(a) Regress $c_j^{m_\omega-l+1}(G_\omega^{m_\omega-l+1}y_t' \quad G_\omega^{m_\omega-l+1}y_{t-1}')'$ on $G_\omega y_t, G_\omega y_{t-1}, \ldots, G_\omega^{m_\omega-1} y_t$, $G_\omega^{m_\omega-l} y_{t-1}$ and a basis of $\mathcal{T}_\omega^{m_\omega-l+1}$ and compute the principal components of the $2N - s_{l,j}$ + one-dimensional residuals. The coordinates of the smallest component denoted $(\hat{a}_{\omega,m_\omega-l+1,(j)}^1 \quad \hat{a}_{\omega,m_\omega-l+1,(j)}^2)$ give $(\hat{\alpha}_{\omega,m_\omega-l+1,(j)}^{1,m_\omega-l+1} \quad \hat{\alpha}_{\omega,m_\omega-l+1,(j)}^{2,m_\omega-l+1}) = (\hat{a}_{\omega,m_\omega-l+1,(j)}^1 \quad \hat{a}_{\omega,m_\omega-l+1,(j)}^2)c_j^{m_\omega-l+1}$ whereas the associated regression coefficients (i.e., premultiplied by $(\hat{a}_{\omega,m_\omega-l+1,(j)}^1 \quad \hat{a}_{\omega,m_\omega-l+1,(j)}^2)$ with the opposite sign) give $(\hat{\alpha}_{\omega,h,(j)}^{1,m_\omega-l+1} \quad \hat{\alpha}_{\omega,h,(j)}^{2,m_\omega-l+1})$ for $h \le m_\omega - l$ and $\hat{\beta}_{\omega,h}^{1,m_\omega-l}$ and $\hat{\beta}_{\omega,h}^{2,m_\omega-l}$ for $0 \le h \le m_\omega - l$.

(b) Build up a $2(N - s_{l,j}) \times 2N$ matrix whose orthonormal rows span the vector space orthogonal to $(\hat{\alpha}_{\omega,m_\omega-l+1,(j)}^{1,m_\omega-l+1} \quad \hat{\alpha}_{\omega,m_\omega-l+1,(j)}^{2,m_\omega-l+1})^{*\omega}$ in the vector space spanned by the rows of $c_j^{m_\omega-l+1}$. If $j = r_{m_\omega-l+1}$ then denote this matrix $c_1^{m_\omega-l}$, put $s_{l+1,1} = s_{l,r_{m_\omega-l+1}} + 1$, and go to (step $(l+1)$, substep 1); otherwise $j < r_{m_\omega-l+1}$, denote the matrix $c_{j+1}^{m_\omega-l+1}$, put $s_{l,j+1} = s_{l,j} + 1$, and go to (step $l$, substep $j+1$).

We have the following result.

THEOREM 3.3. *The principal component estimates* $(\hat{\alpha}_\omega^{1,h}(G_\omega), \hat{\alpha}_\omega^{2,h}(G_\omega), \hat{\beta}_\omega^h)$, $h = 1, \ldots, m_\omega$ *are consistent estimators of* $(\alpha_\omega^{1,h}(G_\omega), \alpha_\omega^{2,h}(G_\omega), \beta_\omega^h)$, $h = 1, \ldots, m_\omega$.

As in the preceding section, to derive the test statistics in Section 4, we need to know the speed of convergence of these estimators. We introduce a particular basis of the $2Nm_\omega$-dimensional vector space of polynomial $2N$-vectors in $G_\omega$ of degree inferior or equal to $m_\omega$ without constant term. First we take the $2k_\omega$ vectors described in Remark 3.2 in Gregoir (1999) that applied to $(y_t' \quad y_{t-1}')'$ yield stationary processes with possibly deterministic trends. We then take the $2(N - r_{\omega,m_\omega})$ terms

$$G_\omega((\alpha_\omega^{1,0}, \alpha_\omega^{2,0})^{*\omega\prime} \quad (\alpha_\omega^{1,1}(G_\omega), \alpha_\omega^{2,1}(G_\omega))^{*\omega\prime} \ \ldots \ (\alpha_\omega^{1,m_\omega-1}(G_\omega), \alpha_\omega^{2,m_\omega-1}(G_\omega))^{*\omega\prime})'$$

that applied to $(y_t' \quad y_{t-1}')'$ yield $I_\omega(1)$ times series with possibly deterministic trends following Remark 3.1 in Gregoir (1999).

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We iterate the procedure. At stage $h$, we take the $2(N - r_{\omega,m_\omega} - \cdots - r_{\omega,m_\omega-h+1})$ terms

$$G_\omega^h((\alpha_\omega^{1,0},\alpha_\omega^{2,0})^{*\omega\prime} \quad (\alpha_\omega^{1,1}(G_\omega),\alpha_\omega^{2,1}(G_\omega))^{*\omega\prime}\ldots(\alpha_\omega^{1,m_\omega-h}(G_\omega),\alpha_\omega^{2,m_\omega-h}(G_\omega))^{*\omega\prime})'$$

that applied to $(y_t' \quad y_{t-1}')'$ yield $I_\omega(h)$ time series with possibly deterministic trends, and so on, until $(\alpha_\omega^{1,0}G_\omega^{m_\omega},\alpha_\omega^{2,0}G_\omega^{m_\omega})^{*\omega}$. From Property 4.1 in Gregoir (1999), we conclude that the resulting matrix $\bar{\alpha}_\omega(G_\omega)$ is a full rank $2Nm_\omega \times 2N$ matrix that forms a basis of the $2N$-vector polynomials in $G_\omega$.

We define similarly $\bar{\beta}_\omega(t)$ as the corresponding $2Nm_\omega \times 1$ deterministic vector obtained by the same operations.[1]

Now let $\Delta_\omega(T)$ be the $(2Nm_\omega \times 2Nm_\omega)$ diagonal matrix that describes the speeds of convergence of the $\bar{\alpha}_\omega(G_\omega)(y_t' \quad y_{t-1}')' + \bar{\beta}_\omega(t)$ series. Its first $2k_\omega$ terms are taken equal to $T^{-1/2}$, and the others are $T^{-h}$ where $h$ is the degree of integration of the corresponding term in $\bar{\alpha}_\omega(G_\omega)(y_t' \quad y_{t-1}')' + \bar{\beta}_\omega(t)$. Similarly, let $\Delta_h^\omega(T)$ be the $2h \times 2h$ diagonal matrix whose $j$th $2 \times 2$ diagonal matrix on the diagonal is equal to $T^{-j}$, when associated with the generic elements $t^j \cos \omega t$ and $t^j \sin \omega t$ of the canonical basis of $\mathcal{T}_\omega^h$. In the following statement we use the same symbol for the vector space and its canonical basis. We have the following result.

THEOREM 3.4. *When the sample size T goes to infinity, the principal component estimates have the following asymptotic behavior*:

$$\hat{\alpha}_\omega^h(G_\omega)\begin{pmatrix} y_t \\ y_{t-1}\end{pmatrix} + \hat{\beta}_\omega^h(t) - \alpha_\omega^h(G_\omega)\begin{pmatrix} y_t \\ y_{t-1}\end{pmatrix} - \beta_\omega^h(t)$$

$$= O_{p,1}^{\omega,h}(1)\Delta_\omega(T)\left[\bar{\alpha}_\omega(G_\omega)\begin{pmatrix} y_t \\ y_{t-1}\end{pmatrix} + \bar{\beta}_\omega(t)\right] + T^{-1/2}O_{p,2}^{\omega,h}(1)\Delta_h^\omega(T)\mathcal{T}_\omega^h(t),$$

*where* $O_{p,1}^{\omega,h}(1)$ *and* $O_{p,2}^{\omega,h}(1)$ *are, respectively,* $(r_{\omega,h} \times 2Nm_\omega)$ *and* $(r_{\omega,h} \times h)$ *matrices that converge weakly.*

Remark 3.1. There are other consistent estimators of the cointegration space. However, from Gregoir and Laroque (1994), we know that a procedure that uses similar principal component computations but goes from the lowest to the highest degree gives inconsistent estimators.

Remark 3.2. The estimators we obtain by such a procedure are not efficient in comparison with estimators we would get by MLE techniques. In the case of monomial seasonal cointegration at frequency $\pi/2$, Johansen and Schaumburg (1997) were led to use complex representation of the VAR process and iterative maximization procedure to obtain the maximum likelihood estimators. In the case of polynomial cointegration, this approach could be computationally difficult because of complex linear cross-restrictions that have to be taken into account in the reduced-rank regression problem. Similarly, we cannot use the efficient estimators obtained by Phillips and Loretan (1991), Saikkonen (1991), or Stock and Watson (1993) under some additional restrictions on the structure of the cointegrating vectors we do not assume. We are basically interested in the tests

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for the ranks and degrees of the relation of cointegration. Nevertheless, following Harris (1997), we can modify the estimators we consider with nonparametric corrections in the spirit of the fully modified estimators proposed by Phillips and Hansen (1990) to obtain efficient estimates.

## 4. TEST OF THE MODEL SPECIFICATION

We consider a model in which there is simultaneous polynomial cointegration at different frequencies. We have to take into account the different directions of cointegration. We then reintroduce the index $j$ for the parameter $\omega$. Once the cointegration relationships $(\hat{\alpha}_{\omega_j}(G_{\omega_j}), \hat{\beta}_{\omega_j})$ have been consistently estimated, it is easy to estimate the other parameters $\Theta_{\omega_j}$ and $\phi$ of the representation theorem (equation (1) in Gregoir, 1999) simply by running ordinary least squares regressions of $y_t$ on the various cointegration terms and the lagged dependent variables. As noted by Engle and Granger (1987), this gives consistent, asymptotically normal estimators of $\Theta_{\omega_j}$ and $\phi$. But this analysis is conditional on the initial null hypothesis, i.e., the values of $r_{\omega_j,1}, \ldots, r_{\omega_j,m_{\omega_j}}$, for each $\omega_j \in \Omega$. It is important to design a procedure to test for these hypotheses against possible alternatives.

Under the maintained hypothesis that $m_{\omega_j}$ is large enough for each $(\omega_j, k_j) \in \Omega$, there are, as considered by Gregoir and Laroque (1994), two possible types of errors. First, the misspecification of $r_{\omega_j,1}, \ldots, r_{\omega_j,m_{\omega_j}}$ for at least one $(\omega_j, k_j) \in \Omega$ may have led to inclusion among the error correction terms of one or several terms that are integrated of order at least one for the corresponding frequency, rather than being stationary. In that circumstance, the corresponding coefficients in $\Theta^1_{\omega_j}(0)$, $\omega_j \in \{0, \pi\}$, (resp. $\Theta^1_{\omega_j}(0)$ and $\Theta^2_{\omega_j}(0)$, $\omega_j \in ]0, \pi[$, (super) converge to zero. This should inform the econometricians about the misspecification because according to the representation theorem, the matrix $\Theta^1_{\omega_j}(0)$ (resp. $(\Theta^1_{\omega_j}(0), \Theta^2_{\omega_j}(0))^{*\omega}$) is of full rank. We are not able to build such a test whose null hypothesis would be that the matrices considered previously are of full rank.

The second type of error corresponds to the situation in which one has forgotten some error correction term, so that $k_j = \sum_{h=1}^{m_{\omega_j}} h r_{\omega_j,h}$ is too small for at least one $\omega_j \in \Omega$. It is possible to design a statistical test of the null against the alternative for each frequency that relies on the same idea as that in the Dickey and Fuller (1979) test but in a multivariate setup. The specification test is built as follows. When $\omega_j \in \{0, \pi\}$, we denote $\gamma_{\omega_j}(G_{\omega_j})$ the $Nm_{\omega_j} - k_j$ last components of $\bar{\alpha}_{\omega_j}(G_{\omega_j})$ and $\delta_{\omega_j}(t)$ the $Nm_{\omega_j} - k_j$ corresponding components of $\bar{\beta}_{\omega_j}$. By construction, $\gamma_{\omega_j}(G_{\omega_j})y_t + \delta_{\omega_j}(t) = \mathcal{Z}_{\omega_j,t}$ is made of variables integrated of order at least one at frequency $\omega_j$ that are not cointegrated. Therefore, if we introduce these $(Nm_{\omega_j} - k_j)$ variables on the right-hand side of the representation equation, the estimator $\hat{\psi}_{\omega_j}$ of the coefficients of these variables superconverges toward 0. When $\omega_j \in ]0, \pi[$, we proceed similarly. We denote $\gamma_{\omega_j}(G_{\omega_j})$ the $Nm_{\omega_j} - k_j$ last components of $\bar{\alpha}_{\omega_j}(G_{\omega_j})$ built from polynomials $(\alpha^{1,h}_{\omega_j}(G_{\omega_j}) \quad \alpha^{2,h}_{\omega_j}(G_{\omega_j}))$ and $\delta_{\omega_j}(t)$ the $Nm_{\omega_j} - k_j$ corresponding components of $\bar{\beta}_{\omega_j}$. By construction, $\gamma_{\omega_j}(G_{\omega_j})(y_t' \quad y_{t-1}')' + \delta_{\omega_j}(t) = \mathcal{Z}_{\omega_j,t}$ is such that $(\mathcal{Z}'_{\omega_j,t} \quad \mathcal{Z}'_{\omega_j,t-1})'$ is made of variables integrated of order one or more at frequency $\omega_j$ that are not cointe-

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grated. Similarly, if we introduce these $(2Nm_{\omega_j} - 2k_j)$ variables on the right-hand side of the representation equation, the estimator $\hat{\psi}_{\omega_j}$ of the coefficients of these variables superconverges toward 0. We can introduce simultaneously these various nonstationary terms in the equation. We have statistical results that state that nonstationary processes at different frequencies are asymptotically orthogonal; this ensures that the analysis is similar for each nonstationary term introduced in the right-hand side of the regression. For sake of simplicity, the statements in Theorems 4.1–4.4 take in turn each set of nonstationary terms for each frequency. We then compute a statistic as the ratio of the variance explained by the integrated regressors for each frequency to the variance of the residuals whose asymptotic nonstandard distribution is free of nuisance parameters.

### 4.1. Notations

To describe these distributions, we need to introduce some pieces of notation. First, we define an operator $G$ on the space of multivariate $(1 \times n)$ continuous time stochastic processes $w(\tau)$, with $w$ almost surely in $C[0,1]^n$, through

$$Gw(\tau) = \int_0^\tau w(\tau_1)\, d\tau_1. \tag{2}$$

The term $Gw$ belongs almost surely to $C[0,1]^n$, and we note $G^d$, the $d$th iterate of $G$. For any positive integer $d$

$$G^d w(\tau) = \int_0^\tau d\tau_d \int_0^{\tau_d} d\tau_{d-1} \ldots \int_0^{\tau_2} w(\tau_1)\, d\tau_1 \tag{3}$$

(by convention, $G^0$ is equal to the identity).

Second, we associate to each $(1 \times n)$ complex continuous time stochastic process $m(\tau)$ with $\Re w$ and $\Im w$ almost surely in $C[0,1]^n$ a real matrix representation with the following form:[2]

$$w^*(\tau) = \begin{pmatrix} \Re w(\tau) & \Im w(\tau) \\ -\Im w(\tau) & \Re w(\tau) \end{pmatrix}$$

and define an operator $G_\omega$, $\omega \in ]0, \pi[$ on this real matrix representation through

$$G_\omega w^*(\tau) = \int_0^\tau w^*(\tau_1)(\Upsilon(\omega) \otimes I_n)\, d\tau_1, \tag{4}$$

where

$$\Upsilon(\omega) = \begin{pmatrix} \sin\omega & \cos\omega \\ -\cos\omega & \sin\omega \end{pmatrix}$$

such that the corresponding processes $\Re G_\omega w = G_{\omega,c} w$ and $\Im G_\omega w = G_{\omega,s} w$ belong almost surely to $C[0,1]^n$. We note $G_\omega^d$, the $d$th iterate of $G_\omega$. For any positive integer

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$$G_\omega^d w^*(\tau) = \int_0^\tau G_\omega^{d-1} w^*(\tau_1)(\Upsilon(\omega) \otimes I_n)\, d\tau_1 = \begin{pmatrix} G_{\omega,c}^d w(\tau) & G_{\omega,s}^d w(\tau) \\ -G_{\omega,s}^d w(\tau) & G_{\omega,c}^d w(\tau) \end{pmatrix} \tag{5}$$

(by convention, $G_\omega^0$ is equal to the identity).

Third, we define a bilinear operator $*$ that associates to two multivariate $(1 \times m)$ and $(1 \times n)$ stochastic processes $v$ and $w$ (resp. Brownian process $dw$) the $(m \times n)$ stochastic matrix $v * w'$ (resp. $v * dw'$), whose element $(j,l)$ is equal to

$$\int_0^1 v_j(\tau) w_l(\tau)\, d\tau$$

(resp. $\int_0^1 v_j(\tau)\, dw_l(\tau)$) and to two multivariate $(1 \times m)$ and $(1 \times m)$ complex stochastic processes $v$ and $w$ (resp. Brownian process $dw$) the complex stochastic matrix whose real and imaginary part are $\Re(v' * w) = \Re v' * \Re w + \Im v' * \Im w$, $\Im(v' * w) = -\Im v' * \Re w + \Re v' * \Im w$, (resp. $\Re(v' * dw) = \Re v' * \Re dw + \Im v' * \Im dw$, $\Im(v' * dw) = -\Im v' * \Re dw + \Re v' * \Im dw$). This also corresponds to the components of the $2m \times 2n$ real stochastic matrix $v^{*\prime} * w^*$ (resp. $v^{*\prime} * dw^*$).

Fourth, we use the same symbol $*$, which associates to a deterministic function $(m \times 1) g(t)$ and a multivariate $(1 \times n)$ stochastic process $w$ (resp. Brownian process $dw$) the $(m \times m)$ stochastic matrix $g(t) * w$ (resp. $g(t) * dw$), whose element $(j,l)$ is equal to

$$\int_0^1 g_j(\tau) w_l(\tau)\, d\tau$$

(resp. $\int_0^1 g_j(\tau)\, dw_l(\tau)$) and to the $m \times 1$ deterministic function $g(t)$ and a multivariate $(1 \times n)$ complex stochastic process $w$ (resp. Brownian process $dw$) the $m \times n$ complex stochastic matrix $g(t) * w$ (resp. $g(t) * dw$) whose real and imaginary part are $\Re(g(t) * w) = g(t) * \Re w$, $\Im(g(t) * w) = g(t) * \Im w$ (resp. $\Re(g(t) * dw) = g(t) * \Re dw$, $\Im(g(t) * dw) = g(t) * \Im dw$).

In the sequel, we use the notations $\tau_0^h$ to represent the $h + 1 \times 1$ vector whose $j$th element is equal to $\tau^{j-1}$.

### 4.2. A Theoretical Statistical Property

As in Gregoir and Laroque (1994), we decompose the analysis into two parts. The first part, Theorems 4.1 and 4.2, which follow, states a statistical result, assuming that the cointegration terms are known at each frequency. But the estimation errors of the cointegration terms introduce nuisance parameters in the test statistics. To get rid of these parameters, Theorems 4.3 and 4.4 in the next subsection present some corrections of the test statistics. The results stated in these theorems are still valid if we introduce simultaneously various sets of nonstationary terms in the right-hand side of the regression. We first give the technical assumption we

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require for the data generating process of the time series under study and its innovation process $\epsilon_t$.

Assumption 1. $y_t$ is a process in $\mathcal{C}_\Omega$ where $\Omega = \{(\omega_j, k_j), j = 1, \ldots, K\}$ with a finite representation and satisfies an equation of the following form:

$$\Phi(B)y_t = B\sum_{\omega_j}\{\Theta^1_{\omega_j}(\Delta_{\omega_j}) + B\Theta^2_{\omega_j}(\Delta_{\omega_j})\}\{\alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(t)\} + \varepsilon_t$$

with the properties described in Theorem 3.1 in Gregoir (1999) and $\phi_i = 0$, for $i > p$.

Assumption 2. $\varepsilon_t$ is an $N$-dimensional martingale difference satisfying $E(\varepsilon_t|\mathcal{F}_{t-1}) = 0$, $\mathrm{Cov}(\varepsilon_t|\mathcal{F}_{t-1}) = \Sigma$, and $\max_{j\in\{1,\ldots,N\}}\sup_t E(|\varepsilon_{j,t}|^{2+\delta}|\mathcal{F}_{t-1}) < +\infty$ a.s. for some $\delta > 0$, where $\mathcal{F}_{t-1}$ is the $\sigma$-field generated by $\{\varepsilon_{t-l}, l = 1,2,\ldots\}$ and $\Sigma$ is a positive definite matrix of full rank.

We now state the first theorem for $\omega \in \{0, \pi\}$: Let $y_t$ be a process satisfying Assumptions 1 and 2. Let $\hat{\psi}_{\omega_j}$, $(\hat{\Theta}^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$ $(\Theta^2_{\omega_j}(\cdot) = 0$ when $\omega_j \in \{0, \pi\})$, $\hat{\phi}_{\omega_j}$, and $\hat{\eta}_{\omega_j}$ be, respectively, the ordinary least squares estimators of the $N \times (Nm_{\omega_j} - k_j)$ coefficient matrix $\psi_{\omega_j}$, of $(\Theta^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\phi$, and of the $(N \times T)$ matrix of the residuals in the $N$ regression equations

$$y_t = \psi_{\omega_j}\mathcal{Z}_{\omega_j,t-1} - \sum_{l=1}^{p'}\phi_l y_{t-l} + \eta_t + B\sum_{\omega_l}\{\Theta^1_{\omega_l}(\Delta_{\omega_l}) + B\Theta^2_{\omega_l}(\Delta_{\omega_l})\}\{\alpha^1_{\omega_l}(G_{\omega_l})y_t + \alpha^2_{\omega_l}(G_{\omega_l})y_{t-1} + \beta_{\omega_l}(t)\}, \quad \textbf{(6)}$$

where $\mathcal{Z}_{\omega_j,t}$ is constructed as described in the beginning of this section and $p' \geq p$. In the following statement, $\mathcal{Z}'_{\omega_j,-1}$ stands for the $T \times (Nm_{\omega_j} - k_j)$ matrix of generic element $\mathcal{Z}'_{\omega_j,t-1}, t = 1,\ldots,T$.

