Visually suggestive but mathematically incorrect

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Mathematical calculation at times may produce erroneous results because of ignoring particular features of the functions involved, such as restricted domains, ranges and discontinuities. In this Article, to highlight the significance of conceptual understanding, we choose six examples and for each we give a flawed solution followed by a correct solution.

Example 1

If x is a real number such that $2x^{3x} + 3x^{2x} = 1$, find the principal value of $\sin^{-1}(x^x)$?

Solution

If we substitue $y = x^x$, hence the equation becomes $2y^3 + 3y^2 - 1 = 0$. Now consider $f(y) = 2y^3 + 3y^2 - 1$. It can be seen that $f(-1) = 0$, hence $(y + 1)$ is a factor of the expression.

$$\begin{aligned} f(y) &= 2y^3 + 3y^2 - 1 \\ &= 2y^2(y + 1) + y(y + 1) - (y + 1) \\ &= (y + 1)(2y^2 + y - 1) \\ &= (y + 1)(y + 1)(2y - 1). \end{aligned}$$

Hence the above equation has three solutions, $y = -1, -1$ and $\frac{1}{2}$.

Since $y = x^x$, y cannot be negative, thus $y = -1$ is ruled out. Thus the only admissible solution is $x^x = \frac{1}{2}$, which on substitution in the expression $\sin^{-1}(x^x)$ gives $\sin^{-1}\frac{1}{2} = \dfrac{\pi}{6}$. The principal value of $\sin^{-1}(x^x)$ is $\dfrac{\pi}{6}$.

The above solution contains a flaw. A correct solution now follows.

To arrive at the correct solution, one must consider the domain and the range of the function x^x. A cursory look may not always reveal the whole picture. Using differential calculus, it can be shown that the function x^x attains its minimum value at $x = \dfrac{1}{e}$, hence the minimum value of the function in its domain is $\left(\dfrac{1}{e}\right)^{1/e}$, which is about 0.69 (see Figure 1). The proof is left to the reader. So, for any real x, $x^x \neq \frac{1}{2}$ (since, $\frac{1}{2} < 0.69$). Hence there is no real x for which the given equation is satisfied.

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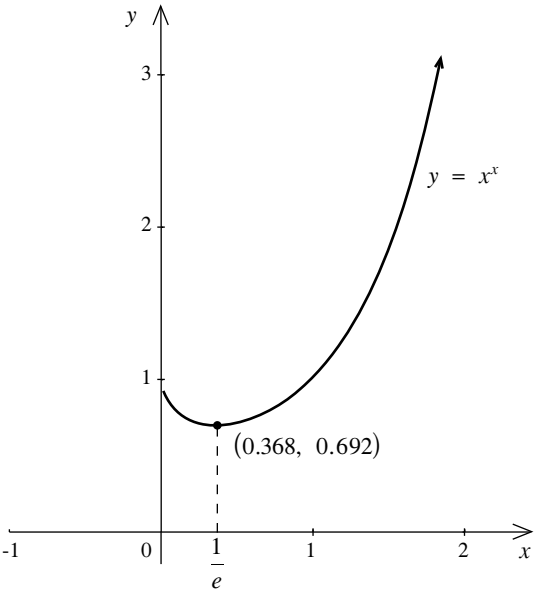


FIGURE 1

Note: There are several ways to show that $\left(\frac{1}{e}\right)^{1/e} > \frac{1}{2}$; one of the approaches is $\left(\frac{1}{e}\right)^{1/e} = e^{-1/e} > 1 + \frac{-\frac{1}{e}}{1!} = \left(1 - \frac{1}{e}\right)$ and $2 < e < 3$, hence $-\frac{1}{e} > -\frac{1}{2}$ and thus $\left(1 - \frac{1}{e}\right) > \frac{1}{2}$. Hence $\left(\frac{1}{e}\right)^{1/e} > \frac{1}{2}$.

Example 2

Evaluate the integral $\int_0^\infty \sin x \; dx$.

The function under the integral sign is an odd function and continuous everywhere in the domain of integration. Now, from 0 to infinity, the domain can be subdivided into $[0, \pi]$, $[\pi, 2\pi]$, $[2\pi, 3\pi]$, etc. It is easy to see that for every segment of positive area there exists an equal segment of negative area, as the Figure 2 indicates. The series of area segments that are alternately arranged above and below the x-axis can make students believe that all these area segments (equal in magnitude but opposite sign) get cancelled if added together and the result will be zero.

