Ergod. Th. & Dynam. Sys., (2021), **41**, 3474–3520 © The Author(s) 2020. Published by Cambridge University Press.
doi:10.1017/etds.2020.101

Topological flows for hyperbolic groups

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(*Received* 1 *April* 2020 *and accepted in revised form* 28 *July* 2020)

Abstract. We show that for every non-elementary hyperbolic group the Bowen–Margulis current associated with a strongly hyperbolic metric forms a unique group-invariant Radon measure class of maximal Hausdorff dimension on the boundary square. Applications include a characterization of roughly similar hyperbolic metrics via mean distortion.

Key words: hyperbolic group, geodesic current, Patterson–Sullivan measure, Cannon's automatic structure, Hausdorff dimension
2020 Mathematics subject classification: 20F67 (Primary); 37D35, 37D40 (Secondary)

1. *Introduction*

Let Γ be a non-elementary hyperbolic group. It is known that the ideal boundary $\partial\Gamma$ admits plenty of finite measures, which give rise to Γ-invariant measures on the boundary square $\partial^2\Gamma := (\partial\Gamma)^2\backslash\{\text{diagonal}\}$, where we consider the diagonal action of Γ on $\partial^2\Gamma$. The Γ-invariant measures on $\partial^2\Gamma$ arise in the study of geodesic flows on negatively curved manifolds since there exists a natural correspondence between geodesic flow-invariant finite measures and Γ-invariant Radon (i.e., locally finite and Borel regular) measures on the boundary square of the universal coverings. For example, one way to obtain an invariant measure of maximal entropy for the geodesic flow on a compact negatively curved manifold is to construct a Γ-invariant Radon measure in the measure class consisting of the product of Patterson–Sullivan measures on $\partial^2\Gamma$. This construction has been generalized beyond the manifold setting such as CAT(-1)-spaces **[Bou93, Bou95]**. For general hyperbolic groups, the study of geodesic flows goes back to the original paper by Gromov **[Gro87,** §8.3**]**. Furman **[Fur02]** has given a framework for a general hyperbolic group Γ to analyze Γ-invariant measures on $\partial^2\Gamma$ without assuming any actions on another reasonable geometric space (see also **[BF17]**). It would be desirable to define the entropy of a Γ-invariant measure on $\partial^2\Gamma$ for a general hyperbolic group such that it gives the measure theoretical entropy for a geodesic flow when the geodesic flow is defined. Instead we discuss the Hausdorff dimension; the entropy coincides with the Hausdorff dimension under an appropriate normalization in the case of negatively curved manifolds as it was

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studied and the approach has been suggested for hyperbolic groups by Kaimanovich **[Kai90,** §3.5**]**. The purpose of this paper is to characterize the Γ-invariant Radon measure class where the maximal entropy is replaced by the maximal Hausdorff dimension.

For any non-elementary hyperbolic group Γ (not necessarily torsion-free), we consider a strongly hyperbolic metric $\widehat{d}$, which is left-invariant and quasi-isometric to a word metric on Γ. For example, one may take the hat metric constructed by Mineyev **[Min05]** and any hyperbolic group admits such a metric. Let us consider the Bowen–Margulis current for $\widehat{d}$,

$$\widehat{\Lambda} := \exp(2\widehat{v}\,\langle \xi|\eta\rangle_o)\,\widehat{\mu}_o \otimes \widehat{\mu}_o \quad \text{on } \partial^2\Gamma,$$

where $\langle \xi|\eta\rangle_o$ is the Gromov product based at o and $\widehat{v}$ is the exponential volume growth rate of $(\Gamma, \widehat{d}\,)$ and $\widehat{\mu}_o$ is the corresponding Patterson–Sullivan measure on $\partial\Gamma$. Letting the gauge $\widehat{\rho}(\xi, \eta) := \exp(-\langle \xi|\eta\rangle_o)$ and $\widehat{\rho}_\times$ be the maximum of $\widehat{\rho}$ on each component on $\partial^2\Gamma$, we define the (lower) Hausdorff dimension of a Borel measure Λ on $\partial^2\Gamma$ by

$$\underline{\dim}_H(\Lambda, \widehat{\rho}_\times) := \inf\{\dim_H(A, \widehat{\rho}_\times) : \Lambda(A) > 0\}.$$

It is known that $\dim_H(\partial^2\Gamma, \widehat{\rho}_\times) = 2\widehat{v}$. We show that the Bowen–Margulis current $\widehat{\Lambda}$ is the unique Γ-invariant Radon measure on $\partial^2\Gamma$ of maximal Hausdorff dimension up to a multiplicative constant.

THEOREM 1.1. *Let Γ be a non-elementary hyperbolic group and $\widehat{d}$ be a left-invariant strongly hyperbolic metric quasi-isometric to a word metric. For any Γ-invariant Radon measure Λ on $\partial^2\Gamma$, if $\underline{\dim}_H(\Lambda, \widehat{\rho}_\times) = \dim_H(\partial^2\Gamma, \widehat{\rho}_\times)$, then Λ is a constant multiple of the Bowen–Margulis current $\widehat{\Lambda}$.*

The result allows us to compare any Γ-invariant Radon measure Λ with the Bowen–Margulis current on $\partial^2\Gamma$. Our proof is based on a topological model of geodesic flow on a general hyperbolic space introduced by Mineyev **[Min05]**. Note that one is able to compare two Bowen–Margulis currents arising from two hyperbolic metrics on Γ in a measure theoretical model introduced by Furman **[Fur02]** and by the double ergodicity result of Kaimanovich **[Kai94,** Theorem 2.8**]**, Bader and Furman **[BF17]** (see also **[Gar16]**) and Coulon *et al*. **[CDST18,** §4**]**. However, since our aim is to deal with any Γ-invariant Radon measure Λ and the question involves the Hausdorff dimension, we require a topological setting independent of the choice of measures.

In the case of free groups with actions on metric trees, Kapovich and Nagnibeda have introduced the *geometric entropy* of a current **[KN10,** Definition 3.1**]**. Analogously, we define

$$h(\Lambda, \widehat{d}\,) := \lim_{R\to\infty} \liminf_{\widehat{d}(x,y)\to\infty,\ x,y\in\Gamma} \left(-\frac{\log \Lambda(O_o(x, R) \times O_o(y, R))}{\widehat{d}(x, y)} \right),$$

where $O_o(x, R)$ is the shadow based at o of the ball centered at x with radius R (§2.1.4) and the limit exists as $R \to \infty$ since it is non-decreasing. The lower Hausdorff dimension of Λ is given by for a large enough $R > 0$,

$$\underline{\dim}_H(\Lambda, \widehat{\rho}_\times) = \inf_{\Lambda\text{-almost every } (\xi,\,\eta)} \liminf_{n\to\infty} \left(-\frac{\log \Lambda(O_o(\phi(-n), R) \times O_o(\phi(n), R))}{n} \right),$$

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where ϕ is a rough geodesic with extreme points (ξ, η) normalized so that $\phi(0)$ realizes a minimum distance to the base point o and $\inf_{\Lambda\text{-almost every } (\xi,\,\eta)}$ stands for the essential infimum relative to Λ (§2.2). Then we have

$$2h(\Lambda, \widehat{d}\,) \le \underline{\dim}_H(\Lambda, \widehat{\rho}_\times).$$

In the case of (simplicial) trees, their result reads if $h(\Lambda, \widehat{d}\,) = \widehat{v}$, then Λ is a constant multiple of the Bowen–Margulis current (in our terminology) **[KN10,** Theorems C and E]. Since geodesic metrics in trees are strongly hyperbolic (§2.4), Theorem 1.1 reproduces their result. We note however that one is not able to replace $\underline{\dim}_H(\Lambda, \widehat{d}\,)$ by the upper Hausdorff dimension $\overline{\dim}_H(\Lambda, \widehat{d}\,)$ (§2.2) to obtain Theorem 1.1 unless Λ is ergodic with respect to the Γ-action, i.e., any Γ-invariant Borel set A in $\partial^2\Gamma$, one has $\Lambda(A) = 0$ or $\Lambda(\partial^2\Gamma \backslash A) = 0$.

1.1. *A connection to mean distortion.* We have the corresponding result for word metrics (Theorem 7.2). Let us illustrate an application to comparison between two word metrics. For any two finite symmetric sets of generators S, $S^\star$, we consider the word norms $|\cdot|_S$ and $|\cdot|_{S^\star}$ for S and $S^\star$, respectively. For each integer $n \ge 0$, let

$$\mathbf{S}_n := \{x \in \Gamma : |x|_S = n\}.$$

We sample an element x_n according to the uniform distribution $\mathrm{Unif}_{S,n}$ on $\mathbf{S}_n$ for each n, and consider the (liminf-) linear growth rate of the average length with respect to $|\cdot|_{S^\star}$,

$$\tau(S^\star/S) := \liminf_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathrm{Unif}_{S,n}}|x_n|_{S^\star} \quad \text{where } \mathbb{E}_{\mathrm{Unif}_{S,n}}|x_n|_{S^\star} := \frac{1}{|\,\mathbf{S}_n|}\sum_{x\in\,\mathbf{S}_n}|x|_{S^\star}.$$

Let us call $\tau(S^\star/S)$ the *mean distortion of* $|\cdot|_{S^\star}$ *relative to* $|\cdot|_S$. Since two word metrics are bi-Lipschitz, the mean distortion is bounded from above and from below by the bi-Lipschitz constants. The quantity $\tau(S^\star/S)$ represents a typical distortion rate between two word norms. If we denote by $\mathrm{gr}(S)$ and $\mathrm{gr}(S^\star)$ the exponential volume growth rates for word metrics relative to S and $S^\star$, respectively, then a simple counting argument yields $\tau(S^\star/S) \ge \mathrm{gr}(S)/\mathrm{gr}(S^\star)$. We show that the equality holds if and only if two word metrics are roughly similar.

THEOREM 1.2. *Let Γ be a non-elementary hyperbolic group. For any pair of finite symmetric sets of generators S and $S^\star$ in Γ, we have that*

$$\tau(S^\star/S) = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\mathrm{Unif}_{S,n}}|x_n|_{S^\star},$$

and for any $\varepsilon > 0$,

$$\lim_{n\to\infty}\mathrm{Unif}_{S,n}\left\{x \in \mathbf{S}_n : \frac{|d_{S^\star}(o,x) - n\tau(S^\star/S)|}{n} > \varepsilon\right\} = 0.$$

Moreover, it holds that

$$\tau(S^\star/S) \ge \frac{\mathrm{gr}(S)}{\mathrm{gr}(S^\star)},$$

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and the equality $\tau(S^\star/S) = \mathrm{gr}(S)/\mathrm{gr}(S^\star)$ *holds if and only if word metrics* $d_{S^\star}$ *and* d_S *are roughly similar, i.e., there exist constants* $\tau > 0$ *and* $D \geq 0$ *such that*

$$|d_{S^\star}(x, y) - \tau d_S(x, y)| \leq D \quad \textit{for all } x, y \in \Gamma.$$

Theorem 1.2 is a consequence of Theorems 7.4 and 7.7 (see also Remark 7.5). We have stated Theorem 1.2 only for word metrics; but this is mainly for the sake of simplicity on the statement—one may discuss more general hyperbolic metrics. The proof is based on showing that the Hausdorff dimension computed by the gauge associated with $d_{S^\star}$ of the Bowen–Margulis current for d_S coincides with $\mathrm{gr}(S)/\tau(S^\star/S)$. The inequality $\tau(S^\star/S) \geq \mathrm{gr}(S)/\mathrm{gr}(S^\star)$ has been obtained by Calegari and Fujiwara **[CF10,** Remark 4.28**]** (and this remark has motivated our result). Moreover, $\tau(S^*/S)$ is an algebraic number **[CF10,** Corollary 4.27**]**. In fact, they have shown a central limit theorem, which implies that $|x_n|_{S^\star} = \tau(S^\star/S)n + O(\sqrt{n})$ with high probability for uniformly chosen x_n in the sphere $\mathbf{S}_n$ as $n \to \infty$ (see also **[Cal13,** Corollary 3.6.4**]**). Our proof indicates that the $O(\sqrt{n})$-fluctuation can not be negligible as soon as two word metrics are not roughly similar. In the case of free groups and word metrics associated with free bases, the characterization of rough similarity has been given by **[KKS07,** Theorem F**]** (where the mean distortion is called the generic stretching factor). We point out that Theorem 1.2 is regarded as a discrete counterpart of a result for compact negatively curved manifolds by Knieper **[Kni95,** Theorem 1.2**]** (where $\tau(S^\star/S)$ corresponds to the geodesic stretch).