THEOREM 4.1. *When the sample size T goes to infinity, the following convergence in distribution holds*:

$$\mathrm{Trace}\{T(\hat{\eta}_{\omega_j}\hat{\eta}'_{\omega_j})^{-1/2}\hat{\psi}_{\omega_j}[\mathcal{Z}_{\omega_j,-1}(\mathcal{Z}_{\omega_j,-1})']\hat{\psi}'_{\omega_j}(\hat{\eta}_{\omega_j}\hat{\eta}'_{\omega_j})^{-1/2}\}$$
$$\Rightarrow \mathrm{Trace}\{(dW'_{\omega_j} * X_{\omega_j})(X'_{\omega_j} * X_{\omega_j})^{-1}(X'_{\omega_j} * dW_{\omega_j})\},$$

*where* $W_{\omega_j} = (W_{\omega_j,0}, W_{\omega_j,1}, \ldots, W_{\omega_j,m_{\omega_j}})$ *is a standard Wiener process of dimension* $(1 \times N)$, *with* $W_{\omega_j,0}$ *of dimension* $(1 \times N - r_{\omega_j})$, *and* $W_{\omega_j,h}$ *of dimension* $(1 \times r_{\omega_j,h})$ *for* $h \geq 1$, *and finally* $X_{\omega_j} = (W_{\omega_j,0}, \ldots, G^{m_{\omega_j}-1}W_{\omega_j,0}, W_{\omega_j,1}, \ldots, G^{m_{\omega_j}-2}W_{\omega_j,1}, \ldots, W_{\omega_j,h}, \ldots, G^{m_{\omega_j}-h-1}W_{\omega_j,h}, \ldots, W_{\omega_j,m_{\omega_j}-1})$ *is of dimension* $(1 \times Nm_{\omega_j} - k_j)$.

We now consider the case $\omega_j \in ]0, \pi[$. Let $y_t$ be a process satisfying Assumptions 1 and 2. Let $\hat{\psi}_{\omega_j}$, $(\hat{\Theta}^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$ $(\Theta^2_{\omega_j}(\cdot) = 0$ when $\omega_j \in \{0, \pi\})$, $\hat{\phi}_{\omega_j}$, and $\hat{\eta}_{\omega_j}$ be, respectively, the ordinary least squares estimators of the $N \times (2Nm_{\omega_j} - 2k_j)$ coefficient matrix $\psi_{\omega_j}$, of $(\Theta^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\phi$, and of the $(N \times T)$ matrix of the residuals in the $N$ regression equations

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$$y_t = \psi_{\omega_j}\begin{pmatrix} \mathcal{Z}_{\omega_j,t-1} \\ \mathcal{Z}_{\omega_j,t-2} \end{pmatrix} - \sum_{l=1}^{p'} \phi_l y_{t-l} + \eta_t$$
$$+ B\sum_{\omega_l}\{\Theta^1_{\omega_l}(\Delta_{\omega_l}) + B\Theta^2_{\omega_l}(\Delta_{\omega_l})\}\{\alpha^1_{\omega_l}(G_{\omega_l})y_t + \alpha^2_{\omega_l}(G_{\omega_l})y_{t-1} + \beta_{\omega_l}(t)\}, \quad \textbf{(7)}$$

where the way $\mathcal{Z}_{\omega_j,t}$ is computed is described in the beginning of this section and $p' \geq p$. In the following theorem, $\mathcal{Z}'_{\omega_j,-1,-2}$ stands for the $T \times (2Nm_{\omega_j} - 2k_j)$ matrix of generic element $(\mathcal{Z}'_{\omega_j,t-1} \quad \mathcal{Z}'_{\omega_j,t-2})$, $t = 1,\ldots,T$.

THEOREM 4.2. *When the sample size T goes to infinity, the following convergence in distribution holds*:

$$\text{Trace}\,\{T(\hat{\eta}_{\omega_j}\hat{\eta}'_{\omega_j})^{-1/2}\hat{\psi}_{\omega_j}[\mathcal{Z}_{\omega_j,-1,-2}(\mathcal{Z}_{\omega_j,-1,-2})']\hat{\psi}'_{\omega_j}(\hat{\eta}_{\omega_j}\hat{\eta}'_{\omega_j})^{-1/2}\}$$
$$\Rightarrow \text{Trace}\left\{(\cos\omega_j I_N \quad \sin\omega_j I_N)(dW^{*\prime}_{\omega_j} * X^*_{\omega_j})(X^{*\prime}_{\omega_j} * X^*_{\omega_j})^{-1}\right.$$
$$\left.\times (X^{*\prime}_{\omega_j} * dW^*_{\omega_j})\begin{pmatrix}\cos\omega_j I_N \\ \sin\omega_j I_N\end{pmatrix}\right\},$$

*where*

$$W^*_{\omega_j} = \begin{pmatrix} W_{c,\omega_j} & W_{s,\omega_j} \\ -W_{s,\omega_j} & W_{c,\omega_j}\end{pmatrix}$$

*where* $W_{c,\omega_j}$ *and* $W_{s,\omega_j}$ *are two independent standard Wiener processes of dimension* $(1 \times N)$, *which can be, respectively, partitioned into* $(W_{f,\omega_j,0}, W_{f,\omega_j,1},\ldots,W_{f,\omega_j,m_{\omega_j}})$, $f \in \{c,s\}$, *with* $W_{f,\omega_j,0}$ *of dimension* $(1\times N - r_{\omega_j})$, *and* $W_{f,\omega_j,h}$ *of dimension* $(1\times r_{\omega_j,h})$ *for* $h \geq 1$; *then*

$$X^*_{\omega_j} = \begin{pmatrix} X_{c,\omega_j} & X_{s,\omega_j} \\ -X_{s,\omega_j} & X_{c,\omega_j}\end{pmatrix}$$

*where* $X_{f,\omega_j} = (W_{f,\omega_j,0},\ldots,G^{m_{\omega_j}-1}_{\omega_j,f}W_{\omega_j,0}, W_{f,\omega_{1j},1},\ldots,G^{m_{\omega_j}-2}_{\omega_j,f}W_{\omega_j,1},\ldots,W_{f,\omega_j,h},\ldots, G^{m_{\omega_j}-h-1}_{\omega_j,f}W_{\omega_j,h},\ldots, W_{f,\omega_j,m_{\omega_j}-1})$, $f \in \{c,s\}$, *and*

$$W_{\omega_j,h} = \begin{pmatrix} W_{c,\omega_j,h} & W_{s,\omega_j,h} \\ -W_{s,\omega_j,h} & W_{c,\omega_j,h}\end{pmatrix}$$

*so that* $X_{f,\omega_j}$, $f \in \{c,s\}$, *is of dimension* $(1 \times Nm_{\omega_j} - k_j)$.

The trace statistics of the preceding theorem provide consistent tests against the alternative hypotheses where $k_j = \sum hr_{\omega_j,h}$ is larger than in the null. Under these alternatives, some linear combinations of $\mathcal{Z}_{\omega_j,t}$ or $(\mathcal{Z}'_{\omega_j,t} \quad \mathcal{Z}'_{\omega_j,t-1})'$ are covariance stationary. The estimates of the (different from zero) coordinates of $\psi_{\omega_j}$ associated with those combinations are consistent: the test statistics is at least $O_p(T)$ (e.g., Phillips and Ouliaris, 1990, Sect. 5).

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### 4.3. The Specification Test

Unfortunately, the first step errors in the estimation of $\alpha_{\omega_j}(G_{\omega_j})$, when $\omega_j \in \{0,\pi\}$, or of $(\alpha^1_{\omega_j}(G_{\omega_j}) \quad \alpha^2_{\omega_j}(G_{\omega_j}))$, when $\omega_j \in ]0,\pi[$, introduce nuisance parameters in the limit distribution of the test statistics presented in Theorems 4.1 and 4.2. A projection on the orthogonal space to $\hat{\Theta}_{\omega_j}(0)$, when $\omega_j \in \{0,\pi\}$, allows us to get rid of these nuisance parameters. When $\omega_j \in ]0,\pi[$, the correction is more complicated because of the involvement of lagged integrated variables in the error correction terms. The correction we propose corresponds to a projection on the orthogonal space to $(\hat{\Theta}^1_{\omega_j}(0), \hat{\Theta}^2_{\omega_j}(0))^{*\omega_j}$, which allows us to eliminate the spreading of the error correction terms at date $t-1$ in the evolution of $y_t$ and $y_{t+1}$. It requires an additional regression. We now have to describe the way the terms $\hat{\mathcal{Z}}_{\omega_j,t}$ must be built.

When $\omega_j \in \{0,\pi\}$, take in turn the cointegration relationships of degree $h, h = m_{\omega_j},\ldots,1$, and $\alpha^h_{\omega_j,h}$ the estimated coefficients of the terms of highest degree. Then apply these coefficients to the residuals of the regression of $G^l_{\omega_j} y$ on $G^{l-1}_{\omega_j} y,\ldots,G_{\omega_j} y$ and a basis of $\mathcal{T}^l_\omega, l = h+1,\ldots,m_{\omega_j}$. Finally, let $\alpha^0_{\omega_j}$ be an $(N - r_{\omega_j} \times N)$ matrix of orthonormal vectors that completes the $\alpha^h_{\omega_j,h}$. Apply the same rule for $h = 0$. This yields an $(Nm_{\omega_j} - k_j \times T)$ estimate of $\hat{\mathcal{Z}}_{\omega_j,t}$.

When $\omega_j \in ]0,\pi[$, similarly, take in turn the cointegration relationships of degree $h, h = m_{\omega_j},\ldots,1$, and $(\alpha^{1,h}_{\omega_j,h} \quad \alpha^{1,h}_{\omega_j,h})$ the estimated coefficients of the terms of highest degree. Then apply these coefficients to the residuals of the regression of $G^l_{\omega_j} y_t$ and $G^l_{\omega_j} y_{t-1}$ on $G^{l-1}_{\omega_j} y_t, G^{l-1}_{\omega_j} y_{t-1},\ldots,G_{\omega_j} y_t, G_{\omega_j} y_{t-1}$ and a basis of $\mathcal{T}^l_\omega, l = h+1,\ldots,m_{\omega_j}$. Finally, let $(\alpha^{1,0}_{\omega_j} \quad \alpha^{2,0}_{\omega_j})$ be an $(N - r_{\omega_j} \times 2N)$ matrix of orthonormal vectors (built sequentially using the procedure described in Section 3.2) that completes the $\alpha^h_{\omega_j,h}$ in an $N \times 2N$ matrix of full rank $N$. Apply the same rule for $h = 0$. This yields an $(Nm_{\omega_j} - k_j \times T)$ estimate of $\hat{\mathcal{Z}}_{\omega_j,t}$.

In the sequel, $\mathcal{T}^{m_{\omega_j}}_{\omega_j}(t)$ stands for the canonical basis of the vector space $\mathcal{T}^{m_{\omega_j}}_{\omega_j}$ at date $t$. We first give a result for $\omega \in \{0,\pi\}$. Let $y_t$ be a process as in Theorem 4.1. Let $\tilde{\psi}_{\omega_j}$, $(\tilde{\Theta}^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$ $(\Theta^2_{\omega_j}(\cdot) = 0$ when $\omega_j \in \{0,\pi\})$, $\tilde{\phi}$, and $\tilde{\eta}$ be, respectively, the ordinary least square estimators of the $N \times ((N+1)m_{\omega_j} - k_j)$ coefficient matrix $\psi_{\omega_j}$, of $(\Theta^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\phi$, and of the $(N \times T)$ matrix of the residuals in the $N$ regression equations

$$y_t = \psi_{\omega_j}\begin{pmatrix}\hat{\mathcal{Z}}_{\omega_j,t-1}\\ \mathcal{T}^{m_{\omega_j}}_{\omega_j}(t)\end{pmatrix} - \sum_{l=1}^{p'} \phi_l y_{t-l} + \eta_t$$

$$+ B\sum_{\omega_l}\{\Theta^1_{\omega_l}(\Delta_{\omega_l}) + B\Theta^2_{\omega_l}(\Delta_{\omega_l})\}\{\hat{\alpha}^1_{\omega_l}(G_{\omega_l})y_t + \hat{\alpha}^2_{\omega_l}(G_{\omega_l})y_{t-1} + \hat{\beta}_{\omega_l}(t)\} \quad \textbf{(8)}$$

where $\hat{\mathcal{Z}}_{\omega_j,t}$ is described previously and $p' \geq p$.

THEOREM 4.3. *Let $\tilde{\Theta}_{\omega_j,\perp}$ be an $(N \times N - r_{\omega_j})$ full rank matrix such that $\tilde{\Theta}'_{\omega_j,\perp}\tilde{\Theta}_{\omega_j}(0) = 0$. Then when the sample size T goes to infinity, the following convergence in distribution holds*:

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$$\text{Trace}\,\{T(\tilde{\Theta}'_{\omega_j,\perp}\tilde{\eta}\tilde{\eta}'\tilde{\Theta}_{\omega_j,\perp})^{-1}\tilde{\Theta}'_{\omega_j,\perp}\tilde{\psi}_{\omega_j}[\hat{\mathcal{Z}}_{\omega_j,-1}(\hat{\mathcal{Z}}_{\omega_j,-1})']\tilde{\psi}'_{\omega_j}\tilde{\Theta}_{\omega_j,\perp}\}$$

$$\Rightarrow \text{Trace}\,\{(dW'_{\omega_j,0} * \tilde{X}_{\omega_j})(\tilde{X}'_{\omega_j} * \tilde{X}_{\omega_j})^{-1}(\tilde{X}'_{\omega_j} * dW_{\omega_j,0})\},$$

*where* $\tilde{X}_{\omega_j} = (X_{\omega} \quad \tau_0^{m_{\omega_j}-1\prime})$ *with* $W_{\omega_j,0}$ *and* $X_{\omega_j}$ *as defined in Theorem* 4.1 *and where* $\hat{\mathcal{Z}}'_{\omega_j,-1}$ *stands for the* $T \times ((N+1)m_{\omega_j} - k_j)$ *matrix of generic element* $(\mathcal{Z}'_{\omega_j,t-1} \quad \mathcal{T}_{\omega_j}^{m_{\omega_j}}(t-1)'), t = 1,\ldots,T.$

We now consider the case $\omega_j \in ]0,\pi[$. Let $y_t$ be a process as in Theorem 4.2. Let $\tilde{\psi}^1_{\omega_j}, (\tilde{\Theta}^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$ ($\Theta^2_{\omega_j}(\cdot) = 0$ when $\omega_j \in \{0,\pi\}$), $\tilde{\phi}$, and $\tilde{\eta}$ be, respectively, the ordinary least squares estimators of the $N \times (2(N+1)m_{\omega_j} - 2k_j)$ coefficient matrix $\psi^1_{\omega_j}$, of $(\Theta^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\phi$, and of the $(N \times T)$ matrix of the residuals in the $N$ regression equations

$$y_t = \psi^1_{\omega_j}\begin{pmatrix} \hat{\mathcal{Z}}_{\omega_j,t-2} \\ \hat{\mathcal{Z}}_{\omega_j,t-3} \\ \mathcal{T}_{\omega_j}^{m_{\omega_j}}(t-2) \end{pmatrix} - \sum_{l=1}^{p'} \phi_l y_{t-l} + \eta_t$$

$$+ B \sum_{\omega_l \in ]0,\pi[} \{\Theta^1_{\omega_l}(\Delta_{\omega_l}) + B\Theta^2_{\omega_l}(\Delta_{\omega_l})\}$$

$$\times \{\hat{\alpha}^1_{\omega_l}(G_{\omega_l})y_t + \hat{\alpha}^2_{\omega_l}(G_{\omega_l})y_{t-1} + \hat{\beta}_{\omega_l}(t)\}, \tag{9}$$

where $\hat{\mathcal{Z}}_{\omega_j,t}$ is described previously and $p' \geq p$. Let $\tilde{\psi}^2_{\omega_j}, (\breve{\Theta}^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\breve{\phi}$, and $\breve{\eta}$ be, respectively, the ordinary least squares estimators of the $N \times (2(N+1)m_{\omega_j} - 2k_j)$ coefficient matrix $\psi^2_{\omega_j}$, of $(\Theta^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\phi$, and of the $(N \times T)$ matrix of the residuals in the $N$ regression equations

$$y_t = \psi^2_{\omega_j}\begin{pmatrix} \hat{\mathcal{Z}}_{\omega_j,t-1} \\ \hat{\mathcal{Z}}_{\omega_j,t-2} \\ \mathcal{T}_{\omega_j}^{m_{\omega_j}}(t-1) \end{pmatrix} - \sum_{l=1}^{p'} \phi_l y_{t-l} + \eta_t$$

$$+ B \sum_{\omega_l \in ]0,\pi[} \{\Theta^1_{\omega_l}(\Delta_{\omega_l}) + B\Theta^2_{\omega_l}(\Delta_{\omega_l})\}$$

$$\times \{\hat{\alpha}^1_{\omega_l}(G_{\omega_l})y_t + \hat{\alpha}^2_{\omega_l}(G_{\omega_l})y_{t-1} + \hat{\beta}_{\omega_l}(t)\}. \tag{10}$$

THEOREM 4.4. *Let* $\tilde{\Theta}_{\omega_j,\perp}$ *be a* $2N \times 2(N - r_{\omega_j})$ *full rank matrix such that* $\tilde{\Theta}'_{\omega_j,\perp}(\tilde{\Theta}^1_{\omega_j}(0), \tilde{\Theta}^2_{\omega_j}(0))^{*\omega_j} = 0$. *Then when the sample size* $T$ *goes to infinity, the following convergence in distribution holds*:

$$\text{Trace}\left\{T\,\tilde{\Sigma}_{\omega_j}^{-1/2}\tilde{\Theta}'_{\omega_j,\perp}\begin{pmatrix}\tilde{\psi}^1_{\omega_j}\\ \tilde{\psi}^2_{\omega_j}\end{pmatrix}[\hat{\mathcal{Z}}_{\omega_j,-1,-2}(\hat{\mathcal{Z}}_{\omega_j,-1,-2})']\begin{pmatrix}\tilde{\psi}^1_{\omega_j}\\ \tilde{\psi}^2_{\omega_j}\end{pmatrix}'\tilde{\Theta}_{\omega_j,\perp}\tilde{\Sigma}_{\omega_j}^{-1/2}\right\}$$

$$\Rightarrow \text{Trace}\,\{(dW^{*\prime}_{\omega_j,0} * \tilde{X}^*_{\omega_j})(\tilde{X}^{*\prime}_{\omega_j} * \tilde{X}^*_{\omega_j})^{-1}(\tilde{X}^{*\prime}_{\omega_j} * dW^*_{\omega_j,0})\},$$

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*where*

$$\tilde{\Sigma}_{\omega_j} = \left( \tilde{\Theta}'_{\omega_j,\perp} \begin{pmatrix} \tilde{\eta}\tilde{\eta}' & \cos\omega_j \tilde{\eta}\tilde{\eta}' \\ \cos\omega_j \tilde{\eta}\tilde{\eta}' & \tilde{\eta}\tilde{\eta}' \end{pmatrix} \tilde{\Theta}_{\omega_j,\perp} \right),$$

$\hat{\mathcal{Z}}'_{\omega_j,-1,-2}$ *stands for the* $T \times (2(N+1)m_{\omega_j} - 2k_j)$ *matrix of generic element* $(\hat{\mathcal{Z}}'_{\omega_j,t-1} \quad \hat{\mathcal{Z}}'_{\omega_j,t-2} \quad \mathcal{T}^{m_{\omega_j}}_{\omega_j}(t-1)')$, $t = 1,\ldots,T$, *with* $\tilde{X}_{c,\omega_j} = (X_{c,\omega_j} \quad 0_{(1\times m_{\omega_j})})$ *and* $\tilde{X}_{s,\omega_j} = (X_{s,\omega_j} \quad \tau_0^{m_{\omega_j}-1})$, *where* $W_{\omega_j,0}$, $X_{c,\omega_j}$, *and* $X_{s,\omega_j}$ *have been defined in Theorem* 4.2.

The first step errors in the estimation of the cointegration terms create a difficulty: the test based on the trace statistics of Theorems 4.3 and 4.4 procedure is only consistent against the alternative hypothesis of larger $k_j = \sum h r_{\omega_j,h}$ such that when $\omega_j \in \{0,\pi\}, \Theta'_{\omega_j,\perp}\psi_{\omega_j}$ is different from zero and when $\omega_j \in ]0,\pi[$, $\Theta'_{\omega_j,\perp}\begin{pmatrix}\psi^1_{\omega_j}\\ \psi^2_{\omega_j}\end{pmatrix}$ is different from zero. For the (measure zero) subset of alternatives under which this matrix product is null, the trace statistic does not tend to infinity with $T$. This situation does not exist in the monomial case where when $\omega_j \in \{0,\pi\}$, $\Theta'_{\omega_j,\perp}\psi_{\omega_j}$ and when $\omega_j \in ]0,\pi[$, $\Theta'_{\omega_j,\perp}\begin{pmatrix}\psi^1_{\omega_j}\\ \psi^2_{\omega_j}\end{pmatrix}$ are always different from zero.

Appendix B presents the quantiles of the corresponding test statistics for the case where the statistician considers twice integrated time series ($m_\omega = 2$) and $\omega = \pi/2$. When $\omega = 0$ or $\omega = \pi$, the useful table is given in Gregoir and Laroque (1994). Five thousand independent drawings of the distribution have been computed for a value of $T = 400$. The rank $N - r_{\pi/2}$ ranges from 1 to 5, with values of $r_{\pi/2,1}$ from 0 to 5, $r_{\pi/2,2}$ being implicitly equal to $r_{\pi/2} - r_{\pi/2,1}$.