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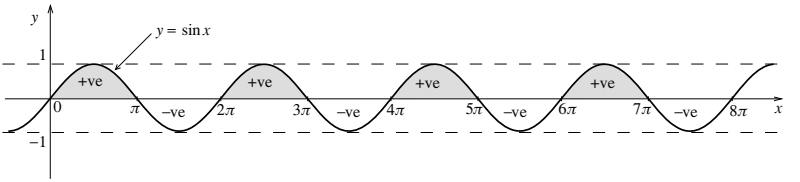


FIGURE 2

Such a solution is incorrect. A correct solution follows.

Let us check the convergence of the integral $\int_0^\infty \sin x \; dx$. It is convergent if $\lim_{t \to \infty} \int_0^t \sin x \; dx$ exists and is finite.

$$\lim_{t\to\infty}\int_0^t \sin x \; dx = \lim_{t\to\infty}\Big[-\cos x\Big]_0^t = \lim_{t\to\infty}\big[\cos 0 - \cos t\big] = \lim_{t\to\infty}\big[1-\cos t\big].$$

Clearly the above limit does not exist since $\cos t$ oscillates between -1 and 1; hence the integral $\int_0^\infty \sin x \; dx$ diverges.

Geometrical meaning

The integral $\int_0^\infty \sin x \; dx$ i.e. the area bounded by the curve $y = \sin x$ (from zero to infinity) oscillates between 0 and 2. Since $-1 \leqslant \cos t \leqslant 1$ for all real t it follows that $0 \leqslant \big[1 - \cos t\big] \leqslant 2$.

Note: I would like to take this opportunity to discuss a few other interesting integrals such as $\int_0^\infty \sin\left(x^2\right) dx$, $\int_0^\infty \sin\left(x^4\right) dx$, $\int_0^\infty \sin\left(x^6\right) dx$, i.e. integrals that are in the form of $\int_0^\infty \sin\left(x^{2n}\right) dx$ where $n = 1, 2, 3, \ldots$. Let us start with $\int_0^\infty \sin\left(x^2\right) dx$. Visually, it may appear that this integral also does not converge as the earlier integral of similar type did, but a conclusion based on mere observation could lead to an error.

Actually $\int_0^\infty \sin\left(x^2\right) dx$ exists and has a finite value. This important integral is called the 'Fresnel Integral' and has widespread applications in Physics. In fact,

$$\int_0^\infty \sin\left(x^2\right) dx = \sqrt{\frac{\pi}{8}} \approx 0.6267.$$

Similarly integrals such as $\int_0^\infty \sin\left(x^4\right) dx$, $\int_0^\infty \sin\left(x^6\right) dx$ converge and have finite value. (These are left for the reader to explore.)

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To get a proper insight, refer to the graph below where it is evident that the integral $\int_0^\infty \sin(x^2)\,dx$ converges because of the fact that as $x \to \infty$, owing to such high frequency of oscillations, the individual segments of positive and negative area tend to zero very quickly thereby contributing almost nothing to the total area. This was not true in case of the integral $\int_0^\infty \sin x\ dx$. Hence, while $\int_0^\infty \sin(x^2)\,dx$ converges, $\int_0^\infty \sin x\ dx$ diverges.

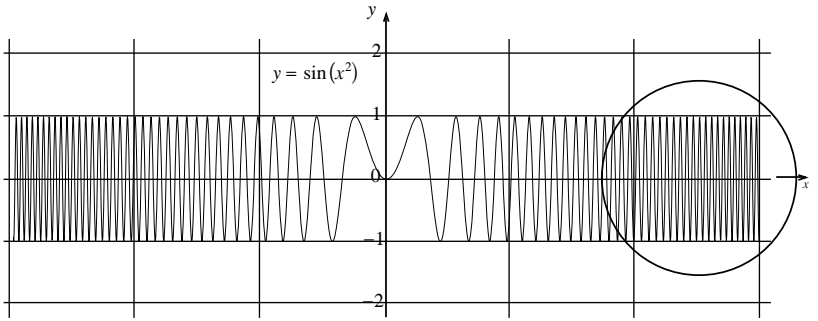


FIGURE 3: Within the circle and beyond the oscillations have increasing frequency. This is also true in the negative direction.

It is now left to the reader to explore whether integrals $\int_0^\infty \sin\left(\frac{1}{x}\right)\,dx$ and $\int_0^\infty \sin\left(\frac{1}{x^2}\right)\,dx$ converge.

Example 3

Evaluate the integral $\int_0^{3\pi/4} \sec^2 x\,dx$.

Solution

Many people will proceed as follows since the definite integral appears to be quite a simple one.

$$\int_0^{3\pi/4} \sec^2 x\,dx \;=\; \left[\tan x\right]_0^{3\pi/4} \;=\; \tan\frac{3\pi}{4} \;-\; \tan 0 \;=\; -1.$$

This approach has a flaw and a correct solution now follows.