1.2. *Outline of the proof.* Let us give an outline of the proof of Theorem 1.1. We consider the flow $\{\widetilde{\Phi}_t\}_{t\in\mathbb{R}}$ on the space $\partial^2\Gamma \times \mathbb{R}$ defined by the translation in the $\mathbb{R}$-coordinate. The space $\partial^2\Gamma \times \mathbb{R}$ admits a Γ-action, which is constructed as in the following. For a strongly hyperbolic metric $\widehat{d}$ on Γ, the corresponding Gromov product and the Busemann functions $\widehat{b}_o(x, \xi)$ extend continuously on the compactification $\Gamma \cup \partial\Gamma$. By using the cocycle

$$\alpha(x, \xi, \eta) := \tfrac{1}{2}(\widehat{b}_o(x^{-1}, \xi) - \widehat{b}_o(x^{-1}, \eta)) \quad \text{for } x \in \Gamma,\ (\xi, \eta) \in \partial^2\Gamma,$$

we define the Γ-action on $\partial^2\Gamma \times \mathbb{R}$ by $x \cdot (\xi, \eta, t) := (x\xi, x\eta, t - \alpha(x, \xi, \eta))$. Let us call this action the (Γ, α)-action. The (Γ, α)-action is properly discontinuous and cocompact (Lemma 3.2). Then we define $\mathcal{F}_\alpha := \Gamma\backslash(\partial^2\Gamma \times \mathbb{R})$ and call it the *topological flow space*. Since the flow $\{\widetilde{\Phi}_t\}_{t\in\mathbb{R}}$ commutes with the (Γ, α)-action, it descends to the flow $\{\Phi_t\}_{t\in\mathbb{R}}$ on $\mathcal{F}_\alpha$. It amounts to considering the geodesic flow on the total space of a unit tangent bundle in the case of manifolds.

This flow $\{\Phi_t\}_{t\in\mathbb{R}}$ on the topological flow space $\mathcal{F}_\alpha$ has potentially a lot of similarities to Axiom A flows. However, a direct connection to the known machinery seems lacking (e.g., **[BCS17,** §6.3**]**). Moreover, the (Γ, α)-action on $\partial^2\Gamma \times \mathbb{R}$ is not necessarily free, and $\mathcal{F}_\alpha$ is far from being a manifold, it is not clear that one could resemble the techniques developed in the manifold setting; in particular, a serious issue arises when one tries to obtain a lower bound of the topological entropy of the flow on $\mathcal{F}_\alpha$. Instead, we follow a classical approach in the Axiom A flows by Bowen and Ruelle **[BR75]**. We construct a symbolic coding of the topological flow space $\mathcal{F}_\alpha$ by using Cannon's automatic structure of hyperbolic group Γ. This allows us to work with a two-sided subshift of finite type (Σ, σ)

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based on the underlying graph structure. Then the suspension flow $\mathrm{Sus}(\Sigma, r)$ with a natural roof function associated with the cocycle α carries a coding map $w_* : \mathrm{Sus}(\Sigma, r) \to \mathcal{F}_\alpha$, which is bounded-to-one and equivariant with flows. The suspension flow $\mathrm{Sus}(\Sigma, r)$ is now the place to work on, but the problem is that it is not clear as to whether the coding map is one-to-one over a residual set, and the two-sided shift space (Σ, σ) is not necessarily topologically transitive. In the CAT(-1)-setting, a coding as satisfactory as in hyperbolic basic sets has been constructed in **[CLT20]**, but we do not know as to whether a similar construction is possible for a general hyperbolic group. An advantage, however, to use Cannon's automatic structure for coding is that it respects the geometry of a Cayley graph. It is actually strong enough to employ thermodynamic formalism on (Σ, σ). The Bowen–Margulis current for a strongly hyperbolic metric is encoded on the shift space by using a Hölder continuous potential associated with the Busemann cocycle. It requires a careful treatment to understand the support of this encoded measure since (Σ, σ) is not transitive and we use a spectral decomposition into transitive components. We show a key proposition, which makes it possible to encode all Γ-invariant Radon measures on $\partial^2\Gamma$ as a shift-invariant probability measure on Σ. Then the problem is basically reduced to the uniqueness of measure of maximal entropy on a transitive subshift of finite type.

Let us mention a direction, which we have not pursued in this paper; given all this framework, it would be interesting to extend results to the product of Patterson–Sullivan measures associated with a Hölder continuous cocycle (e.g., **[Led95]**) for a general hyperbolic group setting.

1.3. *Organization of the paper.* In §2, we review basics on hyperbolic metrics and Patterson–Sullivan measures. In particular, we discuss a strongly hyperbolic metric and an associated cocycle we use. We also give basic facts on the Hausdorff dimension of sets and measures on the boundary of a hyperbolic group. In §3, we construct a topological flow space $\mathcal{F}_\alpha$ and show that the Bowen–Margulis current yields an ergodic flow-invariant measure on $\mathcal{F}_\alpha$ (Theorem 3.6). This implies the double ergodicity of the Patterson–Sullivan measures (Corollary 3.7). We follow **[BF17, Gar16,** Appendix A**]** (see also Coulon *et al*. **[CDST18,** §4**]**) for the proofs in our setting. In §4, we construct a two-sided subshift of finite type based on an automatic structure of the hyperbolic group. We show that there exists a natural coding map from the suspension flow $\mathrm{Sus}(\Sigma, r)$ to $\mathcal{F}_\alpha$ (Proposition 4.4) and a key proposition, which states that all Γ-invariant Radon measures on $\partial^2\Gamma$ arise from shift-invariant probability measures on Σ (Proposition 4.7). In §5, we use thermodynamic formalism to construct a shift-invariant measure, which induces a measure dominated by the Bowen–Margulis current up to a positive multiplicative constant (Lemma 5.8). We also formulate the variational principle based on the subshift of finite type (Σ, σ). In §6, we give a local entropy-dimension estimate, which gives a direct connection between the measure theoretic entropy of an invariant probability measure on the shift space (Σ, σ) and the Hausdorff dimension of a measure on $\partial^2\Gamma$ in the case when the two measures are connected as in Proposition 4.7 (Lemma 6.3). We prove Theorem 6.5 (i.e., Theorem 1.1) for a strongly hyperbolic metric. In §7, we show the corresponding result for a word metric (Theorem 7.2). We prove results on mean distortions (Theorems 7.4 and 7.7) and deduce Theorem 1.2.

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Notation. Throughout this article, we write numerical constants $C, C', C'', \ldots$ whose exact values may change from lines to lines, and we denote by C_δ, etc. to indicate its dependance on δ for a parameter δ. For two real-valued functions $f(t)$ and $g(t)$, we write $f(t) = g(t) + O_\delta$ if and only if there exists a constant C_δ such that $|f(t) - g(t)| \leq C_\delta$ for all $t \in I$. Equivalently, we also sometimes write $f(t) = g(t) \pm C_\delta$ if we want to emphasize the dependence on constants slightly more. Although we might not state it everywhere, all measures on topological spaces are Borel.

2. *Preliminary*

2.1. *Metrics on hyperbolic groups.*

2.1.1. *Hyperbolic groups.* Let Γ be a finitely generated group. We say that a metric d on Γ is *left-invariant* if $d(\gamma x, \gamma y) = d(x, y)$ for all $\gamma, x, y \in \Gamma$, and d is δ-*hyperbolic* if there exists $\delta \geq 0$ such that for any $x, y, z, w \in \Gamma$,

$$(x \mid y)_w \geq \min\{(x|z)_w, (z \mid y)_w\} - \delta,$$

where we define the Gromov product

$$(x \mid y)_w := \frac{d(w, x) + d(w, y) - d(x, y)}{2}.$$

A metric d is called *hyperbolic* if d is δ-hyperbolic for some $\delta \geq 0$. For a finite set of generators S with $S = S^{-1}$ in Γ, the associated word metric for $x, y \in \Gamma$ is defined by

$$d_S(x, y) := |x^{-1}y|_S \quad \text{where } |x|_S := \min\{k \geq 0 : s_1 \cdots s_k = x,\ s_1, \ldots, s_k \in S\}$$

and $|\mathrm{id}|_S = 0$ where id denotes the identity element. We denote the associated Cayley graph by $\mathrm{Cay}(\Gamma, S)$ and regard d_S as the graph distance in $\mathrm{Cay}(\Gamma, S)$. The word metric d_S is left-invariant, and it is proper, i.e., every ball of finite radius consists of finitely many points. A finitely generated group Γ is called a *hyperbolic group* (or a *word hyperbolic group*) if a word metric is δ-hyperbolic for some $\delta \geq 0$. In fact, if Γ is a hyperbolic group, then any word metric is hyperbolic although the constant δ depends on the set of generators S. This follows since for every word metric any two points are joined by an isometric image of a (discrete) path, and any two word metrics are quasi-isometric.

Recall that two metrics d_1 and d_2 on Γ are *quasi-isometric* if for some constants $L > 0$, $C \geq 0$ we have that

$$(1/L)d_1(x, y) - C \leq d_2(x, y) \leq Ld_1(x, y) + C \quad \text{for all } x, y \in \Gamma.$$

Let I be an interval in $\mathbb{R}$. We say that a path $\phi : I \to (\Gamma, d)$ is an (L, C)-*quasi-geodesic* for constants L, C if

$$(1/L)|s - t| - C \leq d(\phi(s), \phi(t)) \leq L|s - t| + C \quad \text{for all } s, t \in I,$$

and $\phi : I \to (\Gamma, d)$ is a C-*rough geodesic* if

$$|s - t| - C \leq d(\phi(s), \phi(t)) \leq |s - t| + C \quad \text{for all } s, t \in I,$$

where we say that ϕ is *geodesic* if $d(\phi(s), \phi(t)) = |s - t|$ for all $s, t \in I$. Note that if d_1 and d_2 are quasi-isometric, then geodesic or C-rough geodesic paths into (Γ, d_1) are

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(L, C')-quasi-geodesics in (Γ, d_2) for some L, C'. A metric space (Γ, d) is called *geodesic* if any two points are joined by a geodesic path, and (Γ, d) is called *C-roughly geodesic* if for any two points $x, y \in \Gamma$ there exists a C-rough geodesic path $\phi : [a, b] \to (\Gamma, d)$ such that $\phi(a) = x$ and $\phi(b) = y$. We say that (Γ, d) is *roughly geodesic* if it is C-roughly geodesic for some $C \geq 0$.