## 5. DETERMINISTIC TRENDS

Johansen (1991) and Johansen and Juselius (1990) have shown that the presence of deterministic trends under the null in the Wold representation of a multivariate process plays a crucial role in the statistical and probabilistic analysis of the model. The subject of this section is to extend the results obtained in the preceding sections to the larger class of similar processes but with deterministic trends we restrict to be a linear combination of various elements in some vector spaces $\mathcal{T}^{d_{j'}}_{\omega_{j'}}$. The representation equation for such a class of processes is in Gregoir (1999). We briefly explain the modification we have to introduce in the estimation procedure and design a new test procedure for the presence of omitted polynomial relations of cointegration. The proofs of these results that are relatively close to the previous ones are not given in the appendix because of lack of space but are available from the author.

We start by introducing our data generating process.

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Assumption 3. $y_t$ is a process in $\mathcal{C}_{\Omega,\Gamma}$ with a finite representation, which satisfies an equation of the following form:

$$\Phi(B)y_t = \Phi_\nu(t) + \varepsilon_t + B\sum_{\omega_j}\{\Theta^1_{\omega_j}(\Delta_{\omega_j}) + B\Theta^2_{\omega_j}(\Delta_{\omega_j})\}$$
$$\times\{\alpha^1_{\omega_j}(G_{\omega_j})y_t + \alpha^2_{\omega_j}(G_{\omega_j})y_{t-1} + \beta_{\omega_j}(\nu)(t)\}$$

with the properties described in Theorem 6.1 in Gregoir (1999), in particular

$$\nu_t \in \bigoplus_{j'=1}^{j'=D} \mathcal{T}^{d_{j'}}_{\varpi_{j'}} = \mathcal{T}_\nu$$

and $\phi_i = 0$, for $i > p$, $\Omega = \{(\omega_j, k_j), j = 1, \ldots, K\}$, and $\Gamma = \{(\varpi_{j'}, d_{j'})j' = 1, \ldots, D\}$.

In this section, we use the notation $\mathcal{T}_{\nu-\tilde{\Gamma}}$ with $\tilde{\Gamma} \subset \Gamma$ to denote the vector space such that $\mathcal{T}_\nu = \mathcal{T}_{\nu-\tilde{\Gamma}} \bigoplus_{(\varpi_{j'}, d_{j'})\in\tilde{\Gamma}} \mathcal{T}^{d_{j'}}_{\varpi_{j'}}$. When $\omega_j$ is not an element of a couple in $\Gamma$, we put $d_j = 0$, so that $\mathcal{T}^{h+d_j}_{\omega_j} = \mathcal{T}^h_{\omega_j}$, $\mathcal{T}^{d_j}_{\omega_j} = \{\vec{0}\}$, and $\nu - \{\omega_j\} = \nu$. From now on, we assume that the innovation process $\epsilon_t$ of the time series under study satisfies Assumption 2.

Using the results of the preceding subsection, it is easy to amend the estimation procedure described in Section 3 to take into account the presence of the deterministic trends in the error correction terms induced by the existence of $\nu_t$ under the null. We have to include at step $h$ in the regressors with $G^{m_{\omega_j}-h}_{\omega_j} y_t, G^{m_{\omega_j}-h-1}_{\omega_j} y_t, \ldots, G_{\omega_j} y_t$ a basis of the vector space of deterministic functions described in Theorem 6.1 in Gregoir (1999). For this modified procedure, results such as Theorems 3.1 and 3.3 can be easily extended, and we have consistency of the estimators. Similarly, Theorems 3.2 and 3.4 are slightly modified. The estimated principal components keep the described asymptotic behavior. They more precisely have weights of order $T^{-h-1/2}$ on each deterministic trend $t^h e^{i\varpi_j t}$ included in the regressions. To describe the behavior of these terms, we need to introduce a basis of the vector polynomials of degree at most $m_{\omega_j}$. We use the basis $\bar{\alpha}_{\omega_j}(G_{\omega_j})$, which is defined in Sections 3.1 and 3.2. The construction of $\beta_{\omega_j}(\nu)(t)$ is derived in the same way as that presented in Section 6 in Gregoir (1999).

THEOREM 5.1. *When the sample size T goes to infinity, the principal component estimates have the following asymptotic behavior.*

1. *When* $\omega_j \in \{0, \pi\}$,

$$\hat{\alpha}^{1,h}_{\omega_j}(G_{\omega_j})y_t + \hat{\beta}^h_{\omega_j}(\nu)(t) - \alpha^{1,h}_{\omega_j}(G_{\omega_j})y_t - \beta^h_{\omega_j}(\nu)(t)$$
$$= O^{\omega_j,h}_{p,1}(1)\tilde{\Delta}_{\omega_j}(T)[\bar{\alpha}_{\omega_j}(G_{\omega_j})y_t + \bar{\beta}_{\omega_j}(\nu)(t)]$$
$$+ T^{-1/2}O^{\omega_j,h}_{p,2}(1)\Delta^{\omega_j}_{h+d_j}(T)\mathcal{T}^{h+d_j}_{\omega_j}(t) + T^{-1/2}O^{\omega_j,h}_{p,3}(1)\Delta^{\nu-\{\omega_j\}}_h(T)\mathcal{T}_{\nu-\{\omega_j\}}(t)$$

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*where* $O_{p,1}^{\omega_j,h}(1)$, $O_{p,2}^{\omega_j,h}(1)$, *and* $O_{p,3}^{\omega_j,h}(1)$ *are, respectively,* $(r_{\omega_j,h} \times Nm_{\omega_j})$, $(r_{\omega_j,h} \times h + d_j)$, *and* $(r_{\omega_j,h} \times \sum_{\varpi_{l'} \neq \omega_j, \varpi_{l'} \in \{0,\pi\}} d_{l'} + \sum_{\varpi_{l'} \in ]0,\pi[} 2d_{l'})$ *matrices that converge weakly. The terms* $\tilde{\Delta}_{\omega_j}(T)$ *and* $\Delta_{h+d_j}^{\omega_j}(T)$ *are defined in Section 3. Here* $\Delta_h^{\nu-\{\omega_j\}}(T)$ *is a diagonal matrix whose terms are equal to* $T^{-l}$ *when they are associated to a monomial of degree* $l$.

2. *When* $\omega_j \in ]0,\pi[$,

$$\hat{\alpha}_{\omega_j}^h(G_{\omega_j})\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \hat{\beta}_{\omega_j}^h(\nu)(t) - \alpha_{\omega_j}^h(G_{\omega_j})\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} - \beta_{\omega_j}^h(\nu)(t)$$

$$= O_{p,1}^{\omega_j,h}(1)\tilde{\Delta}_{\omega_j}(T)\left[\bar{\alpha}_{\omega_j}(G_{\omega_j})\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \bar{\beta}_{\omega_j}(\nu)(t)\right]$$

$$+ T^{-1/2}O_{p,2}^{\omega_j,h}(1)\Delta_{h+d_j}^{\omega_j}(T)\mathcal{T}_{\omega_j}^{h+d_j}(t) + T^{-1/2}O_{p,3}^{\omega_j,h}(1)\Delta_h^{\nu-\{\omega_j\}}(T)\mathcal{T}_{\nu-\{\omega_j\}}(t),$$

*where* $O_{p,1}^{\omega_j,h}(1)$, $O_{p,2}^{\omega_j,h}(1)$, *and* $O_{p,3}^{\omega_j,h}(1)$ *are, respectively,* $(r_{\omega_j,h} \times 2Nm_{\omega_j})$, $(r_{\omega_j,h} \times 2h)$, *and* $(r_{\omega_j,h} \times \sum_{\varpi_{l'} \in \{0,\pi\}} d_{l'} + \sum_{\varpi_{l'} \neq \omega_j, \varpi_{l'} \in ]0,\pi[} 2d_{l'})$ *matrices that converge weakly.*

We want to build a specification test in the same spirit as that in Gregoir and Laroque (1994) and Section 4. We know that we are only able in this framework to test for the omission of polynomial error correction terms. The expression $k_j = \sum_{h=1}^{m_{\omega_j}} hr_{\omega_j,h}$ is too small for at least one $\omega_j \in \Omega$. Under the null, a deterministic term is present; we therefore have to amend the procedures described in Section 4 to take into account this term. The specification test is built as follows.

When $\omega_j \in \{0,\pi\}$, we denote $\gamma_{\omega_j}(G_{\omega_j})$ the $Nm_{\omega_j} - k_j$ last components of $\bar{\alpha}_{\omega_j}(G_{\omega_j})$ and $\delta_{m_j}(\nu)(t)$ the $Nm_{\omega_j} - k_j$ corresponding components of $\bar{\beta}_{\omega_j}(\nu)$. By construction, $\gamma_{\omega_j}(G_{\omega_j})y_t + \delta_{\omega_j}(\nu)(t) = \mathcal{Z}_{\omega_j,t}$ is made of variables integrated of order one or more at frequency $\omega_j$ that are not cointegrated. Therefore, if we introduce these $(Nm_{\omega_j} - k_j)$ variables on the right-hand side of the representation equation, the estimator $\hat{\psi}_{\omega_j}$ of the coefficients of these variables superconverges toward 0.

Practically, these terms are estimated. We know that the estimation errors of these terms and in the error correction terms introduce nuisance parameters in the limit distribution of $\hat{\psi}_{\omega_j}$ that can be removed by a suitable projection. Moreover, these terms are not asymptotically orthogonal to the terms present under the null in the right-hand side of the representation equation. Indeed, there exists a correlation due to the deterministic trend. Two projections therefore are needed to introduce in the equation for test purposes nonstationary terms asymptotically orthogonal to the terms in the right-hand side of the equation and to build a test statistic free of nuisance parameters. The first projection is performed on the nonstationary terms before their introduction in the equation. The second one corresponds to the projection of $\hat{\psi}_{\omega_j}$ on the orthogonal space to $\hat{\Theta}_{\omega_j}(0)$.

Here is the way the divergent terms are built up. Take in turn the cointegration relationships of degree $h$, $h = m_{\omega_j}, \ldots, 1$, and $\alpha_{\omega_j,h}^h$ the estimated coefficients of the terms of highest degree. Then apply these coefficients to the residuals of the

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regression of $G_{\omega_j}^l y$ on $G_{\omega_j}^{l-1} y, \ldots, G_{\omega_j} y$ and a basis of $\mathcal{T}_{\omega_j}^{l+d_j} \oplus \mathcal{T}_{\nu-\{\omega_j\}}$, $l = h+1, \ldots, m_{\omega_j}$. Finally, let $\alpha_{\omega_j}^0$ be an $(N - r_{\omega_j} \times N)$ matrix of orthonormal vectors that completes the $\alpha_{\omega_j,h}^h$. Apply the same rule for $h = 0$. This yields an $(Nm_{\omega_j} - k_j \times T)$ estimate of $\hat{\mathcal{Z}}_{\omega_j,t}$. We complete this vector with the elements of the canonical basis of $\mathcal{T}_{\omega_j}^{m_{\omega_j}+d_j} - \mathcal{T}_{\omega_j}^{d_j} = \{t^{d_j} e^{-i\omega_j t}, \ldots, t^{m_{\omega_j}+d_j-1} e^{-i\omega_j t}\}$. We compute the residuals of this $(N+1)m_{\omega_j} - k_j$ vector in the ordinary least squares regression on the canonical basis of $\mathcal{T}_{\omega_j}^{d_j}$. We then obtain an $(N+1)m_{\omega_j} - k_j \times 1$ vector we denote $\mathcal{P}_{\omega_j,d_j} \hat{\mathcal{Z}}_{\omega_j,t}$ and introduce in the right-hand side of the equation.

When $\omega_j \in ]0, \pi[$, we proceed in a similar way. We denote $\gamma_{\omega_j}(G_{\omega_j})$ the $Nm_{\omega_j} - k_j$ last components of $\bar{\alpha}_{\omega_j}(G_{\omega_j})$ built from polynomials $(\alpha_{\omega_j}^{1,h}(G_{\omega_j}) \quad \alpha_{\omega_j}^{2,h}(G_{\omega_j}))$ and $\delta_{\omega_j}(\nu)(t)$ the $Nm_{\omega_j} - k_j$ corresponding components of $\bar{\beta}_{\omega_j}(\nu)$. By construction, $\gamma_{\omega_j}(G_{\omega_j})(y_t' \quad y_{t-1}')' + \delta_{\omega_j}(\nu)(t) = \mathcal{Z}_{\omega_j,t}$ is such that $(\mathcal{Z}_{\omega_j,t}' \quad \mathcal{Z}_{\omega_j,t-1}')'$ is made of variables integrated of order one or more at frequency $\omega_j$ that are not cointegrated. Similarly, if we introduce these $(2Nm_{\omega_j} - 2k_j)$ variables on the right-hand side of the representation equation, the estimator $\hat{\psi}_{m_j}$ of the coefficients of these variables superconverges toward 0.

Practically, these terms must be estimated, and we have the same problems as that when $\omega_j \in \{0, \pi\}$. The estimation errors of these terms and in the error correction terms introduce nuisance parameters in the limit distribution of $\hat{\psi}_{\omega_j}$, and these terms are not asymptotically orthogonal to the terms present under the null in the right-hand side of the representation equation. We then perform two similar projections. The first projection is performed on the nonstationary terms before their introduction in the equation. The second one corresponds to the projection of $\hat{\psi}_{\omega_j}$ on the orthogonal space to $(\hat{\Theta}_{\omega_j}^1(0), \hat{\Theta}_{\omega_j}^2(0))^{*\omega_j}$.

The divergent terms are built up in the following way. Take in turn the cointegration relationships of degree $h, h = m_{\omega_j}, \ldots, 1$, and $(\alpha_{\omega_j,h}^{1,h} \quad \alpha_{\omega_j,h}^{1,h})$ the estimated coefficients of the terms of highest degree. Then apply these coefficients to the residuals of the regression of $G_{\omega_j}^l y_t$ and $G_{\omega_j}^l y_{t-1}$ on $G_{\omega_j}^{l-1} y_t$, $G_{\omega_j}^{l-1} y_{t-1}, \ldots, G_{\omega_j} y_t, G_{\omega_j} y_{t-1}$ and a basis of $\mathcal{T}_{\omega_j}^{l+d_j} \oplus \mathcal{T}_{\nu-\{\omega_j\}}$, $l = h+1, \ldots, m_{\omega_j}$. Finally, let $(\alpha_{\omega_j}^{1,0} \quad \alpha_{\omega_j}^{2,0})$ be an $(N - r_{\omega_j} \times 2N)$ matrix of orthonormal vectors, built sequentially as in Section 3.2, that completes the $\alpha_{\omega_j,h}^h$ in an $N \times 2N$ matrix of full rank $N$. Apply the same rule for $h = 0$. This yields an $(Nm_{\omega_j} - k_j \times T)$ estimate of $\hat{\mathcal{Z}}_{\omega_j,t}$. We build the $(2Nm_{\omega_j} - 2k_j \times 1)$ vector of these terms and their once lagged value and complete this vector with the elements of the canonical basis of $\mathcal{T}_{\omega_j}^{m_{\omega_j}+d_j} - \mathcal{T}_{\omega_j}^{d_j} = \{t^{d_j} \sin \omega_j t, t^{d_j} \cos \omega_j t, \ldots, t^{m_{\omega_j}+d_j-1} \sin \omega_j t, t^{m_{\omega_j}+d_j-1} \cos \omega_j t\}$. We compute the residuals of this $2(N+1)m_{\omega_j} - 2k_j$ vector in the ordinary least squares regression on the canonical basis of $\mathcal{T}_{\omega_j}^{d_j}$. We then obtain a $2(N+1)m_{\omega_j} - 2k_j \times 1$ vector we denote $\mathcal{P}_{\omega_j,d_j} \hat{\mathcal{Z}}_{\omega_j,t}$ and introduce in the right-hand side of the equation.

As in the procedure described in Section 4, we can introduce simultaneously these various nonstationary terms in the equation. For sake of simplicity, the statements in Theorems 5.2 and 5.3 take in turn each set of nonstationary terms for each frequency. We then compute a statistic as the ratio of the variance ex-

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plained by the integrated regressors for each frequency to the variance of the residuals whose asymptotic nonstandard distribution is free of nuisance parameters.

Before stating the statistical results, we need to introduce some pieces of notation. Let us define two operators $\mathcal{P}_h$ and $\mathcal{Q}_h$ on the space of the multivariate continuous time stochastic processes $w(t)$ through the following procedure.

(i) For $(1 \times n)w(t)$ almost surely in $C[0,1]^n$, we denote

$$\mathcal{P}_h w(t) = w(t) - t_0^{h-1\prime}(\tau_0^{h-1} * \tau_0^{h-1\prime})^{-1}(\tau_0^{h-1} * w),$$

where $t_0^h = (1 \ldots t^h)'$, $(\mathcal{P}_h w(t))$ belongs almost surely to $C[0,1]^n$,$((\tau_0^{h-1} * \tau_0^{h-1\prime})$ is a symmetric invertible matrix, and the other terms of the expression are continuous).

(ii) For a $(1 \times n)$ complex stochastic process $w(t)$ whose real and imaginary parts are almost surely in $C[0,1]^n$, we denote $\mathcal{Q}_h w(t)$ the complex stochastic process such that $\Re\mathcal{Q}_h w(t) = \mathcal{P}_h \Re w(t)$ and $\Im\mathcal{Q}_h w(t) = \mathcal{P}_h \Im w(t)$ that belong almost surely to $C[0,1]^n$. The expression $\mathcal{Q}_h w^*(t)$ is consistently defined.

By convention, $\mathcal{Q}_0$ and $\mathcal{P}_0$ are equal to the identity operator. The same notations are used for similar operations on deterministic functions. We now state the principal result of this section.

We first consider the case $\omega_j \in \{0, \pi\}$. Let $y_t$ be a process satisfying Assumption 3. Let $\bar{\psi}_{\omega_j}$, $(\bar{\Theta}^g_{\omega_l}(\Delta_{\omega_l})$, $g = 1,2)$, $\bar{\Phi}_\nu$, $\bar{\phi}$, and $\bar{\eta}$ be, respectively, the ordinary least squares estimators of the $N \times ((N+1)m_{\omega_j} - k_j)$ coefficient matrix $\psi_{\omega_j}$, of $(\Theta^g_{\omega_l}(\Delta_{\omega_l})$, $g = 1,2)$, $\Phi_\nu$, $\phi$, and of the $(N \times T)$ matrix of the residuals in the $N$ regression equations

$$\begin{aligned} y_t = {} & \Phi_\nu(t) + \psi_{\omega_j} \mathcal{P}_{\omega_j, d_j} \hat{\mathcal{Z}}_{\omega_j, t-1} - \sum_{l=1}^{p'} \phi_l y_{t-l} + \eta_t \\ & + B \sum_{\omega_l} \{\Theta^1_{\omega_l}(\Delta_{\omega_l}) + B\Theta^2_{\omega_l}(\Delta_{\omega_l})\} \\ & \times \{\hat{\alpha}^1_{\omega_l}(G_{\omega_l}) y_t + \hat{\alpha}^2_{\omega_l}(G_{\omega_l}) y_{t-1} + \hat{\beta}_{\omega_l}(\nu)(t)\}, \end{aligned} \tag{11}$$

where $\mathcal{P}_{\omega_j, d_j} \hat{\mathcal{Z}}_{\omega_j, t}$ is described previously and $p' \geq p$.