Let us have a look at the graph of the function $f(x) = \sec^2 x$ (see Figure 4). One needs to note three important things.

(a) The answer obtained above is a negative quantity. However, since the function $\sec^2 x$ is non-negative and real, the area bounded by the curve that is lying completely above x-axis can never be negative.

(b) The integrand, i.e. $\sec^2 x$, is not defined at $x = \frac{1}{2}\pi$. (Note that $x = \frac{1}{2}\pi$ lies within the limits of integration.)

(c) The anti-derivative of $\sec^2 x$, i.e. $\tan x$, is not defined at $x = \frac{1}{2}\pi$.

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Hence the relationship $\frac{d}{dx}(\tan x) = \sec^2 x$ fails to hold at $x = \frac{1}{2}\pi$. That is why the Newton-Leibniz formula (i.e. the fundamental theorem of the calculus) is not applicable.

So this is an improper integral. Let us divide the interval of integration at the point of discontinuity $x = \frac{1}{2}\pi$.

$$\int_0^{3\pi/4} \sec^2 x\,dx = \int_0^{\pi/2} \sec^2 x\,dx + \int_{\pi/2}^{3\pi/4} \sec^2 x\,dx$$

$$= \lim_{t \to \frac{1}{2}\pi^-} \int_0^t \sec^2 x\,dx + \lim_{t \to \frac{1}{2}\pi^+} \int_t^{3\pi/4} \sec^2 x\,dx$$

For the integral to be convergent we will need both integrals to be convergent independently. If one or both are divergent then the whole integral will also be divergent.

Let us check the first integral.

$$\lim_{t \to \frac{1}{2}\pi^-} \int_0^t \sec^2 x\,dx = \lim_{t \to \frac{1}{2}\pi^-} \left[\tan x\right]_0^t = \lim_{t \to \frac{1}{2}\pi^-} \tan t.$$

The above integral diverges to infinity. We conclude that $\int_0^{3\pi/4} \sec^2 x\,dx$ is also divergent.

Geometrical meaning

From graph of $y = \sec^2 x$, it is easy to observe that area bounded by the branches of the curve between the limits $x = 0$ to $x = \frac{3}{4}\pi$ is not finite since the integrand itself blows up as x tends to $\frac{1}{2}\pi$.

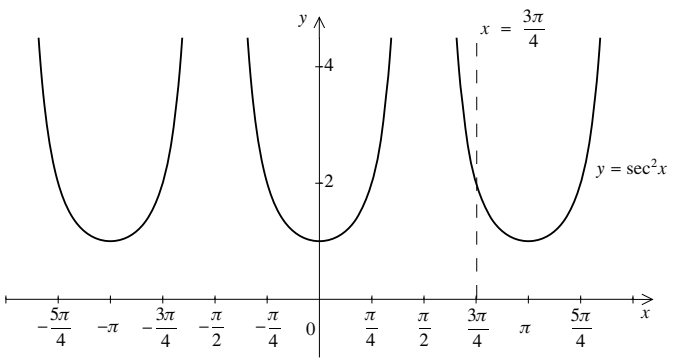


FIGURE 4

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Example 4

Which of these equations

$$\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right)=3.5 \qquad \text{and} \qquad \left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right)=4.5$$

possess real solutions where $0 < x < \frac{1}{2}\pi$?

Solution

We know that, $0 < \sin x < 1$ and $0 < \cos x < 1$ $\left(0 < x < \frac{1}{2}\pi\right)$. Hence, since $\frac{1}{\sin x} > 1$ and $\frac{1}{\cos x} > 1$, this implies the following:

$$\left(1+\frac{1}{\sin x}\right) > 2 \qquad \text{and} \qquad \left(1+\frac{1}{\cos x}\right) > 2.$$

Multiplying these we get that $\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right) > 4$. Hence it is clear that $\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right) = 3.5$ does not have a real solution but there is a possibility that the equation $\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right) = 4.5$ might have real solutions since $4.5 > 4$. This cannot be ascertained without solving the equation. One may follow an alternative approach as mentioned below to determine if either of the equations has any real solution.

$$\begin{aligned}\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right) &= 1+\frac{1}{\sin x}+\frac{1}{\cos x}+\frac{1}{\sin x\cos x}\\ &= 1+\frac{1}{\sin x}+\frac{1}{\cos x}+\frac{2}{\sin 2x}\\ &> 1+1+1+2.\end{aligned}$$

Hence, $\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right) > 5$.

It can be concluded that neither $\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right)=3.5$ nor $\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right)=4.5$ has a real solution.