Let $\mathcal{D}_\Gamma$ be the set of metrics d on Γ such that d is left-invariant, hyperbolic and quasi-isometric to some (equivalently, any) word metric on Γ. Note that every metric $d \in \mathcal{D}_\Gamma$ is proper since it is quasi-isometric to a proper metric. Although $d \in \mathcal{D}_\Gamma$ is not necessarily geodesic, it is roughly geodesic by a result of Bonk and Schramm **[BS00,** Proposition 5.6**]**. We repeatedly use the following lemma called the Morse lemma when we change metrics in $\mathcal{D}_\Gamma$.

LEMMA 2.1. *Let (Γ, d) be a proper C-roughly geodesic δ-hyperbolic space. For any (L, K)-quasi-geodesic ϕ in (Γ, d), there exists a C-rough geodesic ϕ' such that ϕ and ϕ' are within Hausdorff distance D, where D depends only on C, L, K, and the hyperbolic constant δ of (Γ, d).*

See the proof when d is geodesic **[GdlH90,** Théorèmes 21 et 25, Chapitre 5**]**; it is adapted when d is roughly geodesic, cf. **[BS00,** proof of Proposition 5.6**]**.

2.1.2. *Boundary at infinity.* Let us denote by $\partial(\Gamma, d)$ the *geometric boundary* (or *boundary*) of (Γ, d). This is a compact metrizable space consisting of equivalence classes of divergent sequences in (Γ, d). Recall that a sequence $\{x_n\}_{n=0}^\infty$ in (Γ, d) is *divergent* if $(x_n|x_m)_w \to \infty$ as $n, m \to \infty$ for some (equivalently, any) $w \in \Gamma$. Two divergent sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ are equivalent $\{x_n\}_{n=0}^\infty \sim \{y_n\}_{n=0}^\infty$ if $(x_n|y_m)_w \to \infty$ as $n, m \to \infty$. The boundary $\partial(\Gamma, d)$ is the set of equivalence classes of divergent sequences in (Γ, d). For any $d \in \mathcal{D}_\Gamma$, the corresponding boundaries $\partial(\Gamma, d)$ are all homeomorphic each other. We denote by $\partial\Gamma$ the underlying topological space of the boundary for $d \in \mathcal{D}_\Gamma$. The group Γ acts on $\partial\Gamma$ continuously by left multiplications $\Gamma \times \partial\Gamma \to \partial\Gamma$, $(x, \xi) \mapsto x \cdot \xi$.

Since we work with a roughly geodesic hyperbolic metric d in Γ, we record the following lemma, which says that there exists a constant $C \geq 0$ such that any two distinct points in the boundary $\partial\Gamma$ are joined by a C-rough geodesic in (Γ, d).

LEMMA 2.2. **[BS00,** Proposition 5.2(3)**]** *If (Γ, d) is a C-roughly geodesic δ-hyperbolic space, then there exists a constant $C' = C'(\delta, C) \geq 0$ such that for any two points $\xi, \eta \in \partial\Gamma$ with $\xi \neq \eta$, there is a C'-rough geodesic $\phi : \mathbb{R} \to \Gamma$ satisfying that $\phi(-t) \to \xi$ and $\phi(t) \to \eta$ as $t \to \infty$ in $\Gamma \cup \partial\Gamma$.*

2.1.3. *Gauges in the boundary.* Fix a base point o corresponding to the identity element of Γ. We extend the Gromov product relative to $d \in \mathcal{D}_\Gamma$ to $\Gamma \cup \partial\Gamma$ by setting

$$(\xi|\eta)_o := \sup \left\{ \liminf_{n\to\infty} (x_n|y_n)_o : \{x_n\}_{n=0}^\infty \in \xi,\ \{y_n\}_{n=0}^\infty \in \eta \right\}.$$

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If $d \in \mathcal{D}_\Gamma$ is δ-hyperbolic, then for any two pairs of equivalent sequences $\{x_n\}_{n=0}^\infty \sim \{x_n'\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty \sim \{y_n'\}_{n=0}^\infty$,

$$\liminf_{n\to\infty}(x_n'|y_n')_o \geq \limsup_{n\to\infty}(x_n|y_n)_o - 2\delta.$$

Hence if $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ converge to ξ and η, respectively in $\Gamma \cup \partial\Gamma$, then

$$(\xi|\eta)_o - 2\delta \leq \liminf_{n\to\infty}(x_n|y_n)_o \leq \limsup_{n\to\infty}(x_n|y_n)_o \leq (\xi|\eta)_o + 2\delta, \tag{1}$$

and we have the δ-hyperbolic inequality on $\Gamma \cup \partial\Gamma$,

$$(\xi|\eta)_o \geq \min\{(\xi|\zeta)_o, (\zeta|\eta)_o\} - 3\delta \quad \text{for } \xi, \eta, \zeta \in \Gamma \cup \partial\Gamma. \tag{2}$$

For $d \in \mathcal{D}_\Gamma$, let us define

$$\rho(\xi, \eta) := \exp(-(\xi|\eta)_o) \quad \text{for } \xi, \eta \in \partial\Gamma.$$

The ρ defines a *quasi-metric* in $\partial\Gamma$, i.e., ρ satisfies that $\rho(\xi, \eta) = 0$ if and only if $\xi = \eta$, and $\rho(\xi, \eta) = \rho(\eta, \xi)$, moreover there exists a constant $C \geq 1$ such that

$$\rho(\xi, \zeta) \leq C(\rho(\xi, \eta) + \rho(\eta, \zeta)) \quad \text{for } \xi, \eta, \zeta \in \partial\Gamma.$$

In fact, there exists an $\varepsilon \in (0, 1)$ such that ρ^ε is bi-Lipschitz equivalent to a genuine metric on $\partial\Gamma$ (e.g., **[GdlH90,** Proposition 10, §3, Chapitre 7]). However, in order to avoid introducing an additional parameter ε, we mainly use ρ in the boundary $\partial\Gamma$. Let $\partial^2\Gamma := \{(\xi, \eta) \in (\partial\Gamma)^2 : \xi \neq \eta\}$. We define $\rho_\times$ by

$$\rho_\times((\xi_1, \eta_1), (\xi_2, \eta_2)) := \max\{\rho(\xi_1, \xi_2), \rho(\eta_1, \eta_2)\} \quad \text{for } (\xi_1, \eta_1), (\xi_2, \eta_2) \in \partial^2\Gamma.$$

For $d \in \mathcal{D}_\Gamma$, we call ρ and $\rho_\times$ *gauges* associated with d on $\partial\Gamma$ and $\partial^2\Gamma$, respectively.

2.1.4. *Shadows.* For $d \in \mathcal{D}_\Gamma$, we define shadows on $\partial\Gamma$; they behave similarly to balls relative to ρ, but it is better suited to control measures on the boundary. Moreover, shadows are less sensitive to the change of metrics d in $\mathcal{D}_\Gamma$ while ρ vitally depends on d. Let (Γ, d) be a C-roughly geodesic hyperbolic space and fix a point $o \in \Gamma$. For any $x \in \Gamma$ and any $R \geq 0$, let us define $O_o(x, R)$ the *shadow from* o as the set of points $\xi \in \partial\Gamma$ such that some C-rough geodesic ray from o converging to ξ intersects $B_d(x, R) := \{y \in \Gamma : d(x, y) \leq R\}$. The next lemma follows from the definitions of shadows and the gauge ρ by the δ-hyperbolicity.

LEMMA 2.3. (Cf. Proposition 2.1 in **[BHM11]**) *Fix $d \in \mathcal{D}_\Gamma$. Then there exist constants $R_0, C > 0$ such that for all $R \geq R_0$, for all $x \in \Gamma$ and all $\xi \in \partial\Gamma$, we have that*

$$B_\rho(\xi, (1/C)e^{-d(o,x)+R}) \subset O_o(x, R) \subset B_\rho(\xi, Ce^{-d(o,x)+R}),$$

where $B_\rho(\xi, r) := \{\eta \in \partial\Gamma : \rho(\xi, \eta) \leq r\}$.

For $d, d' \in \mathcal{D}_\Gamma$, let $O_o(x, R)$ and $O_o'(x, R)$ be shadows defined in (Γ, d) and (Γ, d'), respectively. Since d and d' are quasi-isometric, any C-rough geodesic ϕ in (Γ, d) is a (L, K)-quasi-geodesic in (Γ, d'), and thus there exists a C'-geodesic in ϕ' in (Γ, d') such that ϕ and ϕ' are within Hausdorff distance at most D in (Γ, d'), where the constant

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D depends only on the constants involved in Lemma 2.1. Therefore, we have $O_o(x, R) \subset O'_o(x, R')$ for $R' = LR + K + D$. Note that shadows around the same point x but different metrics in $\mathcal{D}_\Gamma$ are comparable up to changing the thickness of shadows independent of x.

2.2. *Hausdorff dimension of measures.* Let (X, ρ) be a space endowed with a gauge ρ; we say that ρ is a *gauge* in X if $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$, it holds that $\rho(x, y) = 0$ if and only if $x = y$, and there exists a constant $C > 0$ such that $\rho(x, y) \le C(\rho(x, z) + \rho(z, y))$ for all $x, y, z \in X$. Examples we have in mind are $(\partial\Gamma, \rho)$ and $(\partial^2\Gamma, \rho_\times)$. If ρ^ε is bi-Lipschitz equivalent to a genuine metric d_ε for some $0 < \varepsilon < 1$, then the Hausdorff dimension $\dim_H(A, \rho)$ of a set A relative to the gauge ρ is $\varepsilon \cdot \dim_H(A, d_\varepsilon)$.

For any subset E of X, letting $\rho(E) := \sup\{\rho(x, y) : x, y \in E\}$, we define for every $D \ge 0$ and $\Delta > 0$,

$$\mathcal{H}^D_\Delta(E, \rho) := \inf\left\{ \sum_{i=0}^{\infty} \rho(E_i)^D : E \subset \bigcup_{i=0}^{\infty} E_i \text{ and } \rho(E_i) \le \Delta \right\}.$$

The *D-Hausdorff measure* of a set E is defined by

$$\mathcal{H}^D(E, \rho) := \sup_{\Delta > 0} \mathcal{H}^D_\Delta(E, \rho) = \lim_{\Delta \to 0} \mathcal{H}^D_\Delta(E, \rho).$$

The *Hausdorff dimension* of a set E in (X, ρ) is defined by

$$\dim_H(E, \rho) := \inf\left\{D \ge 0 : \mathcal{H}^D(E, \rho) = 0\right\} = \sup\left\{D \ge 0 : \mathcal{H}^D(E, \rho) > 0\right\}.$$

Definition 2.4. Let ν be a Borel measure on (X, ρ). We define the *lower Hausdorff dimension* of ν by

$$\underline{\dim}_H(\nu, \rho) := \inf\left\{\dim_H(E, \rho) : \nu(E) > 0,\ E \text{ is Borel}\right\},$$

and the *upper Hausdorff dimension* of ν by

$$\overline{\dim}_H(\nu, \rho) := \inf\left\{\dim_H(E, \rho) : \nu(X \backslash E) = 0,\ E \text{ is Borel}\right\}.$$

If we have $\underline{\dim}_H(\nu, \rho) = \overline{\dim}_H(\nu, \rho)$, then we say that the value is the *Hausdorff dimension* of ν and denote it by $\dim_H(\nu, \rho)$.