THEOREM 5.2. *Let $\bar{\Theta}_{\omega_j,\perp}$ be an $(N \times N - r_{\omega_j})$ full rank matrix such that $\bar{\Theta}'_{\omega_j,\perp} \bar{\Theta}_{\omega_j}(0) = 0$. Then when the sample size T goes to infinity, the following convergence in distribution holds*:

$$\begin{aligned} & \text{Trace } \{T \bar{\Sigma}_{\omega_j}^{-1/2} \bar{\Theta}'_{\omega_j,\perp} \bar{\psi}_{\omega_j} [\mathcal{P}_{\omega_j, d_j} \hat{\mathcal{Z}}_{\omega_j, -1} (\mathcal{P}_{\omega_j, d_j} \hat{\mathcal{Z}}_{\omega_j, -1})'] \bar{\psi}'_{\omega_j} \bar{\Theta}_{\omega_j,\perp} \bar{\Sigma}_{\omega_j}^{-1/2}\} \\ & \quad \Rightarrow \text{Trace } \{(dW'_{\omega_j,0} * \mathcal{P}_{d_j} \bar{X}_{\omega_j})((\mathcal{P}_{d_j} \bar{X}_{\omega_j})' * \mathcal{P}_{d_j} \bar{X}_{\omega_j})^{-1}((\mathcal{P}_{d_j} \bar{X}_{\omega_j})' * dW_{\omega_j,0})\}, \end{aligned}$$

*where $\bar{\Sigma}_{\omega_j} = \bar{\Theta}'_{\omega_j,\perp} \bar{\eta}\bar{\eta}' \bar{\Theta}_{\omega_j,\perp}$, $\mathcal{P}_{\omega_j, d_j} \hat{\mathcal{Z}}'_{\omega_j, -1}$ stands for the $T \times ((N+1)m_{\omega_j} - k_j)$ matrix of generic element $\mathcal{P}_{\omega_j, d_j} \hat{\mathcal{Z}}'_{\omega_j, t-1}, t = 1, \ldots, T$, $\bar{X}_{\omega_j} = (X_{\omega_j} \quad \tau^{d_j} \ldots \tau^{m_{\omega_j} + d_j - 1})$, and $W_{\omega_j,0}$ and $X_{\omega_j}$ are defined in Theorem 4.1.*

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We now consider the case $\omega_j \in ]0,\pi[$. Let $y_t$ be a process satisfying Assumption 3. Let $\bar{\psi}^1_{\omega_j}$, $(\bar{\Theta}^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\bar{\Phi}_\nu$, $\bar{\phi}$, and $\bar{\eta}$ be, respectively, the ordinary least squares estimators of the $N \times (2(N+1)m_{\omega_j} - 2k_j)$ coefficient matrix $\psi^1_{\omega_j}$, of $(\Theta^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\Phi_\nu$, $\phi$, and of the $(N \times T)$ matrix of the residuals in the $N$ regression equations

$$
\begin{aligned}
y_t = {} & \Phi_\nu(t) + \psi^1_{\omega_j}\mathcal{P}_{\omega_j,d_j}\hat{\mathcal{Z}}_{\omega_j,t-2} - \sum_{l=1}^{p'}\phi_l y_{t-l} + \eta_t \\
& + B\sum_{\omega_l}\{\Theta^1_{\omega_l}(\Delta_{\omega_l}) + B\Theta^2_{\omega_l}(\Delta_{\omega_l})\} \\
& \times \{\hat{\alpha}^1_{\omega_l}(G_{\omega_l})y_t + \hat{\alpha}^2_{\omega_l}(G_{\omega_l})y_{t-1} + \hat{\beta}_{\omega_l}(\nu)(t)\}, \qquad \textbf{(12)}
\end{aligned}
$$

where $\mathcal{P}_{\omega_j,d_j}\hat{\mathcal{Z}}_{\omega_j,t}$ is described previously and $p' \geq p$. Let $\bar{\psi}^2_{\omega_j}$, $(\dot{\Theta}^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\dot{\Phi}_\nu$, $\dot{\phi}$, and $\dot{\eta}$ be, respectively, the ordinary least squares estimators of the $N \times (2(N+1)m_{\omega_j} - 2k_j)$ coefficient matrix $\psi^2_{\omega_j}$, of $(\Theta^g_{\omega_l}(\Delta_{\omega_l}), g = 1,2)$, $\Phi_\nu$, $\phi$, and of the $(N \times T)$ matrix of the residuals in the $N$ regression equations

$$
\begin{aligned}
y_t = {} & \Phi_\nu(t) + \psi^2_{\omega_j}\mathcal{P}_{\omega_j,d_j}\hat{\mathcal{Z}}_{\omega_j,t-1} - \sum_{l=1}^{p'}\phi_l y_{t-l} + \eta_t \\
& + B\sum_{\omega_l\in]0,\pi[}\{\Theta^1_{\omega_l}(\Delta_{\omega_l}) + B\Theta^2_{\omega_l}(\Delta_{\omega_l})\} \\
& \times \{\hat{\alpha}^1_{\omega_l}(G_{\omega_l})y_t + \hat{\alpha}^2_{\omega_l}(G_{\omega_l})y_{t-1} + \hat{\beta}_{\omega_l}(\nu)(t)\}. \qquad \textbf{(13)}
\end{aligned}
$$

THEOREM 5.3. *Let $\bar{\Theta}_{\omega_j,\perp}$ be a $2N \times 2(N - r_{\omega_j})$ full rank matrix such that $\bar{\Theta}'_{\omega_j,\perp}(\bar{\Theta}^1_{\omega_j}(0), \bar{\Theta}^2_{\omega_j}(0))^{*\omega_j} = 0$. Then when the sample size $T$ goes to infinity, the following convergence in distribution holds*:

$$
\begin{aligned}
&\text{Trace}\left\{T\bar{\Sigma}^{-1}_{\omega_j}\bar{\Theta}'_{\omega_j,\perp}\begin{pmatrix}\bar{\psi}^1_{\omega_j}\\ \bar{\psi}^2_{\omega_j}\end{pmatrix}[\mathcal{P}_{\omega_j,d_j}\hat{\mathcal{Z}}_{\omega_j,-1}(\mathcal{P}_{\omega_j,d_j}\hat{\mathcal{Z}}_{\omega_j,-1})']\begin{pmatrix}\bar{\psi}^1_{\omega_j}\\ \bar{\psi}^2_{\omega_j}\end{pmatrix}'\bar{\Theta}_{\omega_j,\perp}\right\} \\
&\quad \Rightarrow \text{Trace}\,\{(dW^{*\prime}_{\omega_j,0} * \mathcal{Q}_{d_j}\bar{X}^*_{\omega_j})((\mathcal{Q}_{d_j}\bar{X}^*_{\omega_j})' * \mathcal{Q}_{d_j}\bar{X}^*_{\omega_j})^{-1}((\mathcal{Q}_{d_j}\bar{X}^*_{\omega_j})' * dW^*_{\omega_j,0})\},
\end{aligned}
$$

*where*

$$
\bar{\Sigma}_{\omega_j} = \left(\bar{\Theta}'_{\omega_j,\perp}\begin{pmatrix}\bar{\eta}\bar{\eta}' & \cos\omega_j\,\bar{\eta}\bar{\eta}'\\ \cos\omega_j\,\bar{\eta}\bar{\eta}' & \bar{\eta}\bar{\eta}'\end{pmatrix}\bar{\Theta}_{\omega_j,\perp}\right),
$$

$\mathcal{P}_{\omega_j,d_j}\hat{\mathcal{Z}}'_{\omega_j,-1}$ *stands for the* $T \times (2(N+1)m_{\omega_j} - 2k_j)$ *matrix of generic element* $\mathcal{P}_{\omega_j,d_j}\hat{\mathcal{Z}}'_{\omega_j,t-1}, t = 1,\ldots,T$,

$$
\bar{X}^*_{\omega_j} = \begin{pmatrix}\bar{X}_{c,\omega_j} & \bar{X}_{s,\omega_j}\\ -\bar{X}_{s,\omega_j} & \bar{X}_{c,\omega_j}\end{pmatrix}
$$

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*with* $\bar{X}_{c,\omega_j} = (X_{c,\omega_j} \quad 0_{(1\times m_{\omega_j})})$ *and* $\bar{X}_{s,\omega_j} = (X_{s,\omega_j} \quad \tau^{d_j}\ldots\tau^{m_{\omega_j}+d_j-1})$ *and* $W_{\omega_j,0}$, $X_{c,\omega_j}$, *and* $X_{s,\omega_j}$ *are defined in Theorem* 4.2.

As noted in Section 4.2, the first step errors in the estimation of the cointegration terms cause a problem. The test based on the trace statistics of Theorems 5.2 and 5.3 procedure is only consistent against the alternative hypothesis of larger $k_j = \sum hr_{\omega_j,h}$ such that when $\omega_j \in \{0,\pi\}$, $\Theta'_{\omega_j,\perp}\psi_{\omega_j}$ is different from zero and when $\omega_j \in ]0,\pi[$, $\Theta'_{\omega_j,\perp}\begin{pmatrix}\psi^1_{\omega_j}\\ \psi^2_{\omega_j}\end{pmatrix}$ is different from zero. For the (measure zero) subset of alternatives under which these matrix products are null, the trace statistics do not tend to infinity with *T*. Similarly, this difficulty does not exist in the monomial case where these matrix products are always different from zero.

Finally, Appendix B presents the quantiles of the corresponding test statistics for the case where the statistician considers twice integrated time series ($m_\omega = 2$) and $\omega = \pi/2$. When $\omega = 0$ or $\omega = \pi$, the useful table is given in Gregoir and Laroque (1994). The table is similar to that described in Section 4.3: 5,000 independent drawings of the distribution have been computed for a value of $T = 400$. The rank $N - r_{\pi/2}$ ranges from 1 to 5, with values of $r_{\pi/2,1}$ from 0 to 5, $r_{\pi/2,2}$ being implicitly equal to $r_{\pi/2} - r_{\pi/2,1}$.

## 6. CONCLUSION

In this paper, we develop an estimation and test strategy relying on simple statistical procedures such as principal component computation and ordinary least squares to deal with vector error correction models with complicated polynomial error correction terms (with possibly deterministic terms) at different frequencies as described in Gregoir (1999). This specification may occur when we work with nonseasonally adjusted nominal data and price indices that may be well described with models containing more than one unit root at frequency 0 and some seasonal unit roots. Once the null hypothesis is chosen and the stationary polynomially cointegrated terms are estimated, we can test separately for each frequency for the omission of a polynomial error correction term. The result presented here assumes the knowledge of the highest degree of the polynomial error correction terms at each frequency and the dimension decomposition of the cointegration vector space into independent subspaces of polynomials of different highest degrees (between one and the highest value). In practice, following Gregoir and Laroque (1994), it is possible to use iteratively the same simple ideas and statistical procedures to construct an exploratory procedure to build the null hypothesis and then proceed to an overall specification test.

### NOTES

1. We simply have to note that if $\beta^h_\omega(t) \in \mathcal{T}^h_\omega$ then $\tilde{\beta}^h_\omega(t)$ as defined in Remark 3.1 in Gregoir (1999) is also in $\mathcal{T}^h_\omega$.

2. This relies on the existence of an isomorphism between the spaces under study. See Goodman (1963) for an introduction to multivariate complex distribution.

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# APPENDIX A: PROOFS

## PRELIMINARIES

All the proofs rely on the asymptotic behavior of integrated processes. We first introduce two new integral operators, and then we summarize the statistical properties of the processes we have considered up to now.

DEFINITION A.1.

(i) *The integral operator* $\mathcal{S}_{c,\omega}$, $\omega \in ]0,\pi[$ *associates to any sequences* $y_t = (y_t, t = 0,1,\dots)$ *of* $R^N$ *a sequence* $\mathcal{S}_{c,\omega}(y)_t$ *defined by*

$$\mathcal{S}_{c,\omega}(y)_t = \sum_{\tau=1}^{t} \cos(\omega\tau) y_\tau.$$

(ii) *The integral operator* $\mathcal{S}_{s,\omega}$, $\omega \in ]0,\pi[$ *associates to any sequences* $y_t = (y_t, t = 0,1,\dots)$ *of* $R^N$ *a sequence* $\mathcal{S}_{s,\omega}(y)_t$ *defined by*

$$\mathcal{S}_{s,\omega}(y)_t = \sum_{\tau=1}^{t} \sin(\omega\tau) y_\tau.$$

For $d \geq 1$, we denote $\mathcal{S}^d_{c,\omega}$ and $\mathcal{S}^d_{s,\omega}$, the integral operators $\mathcal{S}_{c,\omega}$ and $\mathcal{S}_{s,\omega}$ applied to $G^{d-1}_\omega$:

$$\mathcal{S}^d_{c,\omega} = \mathcal{S}_{c,\omega} G^{d-1}_\omega,$$

$$\mathcal{S}^d_{s,\omega} = \mathcal{S}_{s,\omega} G^{d-1}_\omega.$$

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There exists a simple linear relation between the vectors $(G_\omega^d y_t' \quad G_\omega^d y_{t-1}')'$ on which we work and the terms $(\mathcal{S}_{c,\omega}^d y_t' \quad \mathcal{S}_{s,\omega}^d y_t')'$. After simple trigonometric computations, we get

$$Q(\omega) \otimes I_N \begin{pmatrix} G_\omega^d y_t \\ G_\omega^d y_{t-1} \end{pmatrix} = R^{-t}(\omega) \otimes I_N \begin{pmatrix} \mathcal{S}_{c,\omega}^d y_t \\ \mathcal{S}_{s,\omega}^d y_t \end{pmatrix}, \tag{A.1}$$

where

$$Q(\omega) = \begin{pmatrix} 1 & -\cos\omega \\ 0 & -\sin\omega \end{pmatrix}$$

and

$$R(\omega) = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}.$$

In the sequel, we denote $f(\omega, d) = 1/2^{d+1} \sin^d \omega$.

**Property A.1.** Let $\mu_t$ be an $N$-dimensional martingale difference satisfying $E(\mu_t | \mathcal{F}_{t-1}) = 0$, $\mathrm{Cov}(\mu_t | \mathcal{F}_{t-1}) = I_N$, and $\max_{j\in\{1,\ldots,N\}} \sup_t E(|\mu_{j,t}|^{2+\delta} | \mathcal{F}_{t-1}) < +\infty$ a.s. for some $\delta > 0$, where $\mathcal{F}_{t-1}$ is the $\sigma$-field generated by $\{\mu_{t-l}, l = 1,2,\ldots\}$. Then when $T$, the sample size, goes to infinity, the following convergences in distribution hold where $[T\tau]$ is equal to the integral part of $T\tau$.

(i(a)) $\omega \in \{0,\pi\}, T^{-1/2} e^{i\omega[T\tau]} G_\omega \mu_{[T\tau]} \Rightarrow W_\omega(\tau)$,

where $W_\omega$ is an $N$-dimensional standard Wiener process.

(i(b)) $\omega \in ]0,\pi[, \left(\frac{2}{T}\right)^{1/2} (\mathcal{S}_{c,\omega} \mu_{[T\tau]}' \quad \mathcal{S}_{s,\omega} \mu_{[T\tau]}') \Rightarrow (W_{c,\omega}(\tau) \quad W_{s,\omega}(\tau))$,

where $W_{c,\omega}$ and $W_{s,\omega}$ are two $N$-dimensional independent standard Wiener processes.

For any integer $d_1 \geq 1$, $d_2 \geq 1$, $d_3 \geq 0$, and $d_4 \geq 0$, we have the following joint weak convergence where we denote when $\omega \in ]0,\pi[$, $W_\omega = \begin{pmatrix} W_{c,\omega} & W_{s,\omega} \\ -W_{s,\omega} & W_{c,\omega} \end{pmatrix}$:

(ii(a)) $\omega \in \{0,\pi\}$, $T^{-d_1-d_2} \sum_{t=1}^{T} G_\omega^{d_1} \mu_t G_\omega^{d_2} \mu_t' \Rightarrow G^{d_1-1} W_\omega * G^{d_2-1} W_\omega'$.

(ii(b)) $\omega \in \{0,\pi\}$, $T^{-d_3-d_2-1/2} \sum_{t=1}^{T} e^{i\omega t} t^{d_3} G_\omega^{d_2} \mu_t' \Rightarrow \tau^{d_3} * G^{d_2-1} W_\omega$.

(ii(c)) $\omega \in ]0,\pi[$, $T^{-d_1-d_2} \sum_{t=1}^{T} \begin{pmatrix} G_\omega^{d_1} \mu_t \\ G_\omega^{d_1} \mu_{t-1} \end{pmatrix} \begin{pmatrix} G_\omega^{d_2} \mu_t \\ G_m^{d_2} \mu_{t-1} \end{pmatrix}'$

$$\Rightarrow f(\omega, d_1 - 1) f(\omega, d_2 - 1) Q_N^{-1}(\omega) (G_\omega^{d_1-1} W_\omega' * G_\omega^{d_2-1} W_\omega) Q_N^{-1}(\omega)',$$

where $Q_N^{-1}(\omega) = Q^{-1}(\omega) \otimes I_N$.

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$$\text{(ii(d))}\ \ \omega \in ]0,\pi[,\ T^{-d_3-d_2-1/2}\sum_{t=1}^{T}\begin{pmatrix} t^{d_3}\sin(\omega t)\\ t^{d_3}\cos(\omega t)\end{pmatrix}\begin{pmatrix} G_\omega^{d_2}\mu_t\\ G_\omega^{d_2}\mu_{t-1}\end{pmatrix}'$$

$$\Rightarrow 2^{-1/2}f(\omega,d_2-1)\tau^{d_3} * G_\omega^{d_2-1}W_\omega Q_N^{-1}(\omega)'.$$

$$\text{(iii(a))}\ \ \omega \in \{0,\pi\},\ T^{-d_3-d_4-1}\sum_{t=1}^{T}(e^{i\omega t}t^{d_3})(e^{i\omega t}t^{d_4}) \to \frac{1}{d_3+d_4+1}.$$

$$\text{(iii(b))}\ \ \omega \in ]0,\pi[,\ T^{-d_3-d_4-1}\sum_{t=1}^{T}\begin{pmatrix} t^{d_3}\sin(\omega t)\\ t^{d_3}\cos(\omega t)\end{pmatrix}\begin{pmatrix} t^{d_4}\sin(\omega t)\\ t^{d_4}\cos(\omega t)\end{pmatrix}'$$

$$\to \frac{1}{d_3+d_4+1}\begin{pmatrix} \frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}.$$

$$\text{(iv(a))}\ \ \omega \in \{0,\pi\},\ T^{-d_1}\sum_{t=1}^{T} G_\omega^{d_1}\mu_{t-1}\mu_t' \Rightarrow e^{i\omega}G^{d_1-1}W_\omega * dW_\omega'.$$

$$\text{(iv(b))}\ \ \omega \in \{0,\pi\},\ T^{-d_3-1/2}\sum_{t=1}^{T} e^{i\omega t}t^{d_3}\mu_{t+1}' \Rightarrow e^{i\omega}\tau^{d_3} * dW_\omega'.$$

$$\text{(iv(c))}\ \ \omega \in ]0,\pi[,\ T^{-d_1}\sum_{t=1}^{T}\begin{pmatrix} G_\omega^{d_1}\mu_{t-1}\\ G_\omega^{d_1}\mu_{t-2}\end{pmatrix}\mu_t'$$

$$\Rightarrow f(\omega,d_1-1)Q^{-1}(\omega)\otimes I_N G_\omega^{d_1-1}W_\omega' * dW_\omega\begin{pmatrix}\cos\omega I_N\\ \sin\omega I_N\end{pmatrix}.$$

$$\text{(iv(d))}\ \ \omega \in ]0,\pi[,\ T^{-d_1}\sum_{t=1}^{T}\begin{pmatrix} G_\omega^{d_1}\mu_{t-2}\\ G_\omega^{d_1}\mu_{t-3}\end{pmatrix}\mu_t'$$

$$\Rightarrow f(\omega,d_1-1)Q^{-1}(\omega)\otimes I_N G_\omega^{d_1-1}W_\omega' * dW_\omega\begin{pmatrix}\cos 2\omega I_N\\ \sin 2\omega I_N\end{pmatrix}.$$

$$\text{(iv(e))}\ \ \omega \in ]0,\pi[,\ T^{-d_3-1/2}\sum_{t=1}^{T}\begin{pmatrix} t^{d_3}\cos(\omega t)\\ t^{d_3}\sin(\omega t)\end{pmatrix}\mu_{t+1}' \Rightarrow 2^{-1/2}\tau^{d_3} * dW_\omega\begin{pmatrix}\cos\omega I_N\\ \sin\omega I_N\end{pmatrix}.$$

$$\text{(iv(f))}\ \ \omega \in ]0,\pi[,\ T^{-d_3-1/2}\sum_{t=1}^{T}\begin{pmatrix} t^{d_3}\cos(\omega t)\\ t^{d_3}\sin(\omega t)\end{pmatrix}\mu_{t+2}' \Rightarrow 2^{-1/2}\tau^{d_3} * dW_\omega\begin{pmatrix}\cos 2\omega I_N\\ \sin 2\omega I_N\end{pmatrix}.$$

Furthermore

$$\text{(v(a))}\ \ T^{-d_1-d_2}\sum_{t=1}^{T} G_0^{d_1}\mu_t G_\pi^{d_2}\mu_t' \Rightarrow 0.$$

$$\text{(v(b))}\ \ \omega_1 \in \{0,\pi\},\omega_2 \in ]0,\pi[,\ T^{-d_1-d_2}\sum_{t=1}^{T} G_{\omega_1}^{d_1}\mu_t\begin{pmatrix} G_{\omega_2}^{d_2}\mu_t\\ G_{\omega_2}^{d_2}\mu_{t-1}\end{pmatrix}' \Rightarrow 0.$$

$$\text{(v(c))}\ \ (\omega_1,\omega_2) \in ]0,\pi[^2,\ \omega_1 \neq \omega_2,$$

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$$T^{-d_1-d_2}\sum_{t=1}^{T}\begin{pmatrix} G_{\omega_1}^{d_1}\mu_t \\ G_{\omega_1}^{d_1}\mu_{t-1}\end{pmatrix}\begin{pmatrix} G_{\omega_2}^{d_2}\mu_t \\ G_{\omega_2}^{d_2}\mu_{t-1}\end{pmatrix}' \Rightarrow 0.$$

(v(d)) $\omega \in ]0,\pi[$, $T^{-d_1}\sum_{t=1}^{T}\begin{pmatrix} G_{\omega}^{d_1}\mu_t \\ G_{\omega}^{d_1}\mu_{t-1}\end{pmatrix}(\mu_t - 2\cos\omega\mu_{t-1} + \mu_{t-2})' \Rightarrow 0.$

(v(e)) $T^{-d_3-d_4-1}\sum_{t=1}^{T} t^{d_3}(-1)^t t^{d_4} \to 0.$

(v(f)) $\omega_1 \in \{0,\pi\}, \omega_2 \in ]0,\pi[$, $T^{-d_3-d_4-1}\sum_{t=1}^{T} t^{d_3}e^{-i\omega_1 t}\begin{pmatrix} t^{d_4}\sin\omega_2 t \\ t^{d_4}\cos\omega_2 t\end{pmatrix}' \to 0.$

(v(g)) $(\omega_1,\omega_2) \in ]0,\pi[^2$, $\omega_1 \neq \omega_2$,

$$T^{-d_3-d_4-1}\sum_{t=1}^{T}\begin{pmatrix} t^{d_3}\sin\omega_1 t \\ t^{d_3}\cos\omega_1 t\end{pmatrix}\begin{pmatrix} t^{d_4}\sin\omega_2 t \\ t^{d_4}\cos\omega_2 t\end{pmatrix}' \to 0.$$

(v(h)) $T^{-d_3-d_2-1/2}\sum_{t=1}^{T}(-1)^t t^{d_3} G_0^{d_2}\mu_t' \Rightarrow 0.$

(v(i)) $T^{-d_3-d_2-1/2}\sum_{t=1}^{T} t^{d_3} G_\pi^{d_2}\mu_t' \Rightarrow 0.$

(v(j)) $\omega_1 \in \{0,\pi\}, \omega_2 \in ]0,\pi[$, $T^{-d_3-d_2-1/2}\sum_{t=1}^{T} e^{-i\omega_1 t}t^{d_3}\begin{pmatrix} G_{\omega_2}^{d_2}\mu_t \\ G_{\omega_2}^{d_2}\mu_{t-1}\end{pmatrix}' \Rightarrow 0.$

(v(k)) $\omega_1 \in \{0,\pi\}, \omega_2 \in ]0,\pi[$, $T^{-d_3-d_2-1/2}\sum_{t=1}^{T}\begin{pmatrix} t^{d_3}\sin(\omega_2 t) \\ t^{d_3}\cos(\omega_2 t)\end{pmatrix} G_{\omega_1}^{d_2}\mu_t' \Rightarrow 0.$

(v(l)) $(\omega_1,\omega_2) \in ]0,\pi[^2$, $\omega_1 \neq \omega_2$,

$$T^{-d_3-d_2-1/2}\sum_{t=1}^{T}\begin{pmatrix} t^{d_3}\sin(\omega_2 t) \\ t^{d_3}\cos(\omega_2 t)\end{pmatrix}\begin{pmatrix} G_{\omega_1}^{d_2}\mu_t \\ G_{\omega_1}^{d_2}\mu_{t-1}\end{pmatrix}' \Rightarrow 0.$$

Finally, if $C(B)$ is a matrix series whose radius of convergence is strictly larger than one:

(vi(a)) $\omega \in \{0,\pi\}$, $T^{-d_1}\sum_{t=1}^{T} G_\omega^{d_1}\mu_t[C(B)\mu_t]'$

has a limit.