Example 5

Find the limit $\lim\limits_{x\to 1}\left[\cos\left(\cos^{-1}x\right)\right]$, where $[\ ]$ denotes the greatest integer function.

Solution

We know that $\cos\left(\cos^{-1}x\right) = x$, hence using the above the given limit can be simplified as

$$\lim_{x\to 1}\left[\cos\left(\cos^{-1}x\right)\right] = \lim_{x\to 1}\left[\,x\,\right].$$

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Let's calculate the left-hand limit and the right-hand limit

$$\lim_{x \to 1^+} [\, x \,] = 1 \qquad \text{but} \qquad \lim_{x \to 1^-} [\, x \,] = 0.$$

So the left-hand limit and the right-hand limit are not equal, hence the limit does not exist.

However, the above solution is conceptually flawed. A correct solution is given below.

$$\cos\left(\cos^{-1} x\right) = x \text{ only if } -1 \leqslant x \leqslant 1$$

It is easy to see that the function $\cos\left(\cos^{-1} x\right)$ is not defined for $x > 1$ thus it is not necessary to evaluate the right-hand limit since the function itself is not defined as x approaches 1 from right-hand side i.e. when $x \to 1^+$. Hence, only the left-hand limit can be evaluated, the limit exists and equals zero.

Example 6

Solve for real x: $(\log_a |x|)^{\sin^{-1} x} = 1$, where a is a positive real number.

Solution

$$(\log_a |x|)^{\sin^{-1} x} = 1$$

By general understanding, 1 raised to any power always equals 1 or any number raised to the power zero equals 1. Using these rules, the given equation is satisfied if any of the two following conditions are satisfied:

Case 1: $\log_a |x| = 1$

Case 2: $\sin^{-1} x = 0$ and $\log_a |x| > 0$

Case 1

$\log_a |x| = 1$ implies that $|x| = a$ yields $x = \pm a$.

Case 2

$\sin^{-1} x = 0$ implies that $x = 0$ but again x cannot be equal to zero since $\log_a |x|$ is not defined at $x = 0$. Hence, this case fails to produce any solution.

So the solution is $x = \pm a$ (where a is a positive real number).

The reasoning provided above contains a flaw. The correct solution is provided below.

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Two important things to note

(a) Students must understand that statements such as unity (i.e. 1) raised to any real power always equals to 1 or any real number raised to 0 equal 1 are valid unless functions (dependent variables) are involved in the expression. If functions are involved, due attention must be paid to the domain and range of those functions.

(b) Students often tend to ignore the presence of another variable 'a' in the equation ($a > 0$). There is general sense of belief that 'a' is just a constant.

The equation in question involves two functions i.e. $\log_a|x|$ and $\sin^{-1}x$, hence any solution of the equation must also ensure that these functions exist.

Case 1

$\log_a|x| = 1$ implies that $x = \pm a$ but again if $a > 1$ then $x > 1$ or $x < -1$ which is not possible because, $\sin^{-1}x$ is not defined if $x > 1$ or $x < -1$. Hence $x = \pm a$ is a valid solution if, and only if, $0 < a < 1$. Also note that $a \neq 1$ because $\log_a|x|$ is not defined for $a = 1$.

For all other values of 'a', there is no solution. This emphasises the need to pay attention to each component in a mathematical expression.

Case 2

As already indicated earlier this case does not yield any solution. Hence the answer $x = \pm a$ where $0 < a < 1$ is complete.

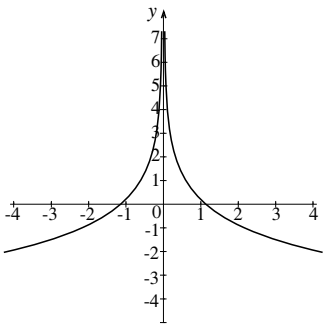


FIGURE 5: Graph of $y = \log_a|x|$ for $a = \frac{1}{2}$

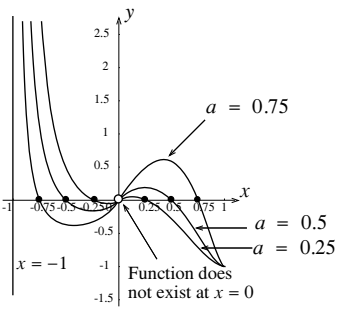


FIGURE 6: Graphs of $y = (\log_a|x|)^{\sin^{-1}x} - 1$ for $a = 0.25,\ 0.5,\ 0.75$

Notes

1. For $0 < a < 1$, $\log_a|x| > 0$ implies that $|x| < 1$ (see Figure 5).
2. It can be seen from the graph that the function does not converge (i.e. not defined) when $|x| > 1$.
3. Figure 6 shows the solution of our equation for three values of a.

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