In order to estimate the Hausdorff dimension of a measure, we use the following Frostman-type lemma, which relates the dimension of ν to the pointwise behavior $\nu(B_\rho(\xi, r))$ as $r \to 0$ at each point $\xi \in X$.

LEMMA 2.5. (Cf. §8.7 in [**Hei01**]) *Let ν be a finite Borel measure on X. If there exist $D_1, D_2 \ge 0$ such that*

$$D_1 \le \liminf_{r \to 0} \frac{\log \nu(B_\rho(\xi, r))}{\log r} \le D_2 \quad \textit{for } \nu\textit{-almost every } \xi \textit{ in } X,$$

where $B_\rho(\xi, r) = \{\eta \in X : \rho(\xi, \eta) \le r\}$, then $D_1 \le \underline{\dim}_H(\nu, \rho) \le \overline{\dim}_H(\nu, \rho) \le D_2$.

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Note that by Lemma 2.5, for every finite Borel measure ν on (X, ρ), we have

$$\underline{\dim}_H(\nu, \rho) = \inf_{\nu} \liminf_{r\to 0} \frac{\log \nu(B_\rho(\xi, r))}{\log r} \quad \text{and}$$
$$\overline{\dim}_H(\nu, \rho) = \sup_{\nu} \liminf_{r\to 0} \frac{\log \nu(B_\rho(\xi, r))}{\log r},$$

where the infimum (respectively, supremum) stands for the essential infimum (respectively, supremum) relative to ν. If ν is a possibly infinite but σ-finite Borel measure on (X, ρ), then taking compact sets X_n of X for $n = 0, 1, \ldots$ such that $X = \bigcup_{n=0}^{\infty} X_n$, and letting $\nu|_{X_n} := \nu(\,\cdot\, \cap X_n)$, we have

$$\underline{\dim}_H(\nu, \rho) = \inf_{n=0,1,\ldots} \underline{\dim}_H(\nu|_{X_n}, \rho) \quad \text{and} \quad \overline{\dim}_H(\nu, \rho) = \sup_{n=0,1,\ldots} \overline{\dim}_H(\nu|_{X_n}, \rho).$$

We will apply these facts to Radon measures on $(\partial^2\Gamma, \rho_\times)$ in oder to estimate their Hausdorff dimensions.

2.3. *Busemann (quasi-)cocycles.* For any $d \in \mathcal{D}_\Gamma$ and for $w \in \Gamma$, let us define the Busemann function based at w by

$$b_w : \Gamma \times \partial\Gamma \to \mathbb{R}, \quad b_w(x, \xi) := \sup \left\{ \limsup_{n\to\infty} (d(x, z_n) - d(w, z_n)) : \{z_n\}_{n=0}^{\infty} \in \xi \right\}.$$

We focus on the Busemann function b_o based at o. Noting the identity

$$d(x, z) - d(o, z) = d(o, x) - 2(x|z)_o,$$

by equation (1), we have that $|b_o(x, \xi) - (d(o, x) - 2(x|\xi)_o)| \le 4\delta$ for $(x, \xi) \in \Gamma \times \partial\Gamma$. This shows that for $x \in \Gamma$ and $\xi \in O_o(x, R)$,

$$-d(o, x) \le b_o(x, \xi) \le -d(o, x) + C_{R,\delta},$$

where in fact the first inequality holds for all $\xi \in \partial\Gamma$ and all $x \in \Gamma$ by the triangle inequality. Moreover, b_o satisfies the (cocycle) identity up to an additive constant 4δ,

$$|b_o(xy, \xi) - (b_o(y, x^{-1}\xi) + b_o(x, \xi))| \le 4\delta \quad \text{for } x, y \in \Gamma, \xi \in \partial\Gamma. \tag{3}$$

Combining the definition of the Gromov product on $\Gamma \cup \partial\Gamma$, we have

$$b_o(x, \xi) + b_o(x, \eta) = 2(\xi|\eta)_x - 2(\xi|\eta)_o + O_\delta \quad \text{and}$$
$$b_o(x, \xi) - b_o(x, \eta) = -2(\xi|x)_o + 2(\eta|x)_o + O_\delta,$$

for $x \in \Gamma$ and two distinct points $\xi, \eta \in \partial\Gamma$.

2.4. *Strongly hyperbolic metrics.* Let us introduce a special class of metrics, which behave regularly at infinity, following **[NŠ16]**.

Definition 2.6. We say that a hyperbolic metric d on Γ is *strongly hyperbolic* if there exist constants $L \ge 0$, $s > 0$ and $R_0 \ge 0$ such that for all $x, x', y, y' \in \Gamma$, and all $R \ge R_0$, if $d(x, y) - d(x, x') - d(y, y') + d(x', y') \ge R$, then

$$|d(x, y) - d(x', y) - d(x, y') + d(x', y')| \le Le^{-sR}.$$

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Nica and Špakula have shown that d is strongly hyperbolic if and only if there exists a constant $\varepsilon > 0$ such that

$$\exp(-\varepsilon(x \mid y)_o) \leq \exp(-\varepsilon(x|z)_o) + \exp(-\varepsilon(z \mid y)_o),$$

for all $x, y, z, o \in \Gamma$ [**NŠ16,** Lemma 6.2, Definition 4.1], whence the strong hyperbolicity implies the (usual) hyperbolicity, furthermore for the strongly hyperbolic metric the corresponding Gromov product extends to $\Gamma \cup \partial\Gamma$ continuously [**NŠ16,** Theorem 4.2].

This property of strong hyperbolicity is actually much stronger than what we expect from word metrics (except for very special cases such as word metrics on finite rank-free groups with the standard set of generators). We are interested in strongly hyperbolic metrics, which are quasi-isometric to a word metric. It is known that every non-elementary hyperbolic group admits a strongly hyperbolic metric, which is left-invariant and quasi-isometric to a word metric, i.e., there is a strongly hyperbolic metric in $\mathcal{D}_\Gamma$. In this case, since d is roughly geodesic and quasi-isometric to any word metric, the strong hyperbolicity of d is equivalent to the following: for any Cayley graph $\mathrm{Cay}(\Gamma, S)$ there exist constants $L \geq 0$, $s > 0$, and $C \geq 0$ depending only on $\mathrm{Cay}(\Gamma, S)$ such that for any $x, x', y, y' \in \Gamma$ and any $n \geq 0$, if two geodesic segments connecting x and y, and x' and y', respectively, have a common geodesic segment of length n in each C-neighborhood, then

$$|d(x, y) - d(x', y) - d(x, y') + d(x', y')| \leq Le^{-sn}, \tag{4}$$

where δ is a hyperbolic constant of $\mathrm{Cay}(\Gamma, S)$.

Example 2.7.

(1) The hat metric $\widehat{d}$ on Γ introduced by Mineyev is a hyperbolic metric, which is strongly hyperbolic, left-invariant and quasi-isometric to a word metric [**Min05,** Theorem 32].

(2) $\mathrm{CAT}(-1)$ spaces are strongly hyperbolic [**NŠ16,** Theorem 5.1], e.g., Riemannian manifolds with sectional curvature at most -1. If a hyperbolic group Γ acts on a $\mathrm{CAT}(-1)$ space isometrically and the action is properly discontinuous, cocompact and *free*, then for any base point o in the space, the metric d_o induced from the orbit $d_o(x, y) := d(xo, yo)$ is strongly hyperbolic, left-invariant and quasi-isometric to a word metric. If the action is not free and the stabilizer at o is non-trivial, then d_o is not a genuine metric on Γ, but up to changing in the rough isometry class one obtains a metric by defining $d'_o(x, y) := d(xo, yo) + \varepsilon$ if $x \neq y$ and $d'_o(x, y) := 0$ if $x = y$ for a fixed $\varepsilon > 0$. Then the metric d'_o on Γ is strongly hyperbolic, left-invariant and quasi-isometric to a word metric.

(3) The Green metric d_G associated with a μ-random walk on Γ is a hyperbolic metric, which is left-invariant and quasi-isometric to a word metric if μ is a finitely supported and symmetric probability measure on Γ such that the support generates the group Γ as a semigroup [**BHM11,** the first part of Corollary 1.2]. Moreover, d_G is strongly hyperbolic, which follows from a special case of [**Gou14, INO08,** Theorem 2.9, §3] (cf. [**NŠ16,** Theorem 6.1]).

We denote by $\widehat{d}$ a hyperbolic metric, which is strongly hyperbolic, left-invariant and quasi-isometric to a word metric on Γ. This notation is indebted to the hat metric, but $\widehat{d}$

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is not intended to be the particular metric in Example 2.7 (1). The property we use is that $\widehat{d} \in \mathcal{D}_\Gamma$ and $\widehat{d}$ is strongly hyperbolic in Definition 2.6.

Since $\widehat{d} \in \mathcal{D}_\Gamma$, the following is a consequence of the Morse lemma for roughly geodesic hyperbolic spaces (Lemma 2.1): there exists a constant $C \geq 0$ such that for any $x, y \in \mathrm{Cay}(\Gamma, S)$ and any z on a geodesic connecting x and y in $\mathrm{Cay}(\Gamma, S)$, we have

$$|\widehat{d}(x, z) + \widehat{d}(z, y) - \widehat{d}(x, y)| \leq C. \tag{5}$$

Intending to clarify and simplify notations, we denote the Gromov product by $(\cdot, \cdot)_o$ relative to a general $d \in \mathcal{D}_\Gamma$, and by $\langle \cdot, \cdot \rangle_o$ relative to $\widehat{d}$, respectively. An important consequence of the strong hyperbolicity is that the Gromov product $\langle \cdot, \cdot \rangle_o$ extends continuously on $\Gamma \cup \partial\Gamma$.

For the metric $\widehat{d}$, the corresponding Busemann function $\widehat{b}_o$ (based at o) is defined as the limit

$$\widehat{b}_o(x, \xi) = \lim_{n \to \infty} (\widehat{d}(x, z_n) - \widehat{d}(o, z_n)) = \widehat{d}(o, x) - 2\langle x | \xi \rangle_o, \tag{6}$$

for any sequence $\{z_n\}_{n=0}^{\infty}$ converging to ξ. Note that for any $x \in \Gamma$, the Busemann function $\widehat{b}_o(x, \cdot)$ is continuous on $\partial\Gamma$. Furthermore, $\widehat{b}$ satisfies the genuine cocycle identity in equation (3),

$$\widehat{b}_o(xy, \xi) = \widehat{b}_o(y, x^{-1}\xi) + \widehat{b}_o(x, \xi) \quad \text{for } x, y \in \Gamma, \xi \in \partial\Gamma.$$

Note that we have the genuine identity

$$\widehat{b}_o(x, \xi) + \widehat{b}_o(x, \eta) = 2\langle \xi | \eta \rangle_x - 2\langle \xi | \eta \rangle_o, \tag{7}$$

for $x \in \Gamma$ and for any two distinct $\xi, \eta \in \partial\Gamma$.