(vi(b)) $\omega \in ]0,\pi[$, $T^{-d_1}\sum_{t=1}^{T}\begin{pmatrix} G_{\omega}^{d_1}\mu_t \\ G_{\omega}^{d_1}\mu_{t-1}\end{pmatrix}[C(B)\mu_t]'$

has a limit.

(vi(c)) $\omega \in \{0,\pi\}$, $T^{-d_3-1/2}\sum_{t=1}^{T} t^{d_3}[C(B)\mu_t]'$

has a limit.

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$$\text{(vi(d))}\ \ \omega \in ]0,\pi[,\ T^{-d_3-1/2}\sum_{t=1}^{T}\begin{pmatrix} t^{d_3}\sin\omega t \\ t^{d_3}\cos\omega t\end{pmatrix}[C(B)\mu_t]'$$

has a limit.

The proofs of the various items can be built from lemmas or theorems that can be found in Chan and Wei (1988), Sims, Stock, and Watson (1990), and Tsay and Tiao (1990). For convenience, throughout we have to condition on $\mu_t = 0, t \leq 0$. As shown by Tsay and Tiao (1990), the assumption of zero starting value does not matter in studying asymptotic results of the processes built with $\mu_t$. Because of lack of space the proofs are not given but can be provided by the author on request.

The equations in Property A.1(v) are important. They greatly simplify the limiting distribution of the test statistics and allow us to separate the analysis for each couple $(\omega_j, k_j) \in \Omega$. We now define the canonical error terms $\mu$ to which we shall apply Property A.1. In fact we have to build one vector for each frequency. When $\omega \in \{0, \pi\}$, we use the Gregoir and Laroque (1994) approach. We start from equation (2) in Theorem 3.2 (identification) in Gregoir (1999), which gives us the $MA(+\infty)$ representation of the $r_\omega$ polynomial error correction terms appearing in the representation theorem completed by a set of $N - r_\omega$ stationary linear combinations of the process under study. The matrix polynomial associated to this representation is denoted $E_\omega(B)$. From the Gram–Schmidt process, there exists a full rank lower triangular matrix $M_\omega$ such that, if we let

$$\mu_{\omega,t} = M_\omega E_\omega(e^{-i\omega})\epsilon_t$$

then $\mu_{\omega,t}$ is a unit variance martingale difference process. Now, write $E_\omega(B) = E_\omega(e^{i\omega}) + \Delta_\omega E^*_\omega(B)$. Partitioning conformably $\mu_{\omega,t}$ and the inverse matrix of $M_\omega$ yields the following lemma.

LEMMA A.1 $\omega \in \{0, \pi\}, \forall h, h = 1, \ldots, m_\omega$

$$\alpha^h_\omega(G_\omega)y_t + \beta^h_\omega(t) = \sum_{l=0}^{h} M^{hl}_\omega \mu^l_{\omega,t} + \Delta_\omega E^*_{\omega,h}(B)\epsilon_t,$$

*where the matrices $M^{hh}_\omega$ are of full rank.*

Likewise, another property of the representation theorem is useful in the proof.

LEMMA A.2. *When $\omega \in \{0, \pi\}$, consider any $N \times N - r_\omega$ matrix $\Theta_{\omega,\perp}$ orthogonal to $\Theta_\omega(0)$ that is such that the $N \times N$ matrix $(\Theta_\omega(0) \quad \Theta_{\omega,\perp})$ is of full rank and $\Theta'_{\omega,\perp}\Theta_\omega(0) = 0$. Then there exists an $(N - r_\omega \times N - r_\omega)$ matrix $M_{\omega\perp}$ such that*

$$M_{\omega\perp}\Theta'_{\omega,\perp}\epsilon_t = \mu^0_{\omega,t}.$$

The proof relies on the algebraic identities behind the representation equation. It can be built from (4) and (8) in Gregoir (1999). It is left to the reader, but it is close to that of Lemma A.4 in the complex case (see the discussion that follows).

When $\omega \in ]0, \pi[$, the construction of $\mu_\omega$ is more complicated. We start in this case from equation (3) in Theorem 3.3 (identification) in Gregoir (1999), which gives us the $MA(+\infty)$ representation of the $r_\omega$ polynomial error correction terms appearing in the representation theorem completed by a set of $N - r_\omega$ covariance stationary linear combinations of the

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process under study. The matrix polynomial associated to this representation is denoted $E_\omega(B)$. It is such that $E_\omega(e^{i\omega})$ is a full rank matrix. Because $E_\omega(e^{i\omega})$ is a complex matrix of full rank, the matrix

$$\begin{pmatrix} \Re E_\omega(e^{i\omega}) & -\Im E_\omega(e^{i\omega}) \\ \Im E_\omega(e^{i\omega}) & \Re E_\omega(e^{i\omega}) \end{pmatrix}$$

is a real matrix of full rank. The $2N \times N$ matrix that corresponds to its $N$ first columns is of rank $N$. Rearrange the rows of this last matrix alternating real part and imaginary part of each row of $E_\omega(e^{i\omega})$ and denote this new matrix $\mathcal{E}_\omega$. The term $\mathcal{E}_\omega \epsilon_t$ is a $2N$-dimensional process whose variance covariance matrix has $N$ eigenvalues equal to zero. There exists a full rank $2N \times 2N$ lower triangular matrix $M_\omega$ such that, if we define

$$e_{\omega,t} = M_\omega \mathcal{E}_\omega \epsilon_t$$

then

$$Ve_{\omega,t} = I_N \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We then define $\mu_{\omega,t}$ as $\mu_{\omega,t} = I_N \otimes (1 \quad 0) e_{\omega,t}$ so that $\mu_{\omega,t}$ is unit variance martingale difference process. We get a result similar to Lemma A.1 on the $I_\omega(d)$ processes after having partitioned $\mu_\omega$ conformably.

LEMMA A.3. *$\omega \in ]0,\pi[$, for all $h$, $h = 0,1,\ldots,m_\omega$, there exist $h+1$ matrices $M_\omega^{*,hl}$ of respective dimension $2r_{\omega,h} \times 2(N - r_\omega)$ when $l = 0$ and $2r_{\omega,h} \times 2r_{\omega,l}$, $l \le h$ such that for all $d \ge 1$*

$$Q(\omega) \otimes I_{r_{\omega,h}} G_\omega^d \begin{pmatrix} \alpha_\omega^{1,h}(G_\omega) y_t + \alpha_\omega^{2,h}(G_\omega) y_{t-1} + \beta_\omega^h(t) \\ \alpha_\omega^{1,h}(G_\omega) y_{t-1} + \alpha_\omega^{2,h}(G_\omega) y_{t-2} + \beta_\omega^h(t-1) \end{pmatrix}$$

$$= \sum_{l=0}^{h} M_\omega^{*,hl} R(\omega)^{-1} \otimes I_{r_{\omega,h}} \left( \begin{pmatrix} \mathcal{S}_{c,\omega}^d \\ \mathcal{S}_{s,\omega}^d \end{pmatrix} \otimes I_{r_{\omega,h}} \right) \mu_{\omega,t}^l + G_\omega^{d-1} E_{\omega,h}^*(B) \epsilon_t,$$

*where $M_\omega^{*,hh}$ is a $2r_{\omega,h} \times 2r_{\omega,h}$ matrix of full rank.*

**Proof of Lemma A.3.** Simple algebraic computations give

$$\begin{pmatrix} E_\omega(B) \\ BE_\omega(B) \end{pmatrix} = Q(\omega)^{-1} \otimes I_N \begin{pmatrix} \Re E_\omega(e^{i\omega}) & -\Im E_\omega(e^{i\omega}) \\ \Im E_\omega(e^{i\omega}) & \Re E_\omega(e^{i\omega}) \end{pmatrix} Q(\omega) \otimes I_N \begin{pmatrix} I_N \\ BI_N \end{pmatrix}$$

$$+ \Delta_\omega E_\omega^*(B).$$

We introduce the matrix

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

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which is such that $J_2' = -J_2$ and $J_2^2 = -I_2$. The preceding equation can be rewritten as

$$\begin{pmatrix} E_\omega(B) \\ BE_\omega(B) \end{pmatrix} = Q(\omega)^{-1} \otimes I_N (I_2 \otimes \Re E_\omega(e^{i\omega}) + J_2 \otimes \Im E_\omega(e^{i\omega})) Q(\omega) \otimes I_N \begin{pmatrix} I_N \\ BI_N \end{pmatrix} + \Delta_\omega E_\omega^*(B).$$

Using (A.1) and the fact that $J_2$ and $R(\omega)$ commute, we first apply to both sides of the equation the integral operators $G_\omega^d$. We then rearrange the rows of the vectors of each side, grouping each component and its lagged value in the left-hand side and simultaneously grouping the corresponding sums in cosine and sine in $\epsilon_t$ in the right-hand side. We get

$$Q(\omega) \otimes I_N Y_{t,t-1}^d = I_N \otimes R(\omega)^{-t} \{\mathcal{S}_{c,\omega}^d I_N \otimes I_2 - \mathcal{S}_{s,\omega}^d I_N \otimes J_2\} \mathcal{E}_\omega \epsilon_t + G_\omega^{d-1} \tilde{E}_\omega^*(B) \epsilon_t,$$

where $Y_{t,t-1}^d$ stands for the rearrangement of the rows of

$$G_\omega^d \begin{pmatrix} \alpha_\omega^{1,0} y_t + \alpha_\omega^{2,0} y_{t-1} \\ \alpha_\omega^1(G_\omega) y_t + \alpha_\omega^2(G_\omega) y_{t-1} + \beta_\omega(t) \\ \alpha_\omega^{1,0} y_{t-1} + \alpha_\omega^{2,0} y_{t-2} \\ \alpha_\omega^1(G_\omega) y_{t-1} + \alpha_\omega^2(G_\omega) y_{t-2} + \beta_\omega(t-1) \end{pmatrix}$$

and $\tilde{E}_\omega^*$ for those of $E_\omega^*$.

By construction, we have

$$M_\omega^{-1} I_N \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e_{\omega,t} = 0$$

almost surely because $V(I_N \otimes (0 \quad 1) e_{\omega,t}) = 0$. It follows that

$$Q(\omega) \otimes I_N Y_{t,t-1}^d = I_N \otimes R(\omega)^{-1} \{\mathcal{S}_{c,\omega}^d I_{2N} - \mathcal{S}_{s,\omega}^d I_N \otimes J_2\} M_\omega^{-1} I_N \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_{\omega,t} + G_\omega^{d-1} \tilde{E}_\omega^*(B) \epsilon_t.$$

Now using the property that for any $2N \times n$ matrix $\mathcal{M}$, with $n$ a nonzero integer,

$$\mathcal{M} = [I_N \otimes (1 \quad 0) \mathcal{M}] \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + [I_N \otimes (0 \quad 1) \mathcal{M}] \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we obtain after some simple manipulations that

$$Q(\omega) \otimes I_N Y_{t,t-1}^d = I_N \otimes R(\omega)^{-t} M_\omega^* \left( I_N \otimes \begin{pmatrix} \mathcal{S}_{c,\omega}^d \\ \mathcal{S}_{s,\omega}^d \end{pmatrix} \right) \mu_{\omega,t} + G_\omega^{d-1} \tilde{E}_\omega^*(B) \epsilon_t,$$

where

$$M_\omega^* = \left[ I_N \otimes (1 \quad 0) M_\omega^{-1} I_N \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \otimes I_2 - \left[ I_N \otimes (0 \quad 1) M_\omega^{-1} I_N \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \otimes J_2.$$

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More precisely, $M_\omega^*$ is a lower block triangular matrix of full rank; each block is a $2 \times 2$ matrix of the form $\begin{pmatrix} m_1^* & -m_2^* \\ m_2^* & m_1^* \end{pmatrix}$ that commutes with $R(\omega)$. A new rearrangement of the rows gives the formulae stated in Lemma A.3. ■

In the spirit of Lemma A.2, we need an algebraic property of the representation theorem to complete the proofs.

LEMMA A.4. *When $\omega \in ]0, \pi[$, consider any $2N \times 2(N - r_\omega)$ matrix $\Theta_{\omega,\perp}$ orthogonal to $\Theta_\omega(0) = (\Theta_\omega^1(0), \Theta_\omega^2(0))^{*\omega}$ that is such that the $2N \times 2N$ matrix $(\Theta_\omega(0) \quad \Theta_{\omega,\perp})$ is of full rank and $\Theta'_{\omega,\perp}\Theta_\omega(0) = 0$. Then there exists a $2(N - r_\omega) \times 2(N - r_\omega)$ matrix $M_{\omega,\perp}$ such that*

$$M_{\omega,\perp}\Theta'_{\omega,\perp}\begin{pmatrix} \epsilon_t \\ 0 \end{pmatrix} = \begin{pmatrix} \mu^0_{\omega,t} \\ 0 \end{pmatrix}$$

*and*

$$M_{\omega,\perp}\Theta'_{\omega,\perp}\begin{pmatrix} 0 \\ \epsilon_t \end{pmatrix} = \begin{pmatrix} -\cos\omega\mu^0_{\omega,t} \\ -\sin\omega\mu^0_{\omega,t} \end{pmatrix}.$$

**Proof of Lemma A.4.** The proof relies on the algebraic properties of the representation equation. First from (4) in Gregoir (1999), after multiplication of the $h$th set of rows by $\Delta_\omega(B)^h, h = 1, \ldots, m_\omega$, we get in $B = e^{i\omega}$

$$A_\omega(e^{i\omega})C(e^{i\omega}) = \begin{pmatrix} E^0_\omega(e^{i\omega}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using the matrices $D_\omega(B)$ defined in Lemma 5.1 in Gregoir (1999), $\zeta_\omega(e^{i\omega})$ of full rank introduced in (7) of the same paper, and the preceding equation, we obtain the following algebraic equation:

$$I_N = e^{i\omega}(0 \quad \Theta^1_\omega(0) + e^{i\omega}\Theta^2_\omega(0))D_\omega(e^{i\omega}) + \zeta_\omega(e^{i\omega})\, A_\omega(e^{i\omega})^{-1}\begin{pmatrix} E^0_\omega(e^{i\omega}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

With the help of the following equality: $\forall (N \times r)$ complex matrix $\mathcal{U}$,

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \otimes I_N \begin{pmatrix} \mathcal{U} & 0 \\ 0 & \bar{\mathcal{U}} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1} \otimes I_r = \begin{pmatrix} \Re\mathcal{U} & -\Im\mathcal{U} \\ \Im\mathcal{U} & \Re\mathcal{U} \end{pmatrix}$$

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and the fact that

$$Q(\omega)^{-1} \otimes I_N \begin{pmatrix} \Theta_\omega^1(0) + \cos\omega\Theta_\omega^2(0) & -\sin\omega\Theta_\omega^2(0) \\ \sin\omega\Theta_\omega^2(0) & \Theta_\omega^1(0) + \cos\omega\Theta_\omega^2(0) \end{pmatrix} Q(\omega) \otimes I_{r_\omega}$$
$$= \begin{pmatrix} \Theta_\omega^1(0) & \Theta_\omega^2(0) \\ -\Theta_\omega^2(0) & \Theta_\omega^1(0) + 2\cos\omega\Theta_\omega^2(0) \end{pmatrix},$$

we obtain the following algebraic identity:

$$Q(\omega)^{-1} \otimes I_N \begin{pmatrix} \Re\zeta_\omega(e^{i\omega})A_\omega(e^{i\omega})^{-1} & -\Im\zeta_\omega(e^{i\omega})A_\omega(e^{i\omega})^{-1} \\ \Im\zeta_\omega(e^{i\omega})A_\omega(e^{i\omega})^{-1} & \Re\zeta_\omega(e^{i\omega})A_\omega(e^{i\omega})^{-1} \end{pmatrix}$$
$$\times \begin{pmatrix} \Re E_\omega^0(e^{i\omega}) & -\Im E_\omega^0(e^{i\omega}) \\ 0 & 0 \\ \Im E_\omega^0(e^{i\omega}) & \Re E_\omega^0(e^{i\omega}) \\ 0 & 0 \end{pmatrix}$$
$$= \Theta_\omega(0)Q(\omega)^{-1} \otimes I_{r_\omega} \begin{pmatrix} \Re e^{i\omega}E_\omega^{1,m_\omega}(e^{i\omega}) & -\Im e^{i\omega}E_\omega^{1,m_\omega}(e^{i\omega}) \\ \Im e^{i\omega}E_\omega^{1,m_\omega}(e^{i\omega}) & \Re e^{i\omega}E_\omega^{1,m_\omega}(e^{i\omega}) \end{pmatrix}$$
$$+ Q(\omega)^{-1} \otimes I_N,$$

where $E_\omega^{1,m_\omega}(e^{i\omega})$ stands for the $N - r_0 \times N$ matrix obtained by stacking up the matrices $E_\omega^j(e^{i\omega})$ for $j = 1$ to $m_\omega$. Then premultiplying by $\Theta'_{\omega,\perp}$ and postmultiplying by $Q(\omega) \otimes I_N$, if we denote

$$\Xi_\omega = \Theta'_{\omega,\perp}Q(\omega)^{-1} \otimes I_N \begin{pmatrix} \Re\zeta_\omega(e^{i\omega})A_\omega(e^{i\omega})^{-1} & -\Im\zeta_\omega(e^{i\omega})A_\omega(e^{i\omega})^{-1} \\ \Im\zeta_\omega(e^{i\omega})A_\omega(e^{i\omega})^{-1} & \Re\zeta_\omega(e^{i\omega})A_\omega(e^{i\omega})^{-1} \end{pmatrix}$$

we get

$$\Xi_\omega \begin{pmatrix} \Re E_\omega^0(e^{i\omega}) & -\Im E_\omega^0(e^{i\omega}) \\ 0 & 0 \\ \Im E_\omega^0(e^{i\omega}) & \Re E_\omega^0(e^{i\omega}) \\ 0 & 0 \end{pmatrix} Q(\omega) \otimes I_N = \Theta'_{\omega,\perp}.$$

If we now consider the expressions of the form

$$\begin{pmatrix} \Re E_\omega^0(e^{i\omega}) & -\Im E_\omega^0(e^{i\omega}) \\ 0 & 0 \\ \Im E_\omega^0(e^{i\omega}) & \Re E_\omega^0(e^{i\omega}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a\epsilon_t \\ b\epsilon_t \end{pmatrix},$$

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where $a$ and $b$ are real numbers, we have

$$\begin{pmatrix} \Re E^0_\omega(e^{i\omega}) & -\Im E^0_\omega(e^{i\omega}) \\ 0 & 0 \\ \Im E^0_\omega(e^{i\omega}) & \Re E^0_\omega e^{i\omega}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a\epsilon_t \\ b\epsilon_t \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \otimes I_N \begin{pmatrix} \Re E^0_\omega(e^{i\omega}) \\ 0 \\ \Im E^0_\omega(e^{i\omega}) \\ 0 \end{pmatrix} \epsilon_t,$$

which can be rewritten after a rearrangement of the rows and according to the definition of $\mu_{\omega,t}$ as

$$\begin{pmatrix} I_{N-r_\omega} & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} a & -b \\ b & a \end{pmatrix} M_\omega^{-1} I_N \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu_{\omega,t},$$

which is similar to the equation we obtained in the proof of the preceding lemma. We then use the same argument to rewrite it under the form

$$\begin{pmatrix} I_{2(N-r_\omega)} & 0 \\ 0 & 0 \end{pmatrix} M_\omega^* I_N \otimes \begin{pmatrix} a \\ b \end{pmatrix} \mu_{\omega,t},$$

where $M_\omega^*$ is the full rank block triangular matrix previously defined. A new arrangement of the rows gives

$$\begin{pmatrix} \Re E^0_\omega(e^{i\omega}) & -\Im E^0_\omega(e^{i\omega}) \\ 0 & 0 \\ \Im E^0_\omega(e^{i\omega}) & \Re E^0_\omega(e^{i\omega}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a\epsilon_t \\ b\epsilon_t \end{pmatrix} = \begin{pmatrix} \widetilde{M}^*_\omega \\ 0 \end{pmatrix} \begin{pmatrix} a\mu^0_{\omega,t} \\ b\mu^0_{\omega,t} \end{pmatrix},$$

where $\widetilde{M}^*_\omega$ is a matrix of full rank $2(N-r_\omega)$. We denote $M_{\omega,\perp} = \left( \Xi_\omega \begin{pmatrix} \widetilde{M}^*_\omega \\ 0 \end{pmatrix} \right)^{-1}$ and obtain the desired result because

$$Q(\omega) \otimes I_N \begin{pmatrix} \epsilon_t \\ 0 \end{pmatrix} = \begin{pmatrix} \epsilon_t \\ 0 \end{pmatrix}$$

and

$$Q(\omega) \otimes I_N \begin{pmatrix} 0 \\ \epsilon_t \end{pmatrix} = \begin{pmatrix} -\cos\omega\epsilon_t \\ -\sin\omega\epsilon_t \end{pmatrix}.$$

■

## PROPERTIES OF THE PRINCIPAL COMPONENT PROCEDURE

We need for the proof of Theorems 4.1–4.4 a precise expansion of the estimated error correction terms that are derived from the principal components associated with the smallest eigenvalues of the cross-product of residuals at each step of the procedure described in Section 3. The main arguments used in the proof of Theorems 3.1–3.4 are close to those used by Gregoir and Laroque (1994). We do not give the proofs of Theorems 3.1 and 3.2

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that are almost similar to theirs. The basic property used in these proofs is given by their Lemma 5.4. It describes the expansion for large $T$ of the eigenvectors of a symmetric matrix whose particular form is that we are led to consider in our situation. The procedure of Section 3 involves a regression at each step. The properties of this regression are described in the following lemma. For sake of simplicity, we omit the deterministic trends in the analysis of the procedure. Their role will be extensively analyzed in Remark A.1.