2.5. *Patterson–Sullivan measures.* For a metric $d \in \mathcal{D}_\Gamma$, let us denote balls relative to d by $B_d(x, R) := \{y \in \Gamma \ : d(x, y) \leq R\}$ for $x \in \Gamma$ and $R \geq 0$. We define the exponential volume growth rate of (Γ, d) by

$$\mathrm{gr}(d) := \limsup_{n \to \infty} \frac{1}{n} \log |B_d(o, n)|,$$

where $|\cdot|$ stands for the cardinality of the set. We have that $0 < \mathrm{gr}(d) < \infty$ as soon as Γ is non-elementary (i.e., non-amenable) and $d \in \mathcal{D}_\Gamma$. The Patterson–Sullivan construction yields a probability measure μ_d associated with d on $\partial\Gamma$ satisfying that for a constant $C > 0$,

$$\frac{1}{C} \exp(-\mathrm{gr}(d)\, b_o(x, \xi)) \leq \frac{dx_* \mu_d}{d\mu_d}(\xi) \leq C \exp(-\mathrm{gr}(d)\, b_o(x, \xi)), \tag{8}$$

for all $x \in \Gamma$ and μ_d-almost every $\xi \in \partial\Gamma$. For the Patterson–Sullivan construction, see **[Coo93,** Théorème 5.4**]** and for the statement in a more general setting, see **[BHM11,** Theorem 2.7**]**, which covers the case for $d \in \mathcal{D}_\Gamma$. We call μ_d a *Patterson–Sullivan measure* for $d \in \mathcal{D}_\Gamma$. The following is a consequence of equation (8) (for which we do not reproduce the proof, see e.g., **[Coo93,** Proposition 6.1**]**).

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PROPOSITION 2.8. *Fix a large enough constant $R > 0$. Then there exists a constant $C_R \geq 1$ such that for all $x \in \Gamma$,*

$$\frac{1}{C_R} \exp(-\mathrm{gr}(d)\, d(o, x)) \leq \mu_d(O_o(x, R)) \leq C_R \exp(-\mathrm{gr}(d)\, d(o, x)).$$

We use the following doubling property of Patterson–Sullivan measures later.

LEMMA 2.9. *Let us fix a metric $d \in \mathcal{D}_\Gamma$. Then the corresponding Patterson–Sullivan measure μ_d is doubling with respect to ρ, i.e., there exists a constant $C \geq 1$ such that for any $\xi \in \partial\Gamma$ and for any $r > 0$,*

$$\mu_d(B_\rho(\xi, 2r)) \leq C \cdot \mu_d(B_\rho(\xi, r)),$$

where $B_\rho(\xi, r) = \{\eta \in \partial\Gamma : \rho(\xi, \eta) < r\}$.

Proof. Proposition 2.8 shows that a large enough $R > 0$, the measures $\mu_d(O_o(x, 2R))$ and $\mu_d(O_o(x, R))$ are comparable with uniform constants independent of $x \in \Gamma$. Applying this estimate repeatedly, Lemma 2.3 yields the claim. □

It is known that the size of (thickened) spheres is comparable with an exponential function. The following is a version in the form of a weighted sum relative to a word metric.

LEMMA 2.10. *For any hyperbolic metric d in $\mathcal{D}_\Gamma$ and any word metric d_S on Γ, there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \leq \sum_{x \in \boldsymbol{S}_n} \exp(-\mathrm{gr}(d)\, d(o, x)) \leq c_2 \quad \textit{for all } n \geq 0,$$

where $\boldsymbol{S}_n := \{x \in \Gamma : d_S(o, x) = n\}$ for integers $n \geq 0$.

Proof. Let us denote by $O_o(x, R)$ shadows relative to d. Fix a large enough $R > 0$. For any integer n, shadows $O_o(x, R)$ for $|x|_S = n$ cover the boundary $\partial\Gamma$, and thus Proposition 2.8 implies that

$$1 = \mu_d(\partial\Gamma) \leq \sum_{x \in \mathbf{S}_n} \mu_d(O_o(x, R)) \leq C_R \sum_{x \in \mathbf{S}_n} \exp(-\mathrm{gr}(d) d(o, x)).$$

We denote by $O_o^w(x, R)$ shadows relative to a word metric. For any $R > 0$, there exists a constant $R' > 0$ such that $O_o(x, R) \subset O_o^w(x, R')$ for all $x \in \Gamma$ (cf. §2.1.4). Note that $O_o^w(x, R')$ for $|x|_S = n$ cover the boundary $\partial\Gamma$ with bounded overlaps, i.e., for an integer $D \geq 1$ each point in $\partial\Gamma$ is included at most D shadows $O_o^w(x, R')$ with $|x|_S = n$. Therefore, by Proposition 2.8 we have that

$$(1/C_R) \sum_{x \in \mathbf{S}_n} \exp(-\mathrm{gr}(d) d(o, x)) \leq \sum_{x \in \mathbf{S}_n} \mu_d(O_o^w(x, R')) \leq D\mu_d(\partial\Gamma) = D.$$

We conclude the claim. □

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PROPOSITION 2.11. *Let* $\partial^2\Gamma := (\partial\Gamma)^2\backslash\{(\xi,\xi) : \xi \in \partial\Gamma\}$. *For every* $d \in \mathcal{D}_\Gamma$, *there exists a* Γ*-invariant Radon measure* Λ_d *on* $\partial^2\Gamma$ *such that*

$$\Lambda_d = \varphi(\xi,\eta)\exp(2\mathrm{gr}(d)\ (\xi|\eta)_o)\mu_d \otimes \mu_d,$$

where there exist constants $c_1, c_2 > 0$ *such that* $c_1 \le \varphi(\xi,\eta) \le c_2$ *for all* $(\xi,\eta) \in \partial^2\Gamma$. *Moreover,* $\Lambda_d(U) > 0$ *for each open set* U *in* $\partial^2\Gamma$ *and* $\Lambda_d(K) < \infty$ *for each compact set* K *in* $\partial^2\Gamma$.

Proof. Let μ_d be a Patterson–Sullivan measure for $d \in \mathcal{D}_\Gamma$. Restricting the measure $\mu_d \otimes \mu_d$ on $\partial^2\Gamma$, we define the measure on $\partial^2\Gamma$ by

$$\nu := \exp(2\mathrm{gr}(d)(\xi|\eta)_o)\mu_d \otimes \mu_d.$$

Since we have for $x \in \Gamma$ and $\xi,\eta \in \partial\Gamma$, $b_o(x,\xi) + b_o(x,\eta) = 2(\xi|\eta)_x - 2(\xi|\eta)_o \pm C_\delta$ (§2.3) and by equation (8), there exist constants $c_1, c_2 > 0$ such that

$$c_1 \le \frac{dx_*\nu}{d\nu}(\xi,\eta) \le c_2 \quad \text{for all } x \in \Gamma \text{ and } \nu\text{-almost every } (\xi,\eta) \in \partial^2\Gamma.$$

For $x, y \in \Gamma$, note that

$$\frac{d(xy)_*\nu}{d\nu}(\xi,\eta) = \frac{dx_*\nu}{d\nu}(\xi,\eta)\frac{dy_*\nu}{d\nu}(x^{-1}\xi, x^{-1}\eta).$$

Letting $\varphi(\xi,\eta) := \sup_{x\in\Gamma}((dx_*\nu/d\nu)(\xi,\eta))$, we have $c_1 \le \varphi(\xi,\eta) \le c_2$ and

$$\varphi(\xi,\eta) = \frac{dx_*\nu}{d\nu}(\xi,\eta)\cdot\varphi(x^{-1}\xi, x^{-1}\eta).$$

Hence if we define $\Lambda_d := \varphi(\xi,\eta)\nu$, then Λ_d is Γ-invariant. Moreover, note that each compact set K in $\partial^2\Gamma$ is included in $\{(\xi,\eta) \in \partial^2\Gamma : (\xi|\eta)_o \le C\}$ for some $C > 0$, and thus $\nu_d(K) < \infty$. For each open set U in $\partial^2\Gamma$, we have $\Lambda_d(U) > 0$ since $\mu_d(B) > 0$ for each open set B in $\partial\Gamma$. □

Remark 2.12. For a strongly hyperbolic metric $\widehat{d}$, the corresponding Γ-invariant measure is given by $\Lambda_{\widehat{d}} = \exp(2\mathrm{gr}(\widehat{d}\,)\ \langle\xi|\eta\rangle_o)\mu_{\widehat{d}} \otimes \mu_{\widehat{d}}$, that is, one may take $\varphi \equiv 1$ in Proposition 2.11. This is because one is able to take the constant $C = 1$ in equation (8) in the Patterson–Sullivan construction and the Busemann function $\widehat{b}_o(x,\cdot)$ defines a genuine cocycle (without an additive constant) (see also **[Nic13,** §7.1**]**).

3. *Topological flow spaces*

3.1. *Construction of a flow space.* Let $\partial^2\Gamma := (\partial\Gamma)^2\backslash\{(\xi,\xi) : \xi \in \partial\Gamma\}$, i.e., the space of ordered pairs of distinct points in $\partial\Gamma$. We consider the diagonal action of Γ on $\partial^2\Gamma$. The space $\partial^2\Gamma \times \mathbb{R}$ is regarded as the set of directed lines with distinct extreme points in $\partial\Gamma$. Here we give a geometric realization of $\partial^2\Gamma \times \mathbb{R}$ by rough geodesics on $(\Gamma, \widehat{d}\,)$. For every $(\xi,\eta) \in \partial^2\Gamma$, there exists a C-rough geodesic $\phi_{\xi,\eta} : \mathbb{R} \to (\Gamma, \widehat{d}\,)$ from ξ to η by Lemma 2.2. We normalize the parametrization of $\phi_{\xi,\eta}(t)$ by shifting $t \mapsto t + T$ for some T if necessary such that

$$\widehat{d}(\phi_{\xi,\eta}(0), o) = \min_{t\in\mathbb{R}} \widehat{d}(\phi_{\xi,\eta}(t), o).$$

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Then we define the map

$$\partial^2\Gamma \times \mathbb{R} \to (\Gamma, \widehat{d}\,), \quad (\xi, \eta, t) \mapsto \phi_{\xi,\eta}(t).$$

This map is not canonical; it depends on choices for rough geodesics. However, note that for any choices $\phi_{\xi,\eta}$ and $\phi'_{\xi,\eta}$ for $(\xi, \eta) \in \partial^2\Gamma$, we have $\sup_{t\in\mathbb{R}} \widehat{d}(\phi_{\xi,\eta}(t), \phi'_{\xi,\eta}(t)) \le C$ for a constant C depending only on $(\Gamma, \widehat{d}\,)$ by the stability of C-rough geodesics.

LEMMA 3.1. *For the metric $\widehat{d} \in \mathcal{D}_\Gamma$, let $\widehat{\rho}$ be the corresponding gauge in $\partial\Gamma$. For any compact set K in $\partial^2\Gamma$, there exists a constant $c_K > 0$ such that for any $\xi, \eta, \eta' \in \partial\Gamma$ and $x \in \Gamma$, if $(\xi, \eta), (\xi, \eta') \in K$ and $T = -\widehat{b}_o(x^{-1}, \xi)$, then*

$$\widehat{\rho}(x\eta, x\eta') \le e^{2c_K - T} \cdot \widehat{\rho}(\eta, \eta').$$

Proof. Since the Gromov product $\langle\cdot,\cdot\rangle_o$ is continuous on $\partial^2\Gamma$ and K is a compact subset in $\partial^2\Gamma$, there exists a positive constant c_K depending only on K such that $\langle\xi|\eta\rangle_o, \langle\xi|\eta'\rangle_o \le c_K$. Hence

$$\widehat{b}_o(x^{-1}, \xi) + \widehat{b}_o(x^{-1}, \eta) \ge -2c_K \quad \text{and} \quad \widehat{b}_o(x^{-1}, \xi) + \widehat{b}_o(x^{-1}, \eta') \ge -2c_K,$$

and $\widehat{b}_o(x, \eta) + \widehat{b}_o(x, \eta') = 2\langle\eta|\eta'\rangle_x - 2\langle\eta|\eta'\rangle_o$ by equation (7), we obtain

$$\langle\eta|\eta'\rangle_{x^{-1}} - \langle\eta|\eta'\rangle_o \ge -\widehat{b}_o(x^{-1}, \xi) - 2c_K.$$

Therefore, letting $T = -\widehat{b}_o(x^{-1}, \xi)$, we have

$$\widehat{\rho}(x\eta, x\eta') = e^{-\langle\eta|\eta'\rangle_{x^{-1}}} \le e^{2c_K} e^{-T - \langle\eta|\eta'\rangle_o} = e^{2c_K} e^{-T} \widehat{\rho}(\eta, \eta'),$$

as desired. □

Let us define the cocycle $\alpha : \Gamma \times \partial^2\Gamma \to \mathbb{R}$ by

$$\alpha(x, \xi, \eta) := \tfrac{1}{2}(\widehat{b}_o(x^{-1}, \xi) - \widehat{b}_o(x^{-1}, \eta)) \quad \text{for } x \in \Gamma, (\xi, \eta) \in \partial^2\Gamma.$$

Note that we have

$$\alpha(x, \xi, \eta) = \langle x^{-1}|\eta\rangle_o - \langle x^{-1}|\xi\rangle_o. \tag{9}$$

Then, we define the action of Γ on $\partial^2\Gamma \times \mathbb{R}$ by for $x \in \Gamma$ and for $(\xi, \eta, t) \in \partial^2\Gamma \times \mathbb{R}$,

$$x \cdot (\xi, \eta, t) := (x\xi, x\eta, t - \alpha(x, \xi, \eta)).$$

The cocycle identity $\alpha(xy, \xi, \eta) = \alpha(x, y\xi, y\eta) + \alpha(y, \xi, \eta)$ shows that it defines indeed a Γ-action on $\partial^2\Gamma \times \mathbb{R}$. We call this Γ-action on $\partial^2\Gamma \times \mathbb{R}$ via α the (Γ, α)-*action*.

LEMMA 3.2. *The (Γ, α)-action of Γ on $\partial^2\Gamma \times \mathbb{R}$ is properly discontinuous and cocompact, i.e., $\Gamma\backslash(\partial^2\Gamma \times \mathbb{R})$ is compact.*

Proof. For $N, T > 0$, let

$$F_{N,T} := \{(\xi, \eta, t) \in \partial^2\Gamma \times \mathbb{R} : |t| \le T,\ \widehat{\rho}(\xi, \eta) \ge \exp(-N)\}.$$

First we shall show that for each $N, T > 0$, the number of x in Γ such that $xF_{N,T} \cap F_{N,T} \ne \emptyset$ is finite. Suppose that for $x \in \Gamma$ there exists $(\xi, \eta, t) \in F_{N,T}$ such

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that $x(\xi, \eta, t) \in F_{N,T}$. Then, we have

$$\langle \xi | \eta \rangle_o \le N \quad \text{and} \quad \langle \xi | \eta \rangle_{x^{-1}} = \langle x\xi | x\eta \rangle_o \le N.$$

Furthermore, since $|t| \le T$ and $|t - \alpha(x, \xi, \eta)| \le T$, we have by equation (9),

$$|\langle x^{-1} | \eta \rangle_o - \langle x^{-1} | \xi \rangle_o| = |\alpha(x, \xi, \eta)| \le 2T,$$

and

$$\begin{aligned}\langle x^{-1} | \xi \rangle_o + \langle x^{-1} | \eta \rangle_o &\le 2 \min\{\langle x^{-1} | \xi \rangle_o, \langle x^{-1} | \eta \rangle_o\} + |\langle x^{-1} | \eta \rangle_o - \langle x^{-1} | \xi \rangle_o| \\ &\le 2\langle \xi | \eta \rangle_o + 6\delta + 2T,\end{aligned}$$

where the last inequality follows from the δ-hyperbolic inequality (2). Therefore, by equation (6) we have

$$\begin{aligned}\widehat{d}(o, x) &= \tfrac{1}{2}(\widehat{b}_o(x^{-1}, \xi) + \widehat{b}_o(x^{-1}, \eta)) + \langle x^{-1} | \xi \rangle_o + \langle x^{-1} | \eta \rangle_o \\ &= \langle \xi | \eta \rangle_{x^{-1}} - \langle \xi | \eta \rangle_o + \langle x^{-1} | \xi \rangle_o + \langle x^{-1} | \eta \rangle_o \\ &\le \langle \xi | \eta \rangle_{x^{-1}} + \langle \xi | \eta \rangle_o + 6\delta + 2T \le 2N + 6\delta + 2T.\end{aligned}$$

This shows that x is included in a ball of a finite radius in the proper metric space $(\Gamma, \widehat{d}\,)$, and thus the number of such x is finite. Since any compact set K in $\partial^2\Gamma \times \mathbb{R}$ is contained in $F_{N,T}$ for some N and T, we conclude that the action of Γ on $\partial^2\Gamma \times \mathbb{R}$ is properly discontinuous.

Second we shall show that for some N and T, every Γ-orbit intersects $F_{N,T}$. For any given $(\xi, \eta, t) \in \partial^2\Gamma \times \mathbb{R}$, take a C_δ-rough geodesic $\phi_{\xi,\eta}$ from ξ to η such that $\phi_{\xi,\eta}(0)$ is one of the nearest points on $\phi_{\xi,\eta}$ to o. Then we have $\widehat{b}_o(\phi_{\xi,\eta}(0), \xi) = -\langle \xi | \eta \rangle_o + O_\delta$ and $\widehat{b}_o(\phi_{\xi,\eta}(0), \eta) = -\langle \xi | \eta \rangle_o + O_\delta$, and thus $\alpha(\phi_{\xi,\eta}(0)^{-1}, \xi, \eta) = O_\delta$. Since for any $r \in \mathbb{R}$,

$$\widehat{b}_o(\phi_{\xi,\eta}(r), \xi) = r + \widehat{b}_o(\phi_{\xi,\eta}(0), \xi) + O_\delta, \quad \widehat{b}_o(\phi_{\xi,\eta}(r), \eta) = -r + \widehat{b}_o(\phi_{\xi,\eta}(0), \eta) + O_\delta,$$

we have that $\alpha(\phi_{\xi,\eta}(r)^{-1}, \xi, \eta) = r + \alpha(\phi_{\xi,\eta}(0)^{-1}, \xi, \eta) + O_\delta = r + O_\delta$. Letting $x := \phi_{\xi,\eta}(t)^{-1}$, we obtain

$$\alpha(x, \xi, \eta) = \alpha(\phi_{\xi,\eta}(t)^{-1}, \xi, \eta) = t \pm C'_\delta.$$

Since now the base point o is on the C_δ-rough geodesic $x\phi_{\xi,\eta}$ from $x\xi$ to $x\eta$, we have $\langle x\xi | x\eta \rangle_o \le 3C_\delta + 2\delta$. Therefore, if $N := 3C_\delta + 2\delta$ and $T := C'_\delta$, then

$$x \cdot (\xi, \eta, t) = (x\xi, x\eta, t - \alpha(x, \xi, \eta)) \in F_{N,T}.$$

The set $F_{N,T}$ is compact, and thus the action Γ on $\partial^2\Gamma \times \mathbb{R}$ is cocompact. □

Let $\mathcal{F}_\alpha := \Gamma\backslash(\partial^2\Gamma \times \mathbb{R})$ be the quotient topological space of $\partial^2\Gamma \times \mathbb{R}$ by the (Γ, α)-action. The space $\mathcal{F}_\alpha$ is compact by Lemma 3.2. We shall define a flow on $\mathcal{F}_\alpha$, namely, a continuous $\mathbb{R}$-action on $\mathcal{F}_\alpha$.

We define the action of $\mathbb{R}$ on $\partial^2\Gamma \times \mathbb{R}$ by for $t \in \mathbb{R}$, $(\xi, \eta, s) \in \partial^2\Gamma \times \mathbb{R}$,

$$\widetilde{\Phi}_t(\xi, \eta, s) := (\xi, \eta, t + s).$$

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Since this $\mathbb{R}$-action and the (Γ, α)-action commute on $\partial^2\Gamma \times \mathbb{R}$, the action $\{\widetilde{\Phi}_t\}_{t\in\mathbb{R}}$ descends to $\mathcal{F}_\alpha$. We define the induced flow by for $t \in \mathbb{R}$, $[\xi, \eta, s] \in \mathcal{F}_\alpha$,

$$\Phi_t : \mathcal{F}_\alpha \to \mathcal{F}_\alpha, \quad [\xi, \eta, s] \mapsto [\xi, \eta, t + s].$$

The action of $\mathbb{R}$ via Φ_t is continuous on $\mathcal{F}_\alpha$. We call $\{\Phi_t\}_{t\in\mathbb{R}}$ a *topological flow* on $\mathcal{F}_\alpha$.

Remark 3.3. The space $\mathcal{F}_\alpha$ plays a role of the (total space of) unit tangent bundle over a compact Riemannian manifold where the geodesic flow is defined in the classical sense. Let us mention a metric structure on $\mathcal{F}_\alpha$ not just the topology although we do not use it for our purpose. Associated with a metric $\widehat{d}$ in Γ, there is a metric d_* in $\partial^2\Gamma \times \mathbb{R}$ such that the (Γ, α)-action on $(\partial^2\Gamma \times \mathbb{R}, d_*)$ is isometric [**Min05**, Theorem 60(d)]. One may define the metric $d_{\mathcal{F}_\alpha}$ in $\mathcal{F}_\alpha$ by

$$d_{\mathcal{F}_\alpha}([x], [x']) := \inf_{\gamma\in\Gamma} d_*(x, \gamma x') \quad \text{for } x, x' \in \partial^2\Gamma \times \mathbb{R},$$

where the infimum is attained since the (Γ, α)-action on $\partial^2\Gamma \times \mathbb{R}$ is cocompact.