LEMMA A.5. *Let $Y_t$ in $\mathbb{R}$ and $X_t^j$ in $\mathbb{R}^{n_j}$, for $j = 0,1,\ldots,\ell$, be processes in $\mathcal{C}_\Omega$, defined, when $\omega \in ]0,\pi[$, through*

$$\Delta_\omega^d Y_t = d_\omega(B)\varepsilon_t, \qquad \Delta_\omega^j X_t^j = d_\omega^j(B)\varepsilon_t,$$

*where $d$ is a non-negative integer, $\varepsilon_t$ is an $N$-dimensional martingale difference process of variance covariance matrix $\Sigma$ ($N$ being larger than $n_j$ for all $j \geq 1$), and $d_\omega^1(e^{i\omega}),\ldots,d_\omega^\ell(e^{i\omega})$ are of full rank ($d_\omega(e^{i\omega})$ and/or $d_\omega^0(e^{i\omega})$ may be of rank smaller than $n_0$, but the variance covariance matrix of $X^0$ is assumed of full rank).*

*Let $\pi = (\pi_\omega^0,\ldots,\pi_\omega^\ell)$ be the coefficient of the regression of $(Y_t' \quad Y_{t-1}')'$ on $X_t = (X_t^0, X_t^1, X_{t-1}^0,\ldots, X_t^\ell, X_{t-1}^\ell)$. Then, when the sample size $T$ goes to infinity,*

$$\pi_m^j(T) = T^{d-j}\Pi_\omega^j(T),$$

*where $\Pi_\omega^j(T)$ converges in distribution to some random matrix.*

**Proof of Lemma A.5.** It is left to the reader.

Some notations have to be introduced before proceeding to the proof itself. First consider the $2N \times 2N$ unitary matrix $\mathcal{B}_\omega$ built the following way. We take the $N(2N \times 1)$ orthonormal vectors $(\alpha_{\omega,h,(q)}^{1,h} \quad \alpha_{\omega,h,(q)}^{2,h})$, $h = 1,\ldots,m_\omega$, $q = 1,\ldots,r_\omega$, and $(\alpha_\omega^{1,0} \quad \alpha_\omega^{2,0})$. We complete this matrix with the same number of orthonormal vectors $v_{h,(q)}$, $h = 1,\ldots,m_\omega$, $q = 1,\ldots,r_\omega$ and $h = 0$, $q = 1,\ldots,N - r_\omega$, obtained by Gram–Schmidt process starting with $(-\alpha_{\omega,m_\omega,(1)}^{2,m_\omega} \quad \alpha_{\omega,m_\omega,(1)}^{1,m_\omega} + 2\cos\omega\alpha_{\omega,m_\omega,(1)}^{2,m_\omega})$, then increasing the index $q$ in the space of degree $m_\omega$, then decreasing the degree $h$, and so on. By construction the vector space spanned by $\{(\alpha_{\omega,l,(q')}^{l}, v_{l,(q')})$ with $1 \leq q' \leq r_{\omega,l}$, $l < h$, and $1 \leq q' \leq q$, $l = h\}$ is equal to the one spanned by $\{(\alpha_{\omega,l,(q')}^{l}, (-\alpha_{\omega,l,(q')}^{2,l} \quad \alpha_{\omega,l,(q')}^{1,l} + 2\cos\omega\alpha_{\omega,l,(q')}^{2,l}))$ with $1 \leq q' \leq r_{\omega,l}$, $l < h$, and $1 \leq q' \leq q$, $l = h\}$. It follows that the polynomial $\mathcal{V}_{h,(q)}(G_\omega)$ of degree $h$ built with the same linear combination of elements of $\bar{\alpha}_\omega(G_\omega)$ as that which defines the vector $v_{h,(q)}$ once applied to $(y_t' \quad y_{t-1}')'$ also defines a covariance stationary process. We denote $\mathcal{B}_{\omega,h,(q)}$ the matrix obtained by discarding the set of couples of vectors $\{(\alpha_{\omega,l,(q')}^{l}, v_{l,(q')})$ with $1 \leq q' \leq r_{\omega,l}$, $l < h$, and $1 \leq q' \leq q$, $l = h\}$. We have by this construction defined a new rule of identification built upon the previous one in which the matrix obtained by stacking up the terms of highest degree is a unitary matrix. We can now adapt the two remarks that were used by Gregoir and Laroque (1994) in the proof of the similar result when $\omega = 0$, namely, as follows.

(i) Although the procedure is described in terms of $(G_\omega^{m_\omega} y_t, G_\omega^{m_\omega} y_{t-1},\ldots, G_\omega y_t, G_\omega y_{t-1})$, we can analyze it in terms of appropriate components built upon the polynomial associated with the definition of $\mathcal{B}_\omega$ (see the preceding discussion). It allows us to build a new basis $\bar{\alpha}_{\mathcal{B}_\omega}(G_\omega)$ of $2N$-polynomial of degree at most $m_\omega$. Replace $(G_\omega^{m_\omega} y_t' \quad G_\omega^{m_\omega} y_{t-1}')'$ with

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$$\begin{pmatrix} G_\omega^{m_\omega} v_{0,(N-r_\omega)} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} \\ G_\omega^{m_\omega}(\alpha_{\omega,(N-r_m)}^{1,0} y_t + \alpha_{\omega,(N-r_\omega)}^{2,0} y_{t-1}) \\ \vdots \\ \mathcal{V}_{m_\omega,(1)}(G_\omega) y_t \\ \alpha_{\omega,m_\omega,(1)}^{m_\omega}(G_\omega) y_t \end{pmatrix}$$

and $\begin{pmatrix} G_\omega^h y_t \\ G_\omega^h y_{t-1} \end{pmatrix}$, $h = 1, \ldots, m_\omega - 1$ with

$$\begin{pmatrix} G_\omega^h v_{0,(N-r_\omega)} \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} \\ G_\omega^h(\alpha_{\omega,(N-r_\omega)}^{1,0} y_t + \alpha_{\omega,(N-r_\omega)}^{2,0} y_{t-1}) \\ \vdots \\ \Delta_\omega(B)^{m_\omega - h} \sum_{l=m_\omega - h+1}^{m_\omega} \mathcal{V}_{m_\omega,l,(1)} G_\omega^l \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} \\ \Delta_\omega(B)^{m_\omega - h} \sum_{l=m_\omega - h+1}^{m_\omega} \alpha_{\omega,l,(1)}^{m_\omega} G_\omega^l \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} \end{pmatrix}.$$

These replacements do not change the results of the procedure. Indeed, first the vector space spanned by $(G_\omega^l y_t, G_\omega^l y_{t-1}, \ldots, G_\omega y_t, G_\omega y_{t-1})$ is unchanged after the substitution. Second, adding a linear combination of the regressors to the regressand does not change the residuals. Third, the principal components of these residuals are not affected by a full rank unitary transformation of the variables. Finally, an upper triangular matrix allows us to transform the coordinates in the basis $\bar{\alpha}_{\mathcal{B}_\omega}(G_\omega)$ into that of $\bar{\alpha}_\omega(G_\omega)$ (when the rows of these matrices are arranged in decreasing order of integration of the processes they define).

(ii) Let $Y$ denote the regressand and $X$ the regressors in a regression stage of the procedure. The residuals are $Y - YX'(XX')^{-1}X$, and their variance covariance matrix is

$$YY' - YX'(X'X)^{-1}XY'.$$

Now, we claim that in our problem the block structure of this matrix when $T$ goes to infinity is similar to that of $YY'$. Indeed, let $H$ be the variance covariance matrix of $(Y, X)$. It satisfies the properties of Lemma 5.4 of Gregoir and Laroque (1994), up to a permutation matrix, and its inverse has the structure shown in this lemma. Now note that $(YY' - YX'(X'X)^{-1}XY')$ is the inverse of the element of $H^{-1}$, corresponding to the $Y$ block. It follows that the orders of divergence in $T$ of the blocks of $YY'$ are equal to those of $YY' - YX'(X'X)^{-1}XY'$ and left unchanged by the regression.

We now proceed to the proofs. We first prove the result for $\alpha_{\omega,(1)}^{m_\omega}(G_\omega)$. We use (i) from the preceding discussion. By Property A.1, the definition of $\mu_\omega$ in Lemma A.3, and (ii) from the preceding discussion, the variance covariance matrix of the residuals has the structure of the matrix $H(T)$ of Lemma 5.4 in Gregoir and Laroque (1994), with $H_{00}(T)$ of

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dimension $2r_{\omega,m_\omega}, H_{11}(T)$ of dimension $2r_{\omega,m_\omega-1},\ldots,H_{mm}(T)$ of dimension $2(N-r_\omega)$. From Lemma 5.4 of Gregoir and Laroque (1994), the principal component estimator is a linear combination of the $2N$ residuals. It can be expressed in terms of the $G_\omega(y_t' \quad y_{t-1}')'$ with the help of $\hat{f}_{\omega,(1)}$, the eigenvector associated with the smallest eigenvalue, as

$$\hat{\alpha}^{m_\omega}_{\omega,(1)}(G_\omega) = \sum_{l=0}^{m_\omega}\left\{ \frac{\hat{f}^l_{\omega,(1)}(T)}{T^{m_\omega-l}} (G_\omega^{m-l}\alpha^l_\omega(G_\omega) - \hat{\pi}^l(T)\bar{\alpha}_\omega(G_\omega))\right\},$$

where the $(\hat{f}^l_{\omega,(1)}(T))_l$ converge weakly toward some limit when $T$ goes to infinity. The $(\hat{\pi}^l(T))_l$ are computed from the regression coefficients of the first step. The components associated with the regressands of the first step of the procedure are equal to zero. From Lemma A.5, $\hat{\pi}^l(T)$ is $O(T^{m_\omega-l-h})$ on the components of $\bar{\alpha}_{\mathcal{B}_\omega}(G_\omega)$ or those of $\bar{\alpha}_\omega(G_\omega)$ that are integrated of order $h$ because according to (i), the transformation of coordinates from one basis to the other is performed by an upper triangular matrix.

Rearranging terms in the preceding equation, we obtain the result stated in Theorem 3.4 for $\hat{\alpha}^{m_\omega}_{\omega,(1)}(G_\omega)$. The consistency of the estimator is a consequence of the definition of $\bar{\alpha}_\omega(G_\omega)$ and the way $\bar{\alpha}_{\mathcal{B}_\omega}(G_\omega)$ is built; in particular all the coefficients $\hat{\pi}(T)$ of the stationary components of $\bar{\alpha}_\omega(G_\omega)$ converge to zero. From the consistency of $\hat{\alpha}^{m_\omega}_{\omega,(1)}(G_\omega)$, we deduce the consistency of $(-\hat{\alpha}^{2,m_\omega}_{\omega,m_\omega,(1)} \quad \hat{\alpha}^{1,m_\omega}_{\omega,m_\omega,(1)} + 2\cos\omega\hat{\alpha}^{2,m_\omega}_{\omega,m_\omega,(1)})$ and therefore that of $\hat{v}_{m_\omega,(1)}$ (the normalization of this last vector is satisfied for the coefficients of the linear combination of the two preceding vectors converge toward their true value at a rate in $T^{-1/2}$). It follows that $\hat{\mathcal{A}}^{m_\omega}_{(1)}$, the matrix of orthonormal rows orthogonal to $(\hat{\alpha}^{1,m_\omega}_{\omega,m_\omega,(1)}, \hat{\alpha}^{2,m_\omega}_{\omega,m_\omega,(1)})^{*\omega}$ or equivalently to $(\hat{\alpha}^{1,m_\omega}_{\omega,m_\omega,(1)} \quad \hat{\alpha}^{2,m_\omega}_{\omega,m_\omega,(1)})$ and $\hat{v}_{m_\omega,(1)}$, is such that

$$\hat{\mathcal{A}}^{m_\omega}_{(1)} = \hat{F}_{\omega,m_\omega,(1)}(T)\mathcal{B}_{m_\omega,(1)} + o(T)\begin{pmatrix}(\alpha^{1,m_\omega}_{\omega,m_\omega,(1)} \quad \alpha^{2,m_\omega}_{\omega,m_\omega,(1)})\\ v_{m_\omega,(1)}\end{pmatrix},$$

where $\hat{F}_{\omega,m_\omega,(1)}(T)$ is a $2(N-1)\times 2(N-1)$ unitary matrix and $o(T)$ converges to zero and is at least of order $T^{-1/2}$ on the rows corresponding to the estimators of the stationary directions.

We then iterate the procedure along the same lines. The expansion of $\hat{\alpha}^{m_\omega}_{\omega,(2)}(G_\omega)$ is derived from the same arguments, because the principal components are not affected by the unitary transformation $\hat{F}_{\omega,m_\omega,(1)}(T)$, and the same computations, except for the dimension of the matrix $H_{00}(T)$ that is now $2(r_{\omega,m_\omega}-1)\times 2(r_{\omega,m_\omega}-1)$. The terms in $o(T)$ due to the preceding step are negligible. The consistency of the estimators results from the definition of $\bar{\alpha}_\omega(G_\omega)$ and $\bar{\alpha}_{\mathcal{B}_\omega}(G_\omega)$. Finally, we use the fact that the vector space spanned by $(\alpha^{1,m_\omega}_{\omega,m_{\omega_2}(1)}, \alpha^{2,m_\omega}_{\omega,m_\omega,(1)})^{*\omega}$ and $(\alpha^{1,m_\omega}_{\omega,m_\omega,(2)}, \alpha^{2,m_\omega}_{\omega,m_\omega,(2)})^{*\omega}$ is equal to that spanned by $(\alpha^{1,m_\omega}_{\omega,m_\omega,(1)} \quad \alpha^{2,m_\omega}_{\omega,m_\omega,(1)})$, $v_{m_\omega,(1)}$, $(\alpha^{1,m_\omega}_{\omega,m_\omega,(2)} \quad \alpha^{2,m_\omega}_{\omega,m_\omega,(2)})$, and $v_{m_\omega,(2)}$, which is consistently estimated by the estimators of the same vectors, to write

$$\hat{\mathcal{A}}^{m_\omega}_{(2)} = \hat{F}_{\omega,m_\omega,(2)}(T)\mathcal{B}_{m_\omega,(2)} + o(T)\begin{pmatrix}(\hat{\alpha}^{1,m_\omega}_{\omega,m_\omega,(1)} \quad \hat{\alpha}^{2,m_\omega}_{\omega,m_\omega,(1)})\\ \hat{v}_{m_\omega,(1)}\\ (\hat{\alpha}^{1,m_\omega}_{\omega,m_\omega,(2)} \quad \hat{\alpha}^{2,m_\omega}_{\omega,m_\omega,(2)})\\ \hat{v}_{m_\omega,(2)}\end{pmatrix},$$

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where $\hat{F}_{\omega, m_\omega,(2)}(T)$ is a $2(N-2)\times 2(N-2)$ unitary matrix and $o(T)$ converges to zero and is at least of order $T^{-1/2}$ on the rows corresponding to the estimators of the stationary directions. A proof by induction using the same arguments can easily be built.

**Remark A.1.** If we now consider a polynomial cointegration model in which error correction terms have nonzero deterministic trends, we can adapt the preceding proof in two different ways. The first one consists of stating a more general lemma than Lemma 5.4 in Gregoir and Laroque (1994) for symmetric matrices such as some blocks on its diagonal diverging at rate $T^d$ with $d$ odd and out of the diagonal at rate $T^{d+(1/2)}$. This would correspond to the analysis of the cross-product of the complete set of integrated processes and of a basis of the deterministic trend vector space. The second approach, simpler, consists of showing that we keep the same structure of the cross-product matrix if we regress in a preliminary step each variable on the appropriate basis of deterministic trends. This property results from Property A.1. If $Y$ is a $T\times 1$ vector of an $I_\omega(d_1)$ process, $X$ is a $T\times 1$ vector of an $I_\omega(d_2)$ process, and $P_{T,\mathcal{T}_\omega^h}$ stands for the projector orthogonal onto the space orthogonal to the basis of $\mathcal{T}_\omega^h$, then the cross-product of the residuals is given by $Y'P_{T,\mathcal{T}_\omega^h}X$ or equivalently, if we denote by the same symbol $\mathcal{T}_\omega^h$ the $T\times h$ matrix of the regressors in the regression on the basis of $\mathcal{T}_\omega^h$, by

$$Y'X - Y'\mathcal{T}_\omega^h(\mathcal{T}_\omega^{h\prime}\mathcal{T}_\omega^h)^{-1}\mathcal{T}_\omega^{h\prime}X.$$

Thus if we introduce $T^{-1/2}\Delta_h^\omega(T)$, the $h\times h$ diagonal matrix whose terms are equal to $T^l$ on the row related to $t^l e^{-i\omega t}$ when $\omega\in\{0,\pi\}$ (resp. $2h\times 2h$ diagonal matrix whose terms are equal to $T^l$ on the rows related to $t^l\cos\omega t$ and $t^l\sin\omega t$ when $\omega\in]0,\pi[$), in the preceding expression, we get

$$\frac{1}{T^{d_1+d_2}}Y'X - \frac{1}{T_1^d}Y'\mathcal{T}_\omega^h T^{-1/2}\Delta_h^\omega(T)[T^{-1/2}\Delta_h^\omega(T)\mathcal{T}_\omega^{h\prime}\mathcal{T}_\omega^h T^{-1/2}\Delta_h^\omega(T)]^{-1}$$

$$\times\, T^{-1/2}\Delta_h^\omega(T)\mathcal{T}_\omega^{h\prime}X\frac{1}{T^{d_2}},$$

where each element of this expression has a limit in distribution according to (ii(b), ii(d), iii(b), iii(d)) in Property A.1. This result can easily be extended to a cross-product of matrices. Therefore the cross-product of the variables once regressed on the basis of $\mathcal{T}_\omega^h$ has the same structure as the matrix of Lemma 5.4 in Gregoir and Laroque (1994). This simple computation has the consequence that in Lemma A.5, by the Frish–Waugh theorem, if we regress regressand and regressors on $\mathcal{T}_\omega^h$, the result obtained in Lemma A.5 remains valid. This allows us to conclude that the expansion previously obtained still holds when we introduce deterministic trends in the error correction terms. The consistency of the estimator is similarly derived from the definition of the basis $\bar{\alpha}_\omega(G_\omega)$ and $\bar{\beta}_\omega(t)$ as described in Section 3.

Nevertheless, we must notice that the estimated principal components have in their expansion weights on the deterministic variables. To find the magnitude of these weights, we need a new lemma. For sake of simplicity, we consider only the case $\omega\in]0,\pi[$; the extension to $\omega\in\{0,\pi\}$ is straightforward.

LEMMA A.6. *Let $Y_t$ in $\mathbb{R}$ and $X_t^j$ in $\mathbb{R}^{n_j}$, for $j=0,1,\ldots,\ell$, be processes in $\mathcal{C}_\Omega$ as defined in Lemma* A.5.

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*Let* $\sigma = (\sigma_c^0, \sigma_s^0, \sigma_c^1, \sigma_s^1, \ldots, \sigma_c^{\ell'})$ *be the coefficient of the elements of the canonical basis of* $\mathcal{T}_\omega^{\ell'}$ *in the regression of* $(Y' \quad Y'_{-1})'$ *on X and the basis. Then when the sample size T tends to infinity,*

$$\sigma_c^j(T) = T^{d-j-1/2}\tilde{\sigma}_c^j(T),$$

$$\sigma_s^j(T) = T^{d-j-1/2}\tilde{\sigma}_s^j(T),$$

*where* $\tilde{\sigma}_c^j(T)$ *and* $\tilde{\sigma}_s^j(T)$ *converge in distribution to some random matrices.*

**Proof of Lemma A.6.** It is left to the reader.

From these results, we conclude that $\forall h = 1, \ldots, m_\omega$

$$\hat{\alpha}_\omega^h(G_\omega)\begin{pmatrix} y_t \\ y_{t-1}\end{pmatrix} + \hat{\beta}_\omega^h(t) - \alpha_\omega^h(G_\omega)\begin{pmatrix} y_t \\ y_{t-1}\end{pmatrix} - \beta_\omega^h(t)$$

$$= O_{p,1}^{\omega,h}(1)\tilde{\Delta}_\omega(T)\left[\bar{\alpha}_\omega(G_\omega)\begin{pmatrix} y_t \\ y_{t-1}\end{pmatrix} + \bar{\beta}_\omega(t)\right] + T^{-1/2}O_{p,2}^{\omega,h}(1)\Delta_h^\omega(T)\mathcal{T}_\omega^h(t).$$

A similar expansion holds when $\omega \in \{0, \pi\}$. The details are left to the reader.