3.2. *Measures on $\partial^2\Gamma \times \mathbb{R}$ and $\mathcal{F}_\alpha$.* Let Λ be a Γ-invariant Radon measure on $\partial^2\Gamma$. (Recall that a measure on a locally compact Hausdorff space is Radon if it is finite on every compact set and Borel regular.) Let dt be the (normalized) Lebesgue measure on $\mathbb{R}$. Then, the measure $\Lambda \otimes dt$ is Radon on $\partial^2\Gamma \times \mathbb{R}$. Furthermore, the measure $\Lambda \otimes dt$ is (Γ, α)-invariant as well as $\{\widetilde{\Phi}_t\}_{t\in\mathbb{R}}$-invariant. For each continuous function with compact support $f \in C_c(\partial^2\Gamma \times \mathbb{R})$, let

$$\overline{f}(\xi, \eta, t) := \sum_{x\in\Gamma} f(x \cdot (\xi, \eta, t)),$$

where the summation runs over a finite number of x for each point by Lemma 3.2. Since $\overline{f}$ is Γ-invariant on $\partial^2\Gamma \times \mathbb{R}$, the function $\overline{f}$ can be defined on $\mathcal{F}_\alpha$. We use the same symbol for the function $\overline{f}$ defined on $\mathcal{F}_\alpha$.

LEMMA 3.4. *For any Γ-invariant Radon measure Λ on $\partial^2\Gamma$, there exists a unique finite Radon measure m on $\mathcal{F}_\alpha$ such that*

$$\int_{\partial^2\Gamma\times\mathbb{R}} f \, d\Lambda \otimes dt = \int_{\mathcal{F}_\alpha} \overline{f} \, dm,$$

for any compactly supported continuous function f on $\partial^2\Gamma \times \mathbb{R}$. Moreover, the measure m is invariant under the topological flow $\{\Phi_t\}_{t\in\mathbb{R}}$.

Proof. Note that for any continuous function φ on $\mathcal{F}_\alpha$, there exists a $f \in C_c(\partial^2\Gamma \times \mathbb{R})$ such that $\overline{f} = \varphi$. Indeed, let us denote the quotient map by $\pi : \partial^2\Gamma \times \mathbb{R} \to \mathcal{F}_\alpha$. There exists a compact set F on $\partial^2\Gamma \times \mathbb{R}$ such that $\pi(F) = \mathcal{F}_\alpha$ by Lemma 3.2. By the Urysohn lemma, there exists a continuous function χ with values in $[0, 1]$ on $\partial^2\Gamma \times \mathbb{R}$ such that $\chi \equiv 1$ on F and $\chi \equiv 0$ outside a relatively compact neighborhood of F. Taking such a

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function χ, define

$$f(x) := \frac{\chi(x)}{\sum_{\gamma\in\Gamma}\chi(\gamma x)}\varphi(\pi(x)).$$

Then f has a compact support and $\overline{f} = \varphi$. Moreover, if $\varphi \geq 0$, then $f \geq 0$.

For any continuous function φ on $\mathcal{F}_\alpha$, the map

$$\varphi \mapsto \int_{\partial^2\Gamma\times\mathbb{R}} f(x)\, d\Lambda \otimes dt,$$

is well defined, independent of the choice of f such that $\overline{f} = \varphi$, and defines a positive bounded linear functional on the space of continuous functions $C(\mathcal{F}_\alpha)$. Therefore, the Riesz representation theorem yields a unique finite Radon measure m.

Since $\Lambda \otimes dt$ is $\{\widetilde{\Phi}_t\}_{t\in\mathbb{R}}$-invariant and the (Γ,α)-action and $\{\widetilde{\Phi}_t\}_{t\in\mathbb{R}}$ commute on $\partial^2\Gamma\times\mathbb{R}$, for any $\varphi \in C(\mathcal{F}_\alpha)$ and any $t \in \mathbb{R}$,

$$\int_{\mathcal{F}_\alpha} \varphi\circ\Phi_t\, dm = \int_{\mathcal{F}_\alpha} \varphi\, dm.$$

This shows that m is invariant under $\{\Phi_t\}_{t\in\mathbb{R}}$. □

Remark 3.5. If we take a Borel fundamental domain D and a measurable section $\iota : \mathcal{F}_\alpha \to D \subset \partial^2\Gamma\times\mathbb{R}$, then we have $\Lambda\otimes dt = \sum_{\gamma\in\Gamma}\gamma.\iota_* m$. If the group Γ acts on $\partial^2\Gamma\times\mathbb{R}$ *freely*, then the measure m in Lemma 3.4 is obtained by the restriction of the measure $\Lambda\otimes dt$ on D, i.e., $\Lambda\otimes dt|_D = \iota_* m$. However, if the Γ-action is not free, then $\iota_* m$ is not the restriction of $\Lambda\otimes dt$ on D. See **[PPS15,** §2.6**]**.

3.3. *Flow-invariant measures.* For any $d \in \mathcal{D}_\Gamma$, let μ_d be a Patterson–Sullivan measure on $\partial^2\Gamma$ and Λ_d be the associated Γ-invariant Radon measure on $\partial^2\Gamma$ constructed in Proposition 2.11. Let us consider the (Γ,α)-invariant measure $\Lambda_d\otimes dt$ on $\partial^2\Gamma\times\mathbb{R}$. Lemma 3.4 implies that there exists a unique finite Radon measure m_d on $\mathcal{F}_\alpha$ such that m_d is $\{\Phi_t\}_{t\in\mathbb{R}}$-invariant and

$$\int_{\partial^2\Gamma\times\mathbb{R}} f\, d\Lambda_d\otimes dt = \int_{\mathcal{F}_\alpha} \overline{f}\, dm_d \quad \text{for } f \in C_c(\partial^2\Gamma\times\mathbb{R}).$$

We normalize m_d with $m_d(\mathcal{F}_\alpha) = 1$.

THEOREM 3.6. *Fix $d \in \mathcal{D}_\Gamma$. Then for any φ in $L^1(\mathcal{F}_\alpha, m_d)$, we have*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi\circ\Phi_t(x)\, dt = \int_{\mathcal{F}_\alpha}\varphi\, dm_d \quad \textit{for } m_d\textit{-almost every } x.$$

In particular, for any $d \in \mathcal{D}_\Gamma$, the measure m_d is ergodic under the topological flow $\{\Phi_t\}_{t\in\mathbb{R}}$ on $\mathcal{F}_\alpha$, i.e., for any Borel set A in $\mathcal{F}_\alpha$ with $\Phi_{-t}(A) = A$ for all $t \in \mathbb{R}$, one has $m_d(A) = 0$ or $m_d(\mathcal{F}_\alpha\backslash A) = 0$.

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Proof. For $d \in \mathcal{D}_\Gamma$, let us write $\Lambda = \Lambda_d$ and $m = m_d$. For any $\varphi \in L^1(\mathcal{F}_\alpha, m)$, the Birkhoff ergodic theorem implies that for m-almost every $x \in \mathcal{F}_\alpha$, there exists a limit

$$\varphi_\infty(x) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi \circ \Phi_t(x)\, dt,$$

and φ_∞ is defined on m-almost everywhere on $\mathcal{F}_\alpha$ such that $\varphi_\infty \circ \Phi_t = \varphi_\infty$ for any $t \in \mathbb{R}$. (This follows by considering a Birkhoff sum of $\int_0^1 \varphi \circ \Phi_t(x)\, dt$ and integers T.) The convergence is also in $L^1(\mathcal{F}_\alpha, m)$. We shall show that φ_∞ is constant m-almost everywhere. Note that if φ_∞ coincides with a constant m-almost everywhere, then it has to be $\int_{\mathcal{F}_\alpha} \varphi\, dm$. By the Fatou lemma, we have $\|\varphi_\infty\|_1 \le \|\varphi\|_1$. Hence it suffices to show the claim for continuous functions $\varphi \in C(\mathcal{F}_\alpha)$.

Let $\pi : \partial^2\Gamma \times \mathbb{R} \to \mathcal{F}_\alpha$ be the quotient map. Taking a Borel fundamental domain D in $\partial^2\Gamma \times \mathbb{R}$ and a measurable section $\iota : \mathcal{F}_\alpha \to D$, we have $\Lambda \otimes dt = \sum_{\gamma\in\Gamma} \gamma.\iota_* m$ (Remark 3.5). Hence for $\Lambda \otimes dt$-almost every $(\xi, \eta, s) \in \partial^2\Gamma \times \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\varphi \circ \pi)(\widetilde{\Phi}_t(\xi, \eta, s))\, dt = \varphi_\infty \circ \pi(\xi, \eta, s). \tag{10}$$

In fact, since this holds for $\Lambda \otimes dt$-almost every (ξ, η, s) and φ is continuous on a compact space $\mathcal{F}_\alpha$, the limit in equation (10) exists for Λ-almost every $(\xi, \eta) \in \partial^2\Gamma$ and for all $s \in \mathbb{R}$. Let $\widetilde{\varphi}_\infty := \varphi_\infty \circ \pi$ on $\partial^2\Gamma \times \mathbb{R}$. Since φ_∞ is $\{\Phi_t\}_{t\in\mathbb{R}}$-invariant, the lift $\widetilde{\varphi}_\infty$ is $\{\widetilde{\Phi}_t\}_{t\in\mathbb{R}}$-invariant. Therefore, $\widetilde{\varphi}_\infty$ is defined on $\partial^2\Gamma$, and we write $\widetilde{\varphi}_\infty(\xi, \eta)$ for $(\xi, \eta) \in \partial^2\Gamma$. Noting that Λ and $\mu \otimes \mu$ are mutually absolutely continuous on $\partial^2\Gamma$, we have for $\mu \otimes \mu$-almost every $(\xi, \eta) \in \partial^2\Gamma$, and all $s \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (\varphi \circ \pi)(\widetilde{\Phi}_t(\xi, \eta, s))\, dt = \widetilde{\varphi}_\infty(\xi, \eta). \tag{11}$$

Let K be any compact set in $\partial^2\Gamma$. Then there exists a constant $c_K > 0$ such that for any $(\xi, \eta), (\xi, \eta') \in K$, $\langle \xi | \eta \rangle_o, \langle \xi | \eta' \rangle_o \le c_K$. For $\xi \in \partial\Gamma$, if we take a C_δ-rough geodesic ϕ from o to ξ, then for any $r \ge 0$,

$$\widehat{b}_o(\phi(r), \xi) = -r \pm 2C_\delta.$$

For any $T > 0$, let $x := \phi(T)^{-1}$, then $\widehat{b}_o(x^{-1}, \xi) = -T \pm 2C_\delta$. Hence Lemma 3.1 implies that

$$\widehat{\rho}(x\eta, x\eta') \le e^{2c_K + 2C_\delta} \cdot e^{-T},$$

and thus for any $\varepsilon > 0$, there exists $x \in \Gamma$ depending only on ξ such that for any $(\xi, \eta), (\xi, \eta') \in K$, we have $\widehat{\rho}(x\eta, x\eta') \le \varepsilon$. Moreover, since

$$\alpha(x, \xi, \eta) = \tfrac{1}{2}(\widehat{b}_o(x^{-1}, \xi) - \widehat{b}_o(x^{-1}, \eta)) = \widehat{b}_o(x^{-1}, \xi) - \tfrac{1}{2}(\widehat{b}_o(x^{-1}, \xi) + \widehat{b}_o(x^{-1}, \eta)),$$

letting

$$s_1 := \tfrac{1}{2}(\widehat{b}_o(x^{-1}, \xi) + \widehat{b}_o(x^{-1}, \eta)) \quad \text{and} \quad s_2 := \tfrac{1}{2}(\widehat{b}_o(x^{-1}, \xi) + \widehat{b}_o(x^{-1}, \eta')),$$

we have $s_1 + \alpha(x, \xi, \eta) = s_2 + \alpha(x, \xi, \eta')$. Since $\varphi \circ \pi$ is uniformly continuous on $\partial^2\Gamma \times \mathbb{R}$, for any $\varepsilon > 0$ and for any $(\xi, \eta), (\xi, \eta') \in K$, there exists a $x \in \Gamma$ such that for any