## PROOF OF THE STATISTICAL PROPERTY

The proof of Theorems 4.1 and 4.2 uses the same kind of argument as that in the previous subsection. We define when $\omega_j \in \{0, \pi\}$ $\mu_{\omega_j} = (G_{\omega_j}\mu_{\omega_j}^0, \ldots, G_{\omega_j}^{m_{\omega_j}}\mu_{\omega_j}^0, G_{\omega_j}\mu_{\omega_j}^1, \ldots, G_{\omega_j}^{m_{\omega_j}-1}\mu_{\omega_j}^1, \ldots, G_{\omega_j}\mu_{\omega_j}^h, \ldots, G_{\omega_j}^{m_{\omega_j}-h}\mu_{\omega_j}^h, \ldots, G_{\omega_j}\mu_{\omega_j}^{m_{\omega_j}-1})$ and when $\omega \in ]0, \pi[$, $\mu_{c,\omega_j} = (\mathcal{S}_{c,\omega_j}\mu_{\omega_j}^0, \ldots, \mathcal{S}_{c,\omega_j}^{m_{\omega_j}}\mu_{\omega_j}^0, \mathcal{S}_{c,\omega_j}\mu_{\omega_j}^1, \ldots, \mathcal{S}_{c,\omega_j}^{m_{\omega_j}-1}\mu_{\omega_j}^1, \ldots, \mathcal{S}_{c,\omega_j}\mu_{\omega_j}^h, \ldots, \mathcal{S}_{c,\omega_j}^{m_{\omega_j}-h}\mu_{\omega_j}^h, \ldots, \mathcal{S}_{c,\omega_j}\mu_{\omega_j}^{m_{\omega_j}-1})$ and $\mu_{s,\omega_j} = (\mathcal{S}_{s,\omega_j}\mu_{\omega_j}^0, \ldots, \mathcal{S}_{s,\omega_j}^{m_{\omega_j}}\mu_{\omega_j}^0, \mathcal{S}_{s,\omega_j}\mu_{\omega_j}^1, \ldots, \mathcal{S}_{s,\omega_j}^{m_{\omega_j}-1}\mu_{\omega_j}^1, \ldots, \mathcal{S}_{s,\omega_j}\mu_{\omega_j}^h, \ldots, \mathcal{S}_{s,\omega_j}^{m_{\omega_j}-h}\mu_{\omega_j}^h, \ldots, \mathcal{S}_{s,\omega_j}\mu_{\omega_j}^{m_{\omega_j}-1})$.

We first derive an expression of the divergent terms introduced in the right-hand side of the representation equation in terms of the preceding processes. When $\omega_j \in \{0, \pi\}$, it is straightforward using the definition of $\mu_{\omega_j}$ from Lemma A.1 to get

$$\mathcal{Z}_{\omega_j,t} = \kappa_{\omega_j}\mu_{\omega_j,t} + \text{“terms of smaller order”}$$

where $\kappa_{\omega_j}$ is an $(Nm_{\omega_j} - k_j \times Nm_{\omega_j} - k_j)$ full rank matrix built with the submatrices $M_{\omega_j}^{h,l}$ of Lemma A.1. When $\omega_j \in ]0, \pi[$, a similar operation gives from Lemma A.3 after a rearrangement of the rows

$$Q(\omega_j) \otimes I_{Nm_{\omega_j}-k_j}\begin{pmatrix} \mathcal{Z}_{\omega_j,t} \\ \mathcal{Z}_{\omega_j,t-1}\end{pmatrix} = \kappa_{\omega_j}R(\omega)^{-1} \otimes I_{Nm_{\omega_j}-k_j}\begin{pmatrix} \mu_{c,\omega_j,t} \\ \mu_{s,\omega_j,t}\end{pmatrix}$$

$$+ \text{“terms of smaller order,”}$$

where $\kappa_{\omega_j}$ is a $(2Nm_{\omega_j} - 2k_j \times 2Nm_{\omega_j} - 2k_j)$ full rank matrix built with the submatrices $M_{\omega_j}^{*,hl}$ of Lemma A.3.

For sake of simplicity, we prove Theorems 4.1 and 4.2 simultaneously under the null that there exist only three different unit roots in the determinant of the Wold representa-

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tion, related to the frequencies $\omega_1 \in \{0, \pi\}$ and $\omega_2 \in ]0, \pi[$. The same arguments allow us to build a proof for a more general null hypothesis with the use of v(a) and v(d) in Property A.1.

**Proof of Theorems 4.1 and 4.2.** To simplify notations, we denote $z_{\omega_1,t}$ the $k_1$-dimensional vector obtained by stacking up the $k_1$ covariance stationary components of $\bar{\alpha}_{\omega_1}(G_{\omega_1})y_t + \bar{\beta}_{\omega_1}(t)$, $z_{\omega_2,t}$ the $2k_2$-dimensional vector obtained by stacking up the $k_2$ covariance stationary components of $\bar{\alpha}_{\omega_2}(G_{\omega_2})\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \bar{\beta}_{\omega_2}(t)$ and $z_{t-1} = (y_{t-1}, \ldots, y_{t-q})$. The ordinary least squares regression coefficients of (6) or (7) satisfy

$$\mathcal{X}\mathcal{X}' \begin{pmatrix} \hat{\psi}'_{\omega_1} \\ \hat{\psi}'_{\omega_2} \\ \hat{\theta}'_{\omega_1} - \theta'_{\omega_1} \\ \hat{\theta}^{1\prime}_{\omega_2} - \theta^{1\prime}_{\omega_1} \\ \hat{\theta}^{2\prime}_{\omega_2} - \theta^{2\prime}_{\omega_2} \\ \hat{\phi}' - \phi \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_{\omega_1,-1} \\ \mathcal{Z}_{\omega_2,-1} \\ \mathcal{Z}_{\omega_2,-2} \\ z_{\omega_1,-1} \\ z_{\omega_2,-1} \\ z_{\omega_2,-2} \\ z_{-1} \end{pmatrix} \epsilon' = \mathcal{X}\epsilon',$$

where the index $-1$ denotes the matrix obtained by stacking up the $T$ observations for the corresponding variables from $t = 0, \ldots, T-1$, $\theta_{\omega_1}$ stands for the expansion of $\Theta_{\omega_1}(\Delta_{\omega_1})$ according to Remark 3.2 in Gregoir (1999), and likewise $\theta^1_{\omega_2}$ and $\theta^2_{\omega_2}$ are obtained by a similar operation applied to $\Theta^1_{\omega_2}(\Delta_{\omega_2})$ and $\Theta^2_{\omega_2}(\Delta_{\omega_2})$. The coefficients are stored equation by equation in columns in an $N(m_{\omega_1} + 2m_{\omega_2} + p') \times N$ matrix and the error terms $\epsilon$ in an $N \times T$ matrix. The proof is similar to that of Theorem 3.1 in Gregoir and Laroque (1994); we sum up the arguments. We build $\mathcal{D}(T)$, an $N(m_{\omega_1} + 2m_{\omega_2} + p') \times N(m_{\omega_1} + 2m_{\omega_2} + p')$ block diagonal matrix, where the first block is equal to $\Delta_{\omega_1}(T)\kappa^{-1}_{\omega_1}$, the second one to $\Delta_{\omega_2}(T)\kappa^{-1}_{\omega_2} Q(\omega_2) \otimes I_{Nm_{\omega_2}-k_2}$, and all the remaining terms to $T^{-1/2}$. We premultiply and postmultiply the matrix of the cross-product of the regressors and perform the necessary operations to keep the preceding identity. We have

$$\{\mathcal{D}(T)\mathcal{X}\mathcal{X}'\mathcal{D}(T)'\}\mathcal{D}(T)^{-1\prime} \begin{pmatrix} \hat{\psi}'_{\omega_1} \\ \hat{\psi}'_{\omega_2} \\ \hat{\theta}'_{\omega_1} - \theta'_{\omega_1} \\ \hat{\theta}^{1\prime}_{\omega_2} - \theta^{1\prime}_{\omega_1} \\ \hat{\theta}^{2\prime}_{\omega_2} - \theta^{2\prime}_{\omega_2} \\ \hat{\phi}' - \phi \end{pmatrix} = \mathcal{D}(T) \begin{pmatrix} \mathcal{Z}'_{\omega_1,-1} \\ \mathcal{Z}'_{\omega_2,-1} \\ \mathcal{Z}'_{\omega_2,-2} \\ z'_{\omega_1,-1} \\ z'_{\omega_2,-1} \\ z'_{-1} \end{pmatrix} \epsilon'.$$

By construction and from Property A.1, we have the following convergences in distribution, when $T$ goes to infinity:

(i) $\Delta_{\omega_1}(T)\kappa^{-1}_{\omega_1}\mathcal{Z}_{\omega_1,-1}\mathcal{Z}'_{\omega_1,-1}\kappa^{-1\prime}_{\omega_1}\Delta_{\omega_1}(T) \Rightarrow X'_{\omega_1} * X_{\omega_1}$ where $X_{\omega_1}$ is defined in the statement of Theorem 4.1,

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(ii) $\Delta_{\omega_2}(T)\kappa_{\omega_2}^{-1}Q(\omega_2)\otimes I_{Nm_{\omega_2}-k_2}\begin{pmatrix}\mathcal{Z}_{\omega_2,-1}\\ \mathcal{Z}_{\omega_2,-2}\end{pmatrix}\begin{pmatrix}\mathcal{Z}_{\omega_2,-1}\\ \mathcal{Z}_{\omega_2,-2}\end{pmatrix}'$

$$Q(\omega_2)'\otimes I_{Nm_{\omega_2}-k_2}\kappa_{\omega_2}^{-1\prime}\Delta_{\omega_2}(T)\Rightarrow I_2\otimes F_{\omega_2}(X_{\omega_2}^{*\prime}*X_{\omega_2}^{*})I_2\otimes F_{\omega_2},$$

where $X_{\omega_2}$ is defined in the statement of Theorem 4.2, $F_{\omega_2}$ is an $Nm_{\omega_2}-k_2\times Nm_{\omega_2}-k_2$ diagonal matrix whose element corresponding to an $I_{\omega_2}(h)$ process is equal to $\sin^{1-h}\omega_2 2^{-h}$,

(iii) $\Delta_{\omega_1}(T)\kappa_{\omega_1}^{-1}\mathcal{Z}_{\omega_1,-1}\begin{pmatrix}\mathcal{Z}_{\omega_2,-1}\\ \mathcal{Z}_{\omega_2,-2}\end{pmatrix}'Q(\omega_2)'\otimes I_{Nm_{\omega_2}-k_2}\kappa_{\omega_2}^{-1\prime}\Delta_{\omega_2}(T)\Rightarrow 0,$

(iv) $T^{-1/2}\Delta_{\omega_1}(T)\kappa_{\omega_1}^{-1}\mathcal{Z}_{\omega_1,-1}(z'_{\omega_1,-1}\quad z'_{\omega_2,-1}\quad z'_{-1})\Rightarrow 0,$

(v) $T^{-1/2}\Delta_{\omega_2}(T)\kappa_{\omega_2}^{-1}Q(\omega_2)\otimes I_{Nm_{\omega_2}-k_2}\begin{pmatrix}\mathcal{Z}_{\omega_2,-1}\\ \mathcal{Z}_{\omega_2,-2}\end{pmatrix}(z'_{\omega_1,-1}\quad z'_{\omega_2,-1}\quad z'_{-1})\Rightarrow 0.$

Finally, because $\epsilon=[M_{\omega_1}E_{\omega_1}(e^{i\omega_1})]^{-1}\mu_{\omega_1}$ and $\epsilon=[I_N\otimes(1\quad 0)M_{\omega_2}\mathcal{E}_{\omega_2}]^{-1}\mu_{\omega_2}$, we get

(vi) $\Delta_{\omega_1}(T)\kappa_{\omega_1}^{-1}\mathcal{Z}_{\omega_1,-1}\epsilon'\Rightarrow X'_{\omega_1}*dW_{\omega_1}[M_{\omega_1}E_{\omega_1}(e^{i\omega_1})]^{-1\prime},$

(vii) $T^{-1/2}\Delta_{\omega_2}(T)\kappa_{\omega_2}^{-1}Q(\omega_2)\otimes I_{Nm_{\omega_2}-k_2}\begin{pmatrix}\mathcal{Z}_{\omega_2,-1}\\ \mathcal{Z}_{\omega_2,-2}\end{pmatrix}\epsilon'$

$$\Rightarrow I_2\otimes F_{\omega_2}(X_{\omega_2}^{*\prime}*dW_{\omega_2}^{*})\begin{pmatrix}\cos\omega_2 I_N\\ \sin\omega_2 I_N\end{pmatrix}[I_N\otimes(1\quad 0)M_{\omega_2}\mathcal{E}_{\omega_2}]^{-1\prime},$$

where $F_{\omega_2}$ is defined in (ii).

The matrix of the cross-product of the stationary regressors converges toward a positive definite variance covariance matrix; therefore

(viii) $\kappa_{\omega_1}\Delta_{\omega_1}^{-1}(T)\hat{\psi}'_{\omega_1}\Rightarrow(X'_{\omega_1}*X_{\omega_1})^{-1}X'_{\omega_1}*dW_{\omega_1}[M_{\omega_1}E_{\omega_1}(e^{i\omega_1})]^{-1\prime},$

(ix) $\kappa_{\omega_2}\Delta_{\omega_2}^{-1}(T)Q(\omega_2)'\otimes I_{Nm_{\omega_2}-k_2}\hat{\psi}'_{\omega_2}\Rightarrow F_{\omega_2}^{-1}\otimes I_2[(X_{\omega_2}^{*\prime}*X_{\omega_2}^{*})]^{-1}(X_{\omega_2}^{*\prime}*dW_{\omega_2}^{*})\times\begin{pmatrix}\cos\omega_2 I_N\\ \sin\omega_2 I_N\end{pmatrix}[I_N\otimes(1\quad 0)M_{\omega_2}\mathcal{E}_{\omega_2}]^{-1\prime}.$

The coefficients $\hat{\theta}_{\omega_1},\hat{\theta}^1_{\omega_2},\hat{\theta}^2_{\omega_2},\hat{\phi}$ are consistent estimators of $\theta_{\omega_1},\theta^1_{\omega_2},\theta^2_{\omega_2}$, and $\phi$. It follows that $\hat{\eta}\hat{\eta}/T$ converges toward

$$V\epsilon=[M_{\omega_1}E_{\omega_1}(e^{i\omega_1})]^{-1}[M_{\omega_1}E_{\omega_1}(e^{i\omega_1})]^{-1\prime}$$

or

$$V\epsilon=[I_N\otimes(1\quad 0)M_{\omega_2}\mathcal{E}_{\omega_2}]^{-1}[I_N\otimes(1\quad 0)M_{\omega_2}\mathcal{E}_{\omega_2}]^{-1\prime}$$

by construction of $\mu_{\omega_1}$ and $\mu_{\omega_2}$. The results of Theorems 4.1 and 4.2 follow from the properties of the trace operator. ■

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## PROOF OF THE SPECIFICATION TEST

The proof proceeds along similar lines. We first have to take into account that the divergent terms that are introduced in the right-hand side of the equation are now estimated. From the way $\hat{\mathcal{Z}}_{\omega_j,t}$ is built, the proofs of the expansion of Theorems 3.2 and 3.4, and the definition of $\mu$'s from Lemmas A.1 and A.3, we get when $\omega_j \in \{0,\pi\}$

$$\begin{pmatrix} \hat{\mathcal{Z}}_{\omega_j,t} \\ \mathcal{T}_{\omega_j}^{m_{\omega_j}}(t) \end{pmatrix} = \hat{\kappa}_{\omega_j}(T) \begin{pmatrix} \mu_{\omega_j,t} \\ \mathcal{T}_{\omega_j}^{m_{\omega_j}}(t) \end{pmatrix} + \text{“terms of smaller order,”}$$

where $\hat{\kappa}_{\omega_j}(T)$ is an $((N+1)m_{\omega_j}-k_j \times (N+1)m_{\omega_j}-k_j)$ full rank random matrix. Similarly, when $\omega_j \in ]0,\pi[$, we obtain

$$\begin{pmatrix} Q(\omega_j) \otimes I_{Nm_{\omega_j}-k_j} \begin{pmatrix} \hat{\mathcal{Z}}_{\omega_j,t} \\ \hat{\mathcal{Z}}_{\omega_j,t-1} \end{pmatrix} \\ \mathcal{T}_{\omega_j}^{m_{\omega_j}}(t) \end{pmatrix} = \hat{\kappa}_{\omega_j}(T) \begin{pmatrix} R(\omega)^{-1} \otimes I_{Nm_{\omega_j}-k_j} \begin{pmatrix} \mu_{c,\omega_j,t} \\ \mu_{s,\omega_j,t} \end{pmatrix} \\ \mathcal{T}_{\omega_j}^{m_{\omega_j}}(t) \end{pmatrix}$$
$$+ \text{“terms of smaller order,”}$$

where $\hat{\kappa}_{\omega_j}(T)$ is a $(2(N+1)m_{\omega_j}-2k_j \times 2(N+1)m_{\omega_j}-2k_j)$ full rank random matrix and $\mathcal{T}_{\omega_j}^{m_{\omega_j}}(t)$ is arranged in the following way:

$$\mathcal{T}_{\omega_j}^{m_{\omega_j}}(t) = \begin{pmatrix} \sin \omega_j t \\ \cos \omega_j t \end{pmatrix} \otimes \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{m_{\omega_j}-1} \end{pmatrix}.$$

As in the preceding subsection, for the sake of simplicity, we prove simultaneously Theorems 4.3 and 4.4 under the null that there exist only three different unit roots in the determinant of the Wold representation, related to the frequencies $\omega_1 \in \{0,\pi\}$ and $\omega_2 \in ]0,\pi[$. With the help of v(a), v(c), v(d), and v(f) of Property A.1, it is easy to prove that the same result holds for a more general null hypothesis.

**Proof of Theorems 4.3 and 4.4.** We use the same notations as that in the proof of Theorem 4.1. The ordinary least squares coefficients of (8) and (9) satisfy

$$\hat{\mathcal{X}}\hat{\mathcal{X}}' \begin{pmatrix} \tilde{\psi}'_{\omega_1} \\ \tilde{\psi}'_{\omega_2} \\ \tilde{\theta}'_{\omega_1} - \theta'_{\omega_1} \\ \tilde{\theta}^{1\prime}_{\omega_2} - \theta^{1\prime}_{\omega_1} \\ \tilde{\theta}^{2\prime}_{\omega_2} - \theta^{2\prime}_{\omega_2} \\ \tilde{\phi}' - \phi \end{pmatrix} = \hat{\mathcal{X}}\epsilon' + \hat{\mathcal{X}}(z_{\omega_1,-1} - \hat{z}_{\omega_1,-1})'\theta'_{\omega_1}$$
$$+ \hat{\mathcal{X}}(z'_{\omega_2,-1} - \hat{z}'_{\omega_2,-1} \quad z'_{\omega_2,-2} - \hat{z}'_{\omega_2,-2}) \begin{pmatrix} \theta^{1\prime}_{\omega_2} \\ \theta^{2\prime}_{\omega_2} \end{pmatrix},$$

where $\hat{\mathcal{X}}$ stands for

$$(\hat{\mathcal{Z}}_{\omega_1,-1} \quad \mathcal{T}_{\omega_1,-1}^{m_{\omega_1}} \quad \hat{\mathcal{Z}}_{\omega_2,-2} \quad \hat{\mathcal{Z}}_{\omega_2,-3} \quad \mathcal{T}_{\omega_2,-\omega_2,-2}^{m_{\omega_2}} \quad \hat{z}_{\omega_1,-1} \quad \hat{z}_{\omega_2,-1} \quad \hat{z}_{\omega_2,-2} \quad z_{-1})'.$$

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The same kind of identity holds for the second regression (8) and (10) in which

$$(\hat{\mathcal{Z}}_{\omega_2,-2} \quad \hat{\mathcal{Z}}_{\omega_2,-3} \quad \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-2})'$$

is replaced by

$$(\hat{\mathcal{Z}}_{\omega_2,-1} \quad \hat{\mathcal{Z}}_{\omega_2,-2} \quad \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-1})'.$$

In this equation, the estimators are denoted with a $\check{}$ symbol. The arguments of the proof of Theorems 4.1 and 4.2 are still valid; the asymptotic behavior of the elements of the cross-product of the regressors is not affected by the estimation errors introduced in $\hat{\mathcal{X}}$. On the other hand, we have to deal with the extra terms on the right-hand side. First, we give the limit distribution of the cross-product of the regressors. We define $\mathcal{D}(T)$, an $(N+1)(m_{\omega_1}+2m_{\omega_2})+Np' \times (N+1)(m_{\omega_1}+2m_{m_2})+Np'$ block diagonal matrix whose first block is equal to

$$\mathcal{D}_1(T) = \begin{pmatrix} \Delta_{\omega_1}(T) & 0 \\ 0 & T^{-1/2}\Delta^{\omega_1}_{m_{\omega_1}-1}(T) \end{pmatrix} \hat{\kappa}^{-1}_{\omega_1}(T),$$

the second one to

$$\mathcal{D}_2(T) = \begin{pmatrix} \Delta_{\omega_2}(T) & 0 \\ 0 & T^{-1/2} I_2 \otimes \Delta^{\omega_2}_{\omega_{\omega_2}-1}(T) \end{pmatrix} \hat{\kappa}^{-1}_{\omega_2}(T) \begin{pmatrix} Q(\omega_2) \otimes I_{Nm_{\omega_2}-k_2} & 0 \\ 0 & I_{2m_{\omega_2}} \end{pmatrix},$$

and the remaining terms to $T^{-1/2}$. From Property A.1 and the expansions obtained in Theorems 3.2 and 3.4, we get the following result when $T$ goes to infinity.

(i) $$\mathcal{D}_1(T) \begin{pmatrix} \hat{\mathcal{Z}}_{\omega_1,-1} \\ \mathcal{T}^{m_{\omega_1}}_{\omega_1,-1} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{Z}}_{\omega_1,-1} \\ \mathcal{T}^{m_{\omega_1}}_{\omega_1,-1} \end{pmatrix}' \mathcal{D}_1'(T) \Rightarrow \begin{pmatrix} X'_{\omega_1} * X_{\omega_1} & X'_{\omega_1} * \tau_0^{m_{\omega_1}-1\prime} \\ \tau_0^{m_{\omega_1}-1} * X_{\omega_1} & \tau_0^{m_{\omega_1}-1} * \tau_0^{m_{\omega_1}-1\prime} \end{pmatrix},$$

where $X_{\omega_1}$ is defined in the statement of Theorem 4.1.

(ii) $$\mathcal{D}_2(T) \begin{pmatrix} \hat{\mathcal{Z}}_{\omega_2,-2} \\ \hat{\mathcal{Z}}_{\omega_2,-3} \\ \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-2} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{Z}}_{\omega_2,-2} \\ \hat{\mathcal{Z}}_{\omega_2,-3} \\ \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-2} \end{pmatrix}' \mathcal{D}_2'(T)$$
$$\Rightarrow \begin{pmatrix} I_2 \otimes F_{\omega_2}(X^{*\prime}_{\omega_2} * X^*_{\omega_2}) I_2 \otimes F'_{\omega_2} & 2^{-1/2} I_2 \otimes F_{\omega_2} X^{*\prime}_{\omega_2} * \tau_0^{m_{\omega_2}-1\prime} \\ 2^{-1/2} \tau_0^{m_{\omega_2}-1} * X^*_{\omega_2} I_2 \otimes F'_{\omega_2} & \frac{1}{2} I_2 \otimes \tau_0^{m_{\omega_2}-1} * \tau_0^{m_{\omega_2}-1\prime} \end{pmatrix},$$

where $X_{\omega_2}$ is defined in the statement of Theorem 4.2 and $F_{\omega_2}$ in (ii) in the preceding proof.

(iii) $$\mathcal{D}_1(T) \begin{pmatrix} \hat{\mathcal{Z}}_{\omega_1,-1} \\ \mathcal{T}^{m_{\omega_1}}_{\omega_1,-1} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{Z}}_{\omega_2,-2} \\ \hat{\mathcal{Z}}_{\omega_2,-3} \\ \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-2} \end{pmatrix}' \mathcal{D}_2'(T) \Rightarrow 0$$

by v(b), v(f), v(j), v(k), in Property A.1.

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(iv) $T^{-1/2}\mathcal{D}_1(T)\begin{pmatrix}\hat{\mathcal{Z}}_{\omega_1,-1}\\ \mathcal{T}^{m_{\omega_1}}_{\omega_1,-1}\end{pmatrix}(\hat{z}'_{\omega_1,-1}\quad \hat{z}'_{\omega_2,-1}\quad z'_{-1}) \Rightarrow 0$

by iv(a), vi(c) in Property A.1 and Theorems 3.2 and 3.4.

(v) $T^{-1/2}\mathcal{D}_2(T)\begin{pmatrix}\hat{\mathcal{Z}}_{\omega_2,-2}\\ \hat{\mathcal{Z}}_{\omega_2,-3}\\ \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-2}\end{pmatrix}(\hat{z}'_{\omega_1,-1}\quad \hat{z}'_{\omega_2,-1}\quad z'_{-1}) \Rightarrow 0$

by iv(b) and vi(d) in Property A.1 and Theorems 3.2 and 3.4.

We now consider the extra terms of the right-hand side of the equation. The first one is

$$\mathcal{D}(T)\hat{\mathcal{X}}(z_{\omega_1,-1}-\hat{z}_{\omega_1,-1})'\theta'_{\omega_1}.$$

The first $r_{\omega_1}$ components of $z_{\omega_1,-1}-\hat{z}_{\omega_1,-1}$ are according to the expansion described in Theorem 3.4 equal to

$$(I_{r_{\omega_1}}-\hat{\Pi}_{\omega_1,1}(T))[\alpha_{\omega_1}(G_{\omega_1})y_{-1}+\beta_{\omega_1}(-1)]-\hat{\Pi}_{\omega_1,2}(T)\begin{pmatrix}\Delta_{\omega_1}(T) & 0\\ 0 & T^{-1/2}\Delta^{\omega_1}_{m_{\omega_1}-1}(T)\end{pmatrix}$$
$$\times\begin{pmatrix}\bar{\alpha}_{\omega_1}(G_{\omega_1})y_{-1}+\bar{\beta}_{\omega_1}(-1)\\ \mathcal{T}^{m_{\omega_1}}_{\omega_1}(-1)\end{pmatrix},$$

where $\hat{\Pi}_{\omega_1,1}(T)$ converges to $I_{r_{\omega_1}}$ and $\hat{\Pi}_{\omega_1,2}(T)$ has a limit distribution $\Pi_{\omega_1,2}$ when $T$ goes to infinity. The remaining rows are similar processes but premultiply by the operator $\Delta_{\omega_1}$. It follows from Property A.1 that these extra terms converge to zero except the $r_{\omega_1}$ top rows that converge to

$$\begin{pmatrix}X'_{\omega_1}*X_{\omega_1} & X'_{\omega_1}*\tau_0^{m_{\omega_1}-1\prime}\\ \tau_0^{m_{\omega_1}-1}*X_{\omega_1} & \tau_0^{m_{\omega_1}-1}*\tau_0^{m_{\omega_1}-1\prime}\end{pmatrix}\Pi_{\omega_1,2}\Theta_{\omega_1}(0)'.$$

To get rid of this term, we multiply the top equations from the right by $\hat{\Theta}_{\omega_1,\perp}(0)$. Using Lemma A.2 to replace $\Theta_{\omega_1,\perp}(0)\epsilon_t$ by $M^{-1}_{\omega_1,\perp}\mu^0_{\omega_1,t}$, we obtain through similar computations as those in the end of the proof of Theorems 4.1 and 4.2 the desired result.

The same approach for $\omega_2\in]0,\pi[$ leads to the fact that in the first equation, all the extra terms related to the estimation errors at frequency $\omega_2$ in the right-hand side of the equation converge to zero except the cross-product of

$$\mathcal{D}_2(T)(\hat{\mathcal{Z}}_{\omega_2,-2}\quad \hat{\mathcal{Z}}_{\omega_2,-3}\quad \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-2})'$$

and the $2r_{\omega_2}$ first rows of $(z'_{\omega_2,-1}-\hat{z}'_{\omega_2,-1}\quad z'_{\omega_2,-2}-\hat{z}'_{\omega_2,-2})\begin{pmatrix}\Theta^{1\prime}_{\omega_2}(0)\\ \Theta^{2\prime}_{\omega_2}(0)\end{pmatrix}$ for the same kind of expansion holds for $\hat{z}_{\omega_2,-1}$ according to Theorem 3.4. We denote the limit distribution of this term $\mathcal{Z}_{\omega_2}(W_{\omega_2})\begin{pmatrix}\theta^{1\prime}_{\omega_2}\\ \theta^{2\prime}_{\omega_2}\end{pmatrix}$. Nevertheless, this result ensures the consistency of $\hat{\theta}^1_{\omega_2}$ and $\hat{\theta}^2_{\omega_2}$. Now remark that

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$$(\Theta^1_{\omega_2}(\Delta_{\omega_2}) \quad \Theta^2_{\omega_2}(\Delta_{\omega_2}))\begin{pmatrix} z_{\omega_2,t-1} \\ z_{\omega_2,t-2} \end{pmatrix}$$

$$= (-\Theta^2_{\omega_2}(\Delta_{\omega_2}) \quad (\Theta^1_{\omega_2} + \cos\omega_2\Theta^2_{\omega_2})(\Delta_{\omega_2})\begin{pmatrix} z_{\omega_2,t} \\ z_{\omega_2,t-1} \end{pmatrix} + \Delta_{\omega_2}\Theta^2_{\omega_2}(\Delta_{\omega_2})z_{\omega_2,t}.$$

The identity holds of course if we replace $z_{\omega_2,t}$ by $\hat{z}_{\omega_2,t}$. It follows that in the second equation (10), the ordinary least squares regression coefficients satisfy

$$\check{\mathcal{X}}\check{\mathcal{X}}'\begin{pmatrix} \check{\psi}'_{\omega_1} \\ \check{\psi}'_{\omega_2} \\ \check{\theta}'_{\omega_1} - \theta'_{\omega_1} \\ \check{\theta}^{1\prime}_{\omega_2} - \theta^{1\prime}_{\omega_1} \\ \check{\theta}^{2\prime}_{\omega_2} - \theta^{2\prime}_{\omega_2} \\ \check{\phi}' - \phi \end{pmatrix} = \check{\mathcal{X}}\epsilon' + \check{\mathcal{X}}(z_{\omega_1,-1} - \hat{z}_{\omega_1,-1})'\theta'_{\omega_1}$$

$$+ \check{\mathcal{X}}[(1 - 2\cos\omega_2 B + B^2)(z_{\omega_2,0} - \hat{z}_{\omega_2,0})]'\theta^{2\prime}_{\omega_2}$$

$$+ \check{\mathcal{X}}(z'_{\omega_2,0} - \hat{z}'_{\omega_2,0} \quad z'_{\omega_2,-1} - \hat{z}'_{\omega_2,-1})\begin{pmatrix} -\theta^{2\prime}_{\omega_2} \\ \theta^{1\prime}_{\omega_2} + 2\cos\omega_2\theta^{2\prime}_{\omega_2} \end{pmatrix},$$

where $\check{\mathcal{X}}$ stands for

$$(\hat{\mathcal{Z}}_{\omega_1,-1} \quad \mathcal{T}^{m_{\omega_1}}_{\omega_1,-1} \quad \hat{\mathcal{Z}}_{\omega_2,-1} \quad \hat{\mathcal{Z}}_{\omega_2,-2} \quad \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-1} \quad \hat{z}_{\omega_1,-1} \quad \hat{z}_{\omega_2,-1} \quad \hat{z}_{\omega_2,-2} \quad z_{-1})'.$$

The same analysis of the behavior of the extra terms due to the estimation errors at frequency $\omega_2$ in this second equation shows that all these terms converge to zero except the cross-product of

$$\mathcal{D}_2(T)(\hat{\mathcal{Z}}_{\omega_2,-1} \quad \hat{\mathcal{Z}}_{\omega_2,-2} \quad \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-1})'$$

and the $2r_{\omega_2}$ first rows of

$$(z'_{\omega_2,0} - \hat{z}'_{\omega_2,0} \quad z'_{\omega_2,-1} - \hat{z}'_{\omega_2,-1})\begin{pmatrix} \theta^{2\prime}_{\omega_2} \\ \theta^{1\prime} + 2\cos\omega_2\theta^{2\prime}_{\omega_2} \end{pmatrix},$$

which has the following limit distribution:

$$\mathcal{Z}_{\omega_2}(W_{\omega_2})\begin{pmatrix} \Theta^{2\prime}_{\omega_2}(0) \\ \Theta^{1\prime}_{\omega_2}(0) + 2\cos\omega_2\Theta^{2\prime}_{\omega_2}(0) \end{pmatrix}.$$

It follows that if we store in columns the two preceding identities satisfied by the least squares regression coefficients and multiply the equations corresponding to $\theta_{\omega_2}$ from the

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right by $\hat{\Theta}_{\omega_2,\perp}$, we eliminate the nuisance parameter $\mathcal{Z}_{\omega_2}(W_{\omega_2})$. We now describe the consequences of this operation. Using Lemma A.4 and Property A.1, it is easy to show that

$$\text{(vi)}\quad \mathcal{D}_{\omega_2}(T)\begin{pmatrix} \hat{\mathcal{Z}}_{\omega_2,-2} \\ \hat{\mathcal{Z}}_{\omega_2,-3} \\ \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-2} \end{pmatrix}(\epsilon' \quad 0)\Theta'_{\omega_2,\perp}$$

$$\Rightarrow \begin{pmatrix} I_2 \otimes F_{\omega_2} & 0 \\ 0 & 2^{-1/2} \end{pmatrix}\begin{pmatrix} X^{*\prime}_{\omega_2} * dW^*_{\omega_2,0} \\ \tau_0^{m_{\omega_2}-1} * dW^*_{\omega_2,0} \end{pmatrix}\begin{pmatrix} \cos 2\omega_2 & 0 \\ \sin 2\omega_2 & 0 \end{pmatrix} \otimes I_N M^{-1}_{\omega_2,\perp},$$

$$\text{(vii)}\quad \mathcal{D}_{\omega_2}(T)\begin{pmatrix} \hat{\mathcal{Z}}_{\omega_2,-1} \\ \hat{\mathcal{Z}}_{\omega_2,-2} \\ \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-1} \end{pmatrix}(0 \quad \epsilon')\Theta'_{\omega_2,\perp}$$

$$\Rightarrow \begin{pmatrix} I_2 \otimes F_{\omega_2} & 0 \\ 0 & 2^{-1/2} \end{pmatrix}\begin{pmatrix} X^{*\prime}_{\omega_2} * dW^*_{\omega_2,0} \\ \tau_0^{m_{\omega_2}-1} * dW^*_{\omega_2,0} \end{pmatrix}$$

$$\times \begin{pmatrix} -\cos^2\omega_2 & -\cos\omega_2 \sin\omega_2 \\ -\cos\omega_2 \sin\omega_2 & -\sin^2\omega_2 \end{pmatrix} \otimes I_N M^{-1}_{\omega_2,\perp}.$$

Thus, when we mix the preceding results, we obtain that

$$\mathcal{D}_{\omega_2}(T)\left\{\begin{pmatrix} \hat{\mathcal{Z}}_{\omega_2,-2} \\ \hat{\mathcal{Z}}_{\omega_2,-3} \\ \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-2} \end{pmatrix}(\epsilon' \quad 0) + \begin{pmatrix} \hat{\mathcal{Z}}_{\omega_2,-1} \\ \hat{\mathcal{Z}}_{\omega_2,-2} \\ \mathcal{T}^{m_{\omega_2}}_{\omega_2,-\omega_2,-1} \end{pmatrix}(0 \quad \epsilon')\right\}\Theta'_{\omega_2,\perp}$$

$$\Rightarrow -\sin\omega_2 \begin{pmatrix} I_2 \otimes F_{\omega_2} & 0 \\ 0 & 2^{-1/2} \end{pmatrix}\begin{pmatrix} X^{*\prime}_{\omega_2} * dW^*_{\omega_2,0} \\ \tau_0^{m_{\omega_2}-1} * dW^*_{\omega_2,0} \end{pmatrix}\Upsilon(\omega_2) \otimes I_N M^{-1}_{\omega_2,\perp}.$$

The last step consists of providing a consistent estimator of $M^{-1}_{\omega_2,\perp}M^{-1\prime}_{\omega_2,\perp}$, but by construction, we have

$$\Theta_{\omega_2,\perp}\begin{pmatrix} \epsilon\epsilon' & \cos\omega_2\epsilon\epsilon' \\ \cos\omega_2\epsilon\epsilon' & \epsilon\epsilon' \end{pmatrix}\Theta'_{\omega_2,\perp} = \sin^2\omega_2 M^{-1}_{\omega_2,\perp}M^{-1\prime}_{\omega_2,\perp},$$

which because of the consistency of $\tilde{\theta}_{\omega_1}$, $\tilde{\theta}^1_{\omega_2}$, $\tilde{\theta}^2_{\omega_2}$, and $\tilde{\phi}$ can be estimated by

$$\hat{\Theta}_{\omega_2,\perp}\begin{pmatrix} \hat{\eta}\hat{\eta}' & \cos\omega_2\hat{\eta}\hat{\eta}' \\ \cos\omega_2\hat{\eta}\hat{\eta}' & \hat{\eta}\hat{\eta}' \end{pmatrix}\hat{\Theta}'_{\omega_2,\perp}.$$

Finally, joining all these results, we obtain through similar computations as those in the end of the proof of Theorems 4.1 and 4.2 the desired result. ■

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# APPENDIX B

**TABLE B1.** Quantiles of the test statistics presented in Theorem 4.4: $\omega = \pi/2$, 5,000 drawings, $T = 400$

| | | $r_{\pi/2,1}$ | | | | |
|---|---|---|---|---|---|---|
| $N - r_{\pi/2}$ | Level | 0 | 1 | 2 | 3 | 4 |
| 1 | 50 | 46.32 | 54.09 | 61.48 | 69.24 | 76.71 |
| | 80 | 57.19 | 66.06 | 75.29 | 83.55 | 91.89 |
| | 90 | 63.99 | 73.70 | 82.53 | 91.41 | 100.23 |
| | 95 | 69.60 | 80.34 | 89.23 | 97.86 | 107.09 |
| | 97.5 | 74.67 | 85.07 | 94.12 | 104.75 | 113.22 |
| | 99 | 81.60 | 91.53 | 103.33 | 111.63 | 122.21 |
| 2 | 50 | 139.93 | 154.85 | 170.00 | 185.16 | 200.02 |
| | 80 | 157.19 | 173.33 | 190.11 | 206.30 | 222.08 |
| | 90 | 166.37 | 183.83 | 201.53 | 218.35 | 234.98 |
| | 95 | 175.56 | 192.79 | 211.33 | 229.23 | 245.78 |
| | 97.5 | 182.56 | 201.58 | 220.86 | 238.93 | 256.03 |
| | 99 | 191.44 | 214.58 | 231.56 | 250.26 | 269.48 |
| 3 | 50 | 278.09 | 301.13 | 323.34 | 345.51 | 367.33 |
| | 80 | 302.55 | 326.97 | 351.04 | 374.97 | 398.08 |
| | 90 | 315.35 | 341.08 | 366.09 | 390.21 | 413.95 |
| | 95 | 326.82 | 352.75 | 378.20 | 403.36 | 428.78 |
| | 97.5 | 336.44 | 364.21 | 389.46 | 416.89 | 439.24 |
| | 99 | 348.97 | 375.91 | 403.36 | 430.13 | 456.80 |
| 4 | 50 | 461.15 | 490.49 | 519.77 | 549.24 | 578.54 |
| | 80 | 491.80 | 523.77 | 555.60 | 584.98 | 615.43 |
| | 90 | 509.94 | 541.94 | 572.53 | 604.52 | 635.47 |
| | 95 | 523.89 | 556.51 | 587.76 | 620.26 | 652.43 |
| | 97.5 | 535.40 | 568.43 | 600.90 | 632.37 | 664.83 |
| | 99 | 548.21 | 582.84 | 615.01 | 647.32 | 679.85 |
| 5 | 50 | 687.44 | 723.59 | 760.04 | 795.94 | 831.66 |
| | 80 | 722.65 | 760.85 | 799.06 | 837.40 | 874.90 |
| | 90 | 744.06 | 782.21 | 820.59 | 860.07 | 899.40 |
| | 95 | 759.56 | 799.83 | 838.95 | 879.28 | 919.35 |
| | 97.5 | 776.50 | 817.66 | 855.53 | 896.68 | 934.49 |
| | 99 | 795.47 | 837.01 | 877.25 | 914.97 | 953.15 |

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**TABLE B2.** Quantiles of the test statistics presented in Theorem 5.3: $\omega = \pi/2$, 5,000 drawings, $T = 400$

| | | $r_{\pi/2,1}$ | | | | |
|---|---|---|---|---|---|---|
| $N - r_{\pi/2}$ | Level | 0 | 1 | 2 | 3 | 4 |
| 1 | 50 | 63.49 | 70.81 | 78.42 | 85.93 | 93.51 |
| | 80 | 75.33 | 83.93 | 92.44 | 100.97 | 109.23 |
| | 90 | 82.23 | 91.63 | 100.25 | 109.38 | 118.71 |
| | 95 | 87.69 | 97.79 | 107.17 | 117.34 | 126.70 |
| | 97.5 | 93.90 | 103.36 | 113.54 | 124.00 | 132.72 |
| | 99 | 100.89 | 110.63 | 121.87 | 131.91 | 141.11 |
| 2 | 50 | 172.40 | 187.36 | 202.42 | 217.47 | 231.86 |
| | 80 | 190.84 | 207.41 | 223.19 | 239.60 | 255.23 |
| | 90 | 201.01 | 218.33 | 234.85 | 251.71 | 268.61 |
| | 95 | 210.34 | 227.63 | 245.97 | 262.87 | 279.68 |
| | 97.5 | 218.09 | 235.24 | 255.35 | 272.19 | 290.40 |
| | 99 | 229.02 | 246.68 | 266.49 | 282.34 | 301.81 |
| 3 | 50 | 325.75 | 347.77 | 370.27 | 392.36 | 413.84 |
| | 80 | 350.24 | 373.95 | 398.41 | 421.37 | 444.19 |
| | 90 | 364.32 | 389.19 | 414.77 | 437.95 | 460.95 |
| | 95 | 376.47 | 401.87 | 427.76 | 450.76 | 474.90 |
| | 97.5 | 387.12 | 414.61 | 439.89 | 463.19 | 487.58 |
| | 99 | 399.16 | 426.97 | 456.35 | 481.55 | 506.48 |
| 4 | 50 | 522.17 | 551.72 | 580.74 | 610.34 | 638.50 |
| | 80 | 554.03 | 585.18 | 614.95 | 644.65 | 675.34 |
| | 90 | 570.60 | 602.76 | 634.58 | 664.44 | 694.27 |
| | 95 | 585.88 | 618.26 | 649.01 | 679.32 | 710.05 |
| | 97.5 | 598.38 | 631.26 | 663.89 | 695.54 | 725.92 |
| | 99 | 614.91 | 649.81 | 679.54 | 713.78 | 744.56 |
| 5 | 50 | 761.17 | 797.81 | 832.73 | 868.55 | 904.42 |
| | 80 | 798.69 | 835.33 | 873.74 | 911.60 | 948.66 |
| | 90 | 819.39 | 857.23 | 895.40 | 933.10 | 971.10 |
| | 95 | 837.85 | 876.14 | 913.82 | 951.42 | 991.28 |
| | 97.5 | 851.69 | 892.93 | 931.14 | 971.90 | 1,009.98 |
| | 99 | 872.19 | 914.97 | 955.68 | 995.39 | 1,031.25 |

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