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$t \in \mathbb{R}$,

$$|\varphi \circ \pi(x \cdot (\xi, \eta, s_1 + t)) - \varphi \circ \pi(x \cdot (\xi, \eta', s_2 + t))| \leq \varepsilon.$$

Therefore, since $\varphi \circ \pi = \varphi \circ \pi(x \cdot\ )$ for all $x \in \Gamma$, by equation (11), we obtain for $\mu \otimes \mu$-almost every $(\xi, \eta), (\xi, \eta') \in K$,

$$|\widetilde{\varphi}_\infty(\xi, \eta) - \widetilde{\varphi}_\infty(\xi, \eta')| \leq \varepsilon.$$

This shows that for $\mu \otimes \mu$-almost every $(\xi, \eta), (\xi, \eta') \in K$, we have $\widetilde{\varphi}_\infty(\xi, \eta) = \widetilde{\varphi}_\infty(\xi, \eta')$. Since $\partial^2\Gamma$ is σ-compact, for $\mu \otimes \mu$-almost every $(\xi, \eta), (\xi, \eta') \in \partial^2\Gamma$, we have $\widetilde{\varphi}_\infty(\xi, \eta) = \widetilde{\varphi}_\infty(\xi, \eta')$, i.e., $\widetilde{\varphi}_\infty(\xi, \cdot)$ is μ-almost everywhere constant for μ-almost every $\xi \in \partial\Gamma$. Changing the role of ξ and η in $\partial^2\Gamma$, we also show that $\widetilde{\varphi}_\infty(\cdot\ , \eta)$ is μ-almost everywhere constant for μ-almost every $\eta \in \partial\Gamma$. We conclude that $\widetilde{\varphi}_\infty$ is $\mu \otimes \mu$-almost everywhere constant on $\partial^2\Gamma$, and thus Λ-almost everywhere constant on $\partial^2\Gamma$. Therefore, since $\widetilde{\varphi}_\infty = \varphi_\infty \circ \pi$, the limiting function φ_∞ is m-almost everywhere constant and we obtain the claim. □

COROLLARY 3.7. *For any $d \in \mathcal{D}_\Gamma$, the corresponding Γ-invariant Radon measure Λ_d on $\partial^2\Gamma$ is ergodic with respect to the Γ-action on $\partial^2\Gamma$.*

Proof. Let A be any Γ-invariant Borel set in $\partial^2\Gamma$ and consider $A \times \mathbb{R}$. Since A is Γ-invariant, $A \times \mathbb{R}$ is (Γ, α)-invariant on $\partial^2\Gamma \times \mathbb{R}$ and thus $A \times \mathbb{R}$ defines the set $[A \times \mathbb{R}]$ in $\mathcal{F}_\alpha$. The set $[A \times \mathbb{R}]$ is $\{\Phi_t\}_{t\in\mathbb{R}}$-invariant, and it has m_d-measure 0 or 1 since m_d is ergodic by Theorem 3.6. This shows that $A \times \mathbb{R}$ has either null or co-null $\Lambda_d \otimes dt$-measure and A has either null or co-null Λ_d-measure. Therefore, Λ_d is ergodic. □

4. *Coding via automatic structures*

4.1. *Automatic structures.* Let S be a finite set of generators such that $S = S^{-1}$ in Γ. An *automaton* $(\mathcal{A}, w, S)$ is a directed graph $\mathcal{A} = (V, E, s_*)$ where s_* is a vertex called the *initial state*, together with a *labeling* $w : E \to S$ on edges by S. For a directed path $\omega = (e_0, e_1, \ldots, e_{n-1})$ in the graph $\mathcal{A}$ where the terminus of e_i is the origin of e_{i+1}, we associate a path $w(\omega)$ in the Cayley graph of (Γ, S) issuing from the identity $o, w(e_0), w(e_0)w(e_1), \ldots, w(e_0) \cdots w(e_{n-1})$. Let $w_*(\omega)$ be the terminus of the path $w(\omega)$, namely, $w_*(\omega) = w(e_0) \cdots w(e_{n-1})$.

Definition 4.1. We say that an automaton $(\mathcal{A}, w, S)$, where $\mathcal{A} = (V, E, s_*)$ and a labeling $w : E \to S$ is a *strongly Markov automatic structure* if the following hold:

(1) each vertex in V can be reached by a directed path from the initial state s_*;
(2) for every directed path ω in $\mathcal{A}$, the path $w(\omega)$ is a geodesic in the Cayley graph of (Γ, S); and
(3) the map w_* assigning the terminus of a path defines a bijection from the set of directed paths from s_* in $\mathcal{A}$ to Γ.

Every hyperbolic group admits a strongly Markov automatic structure for any finite set of generators S with $S = S^{-1}$ **[Can84, Cal13,** §7, §3.2**]**. We fix an automaton $(\mathcal{A}, w, S)$ for (Γ, S). Let Σ^* be the set of finite directed paths (not necessarily from s_*) in the

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graph $\mathcal{A}$, and Σ^+ be the set of semi-infinite paths $(e_i)_{n=0}^{\infty}$. Let $\overline{\Sigma}^+ := \Sigma^* \cup \Sigma^+$ be the set of unilateral paths. We define the map $w_* : \overline{\Sigma}^+ \to \Gamma \cup \partial\Gamma$, where to every path ω in $\overline{\Sigma}^+$, we associate a point in $\Gamma \cup \partial\Gamma$ as the terminus of geodesic segment or a geodesic ray $w(\omega)$ starting from the identity o in $\mathrm{Cay}(\Gamma, S)$.

Actually we mainly use the space of bilateral paths based on $\mathcal{A}$. We will work with the space $\overline{\Sigma}^+$ to construct an appropriate measure in §5.1.

4.2. *Bilateral paths.* Let us define Σ the space of bilateral directed paths in $\mathcal{A}$. Namely, letting $A = (A(e, e'))_{e,e' \in E}$ be the adjacency matrix (for directed edges) of $\mathcal{A}$, where $A(e, e') = 1$ if (e, e') is a directed path in $\mathcal{A}$, and 0 otherwise, we define

$$\Sigma := \{(\omega_i)_{i\in\mathbb{Z}} : A(\omega_i, \omega_{i+1}) = 1 \quad \text{for all } i \in \mathbb{Z}\}.$$

Let us define the shift σ on Σ,

$$\sigma : \Sigma \to \Sigma, \quad \sigma(\omega_i)_{i\in\mathbb{Z}} := (\omega_{i+1})_{i\in\mathbb{Z}}.$$

Then we obtain a subshift of finite type (Σ, σ) with the set of alphabets E. Let us define the metric in Σ by

$$d_\Sigma(\omega, \omega') := e^{-n} \quad \text{where } n := \sup\{k \geq 0 : \omega_i = \omega'_i \quad \text{for all } |i| \leq k\} \text{ and } e^{-\infty} = 0.$$

We consider the map assigning to a bilateral path $\omega = (\omega_i)_{i\in\mathbb{Z}}$ in Σ a pair of points $\xi_-(\omega)$ and $\xi_+(\omega)$ in $\partial\Gamma$, where $\xi_-(\omega)$ and $\xi_+(\omega)$ are the extremes of geodesic rays

$$(o, w(\omega_{-1})^{-1}, w(\omega_{-1})^{-1}w(\omega_{-2})^{-1}, \ldots) \quad \text{and} \quad (o, w(\omega_0), w(\omega_0)w(\omega_1), \ldots),$$

respectively. Abusing the notation, we denote this map by

$$w_* : \Sigma \to \partial^2\Gamma, \quad \omega \mapsto (\xi_-(\omega), \xi_+(\omega)).$$

The map w_* is continuous; although it is not injective nor surjective, it is C-to-1 for some constant $C \geq 1$.

LEMMA 4.2. *Let $\rho(\xi, \xi') = e^{-(\xi|\xi')_o}$ for a word metric d_S and $\rho_\times((\xi_-, \xi_+), (\xi'_-, \xi'_+)) = \max\{\rho(\xi_-, \xi'_-), \rho(\xi_+, \xi'_+)\}$.*

(i) *There exists a constant $L > 0$ such that*

$$\rho_\times(w_*(\omega), w_*(\omega')) \leq L d_\Sigma(\omega, \omega') \quad \textit{for all } \omega, \omega' \in \Sigma.$$

(ii) *There exists an integer $C \geq 1$ such that for every $(\xi_-, \xi_+) \in \partial^2\Gamma$, the number of pre-image $w_*^{-1}(\xi_-, \xi_+)$ is at most C.*

Proof. For any $\omega, \omega' \in \Sigma$, let $w_*(\omega) = (\xi_-, \xi_+)$ and $w_*(\omega') = (\xi'_-, \xi'_+)$. If $d_\Sigma(\omega, \omega') = e^{-n}$ for $n \geq 0$, then $(\xi_+|\xi'_+)_o \geq n - 2\delta$ and $(\xi_-|\xi'_-)_o \geq n - 2\delta$ by the δ-hyperbolicity of the Cayley graph $\mathrm{Cay}(\Gamma, S)$, and this shows (i).

Note that the δ-hyperbolicity of $\mathrm{Cay}(\Gamma, S)$ implies that two geodesics with the same extreme points are within C_δ-Hausdorff distance for some constant $C_\delta \geq 0$. For $(\xi_-, \xi_+) \in \partial^2\Gamma$, if there exists an $\omega \in \Sigma$ such that $w_*(\omega) = (\xi_-, \xi_+)$, then for any other $\omega' \in \Sigma$ with $w_*(\omega') = (\xi_-, \xi_+)$, the corresponding geodesic $w(\omega')$ in $\mathrm{Cay}(\Gamma, S)$ passes through o from

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ξ_- to ξ_+. Since ω' is a path in the automaton $\mathcal{A}$, the number of geodesic rays from o of the form $w((\omega')_{i=0,1,\dots})$ following $w(\omega)$ within C_δ-distance is at most $|B_S(o, C_\delta)|$. The same is true for geodesic rays from o of the form $w((\omega')_{i=-1,-2,\dots})$. Therefore, the number of such ω' is at most $|B_S(o, C_\delta)|^2$, and this implies (ii). □

Associated to the cocycle $\alpha : \Gamma \times \partial^2\Gamma \to \mathbb{R}$, we define a function on Σ by

$$\widetilde{\alpha} : \Sigma \to \mathbb{R} \quad \omega \mapsto \alpha(s_0^{-1}, w_*(\omega)) \quad \text{where } s_0 := w(\omega_0) \text{ for } \omega = (\omega_i)_{i\in\mathbb{Z}}.$$

LEMMA 4.3. *There exist an $N \geq 1$ and positive constants $c_1, c_2 > 0$ such that*

$$S_N\widetilde{\alpha}(\omega) := \sum_{n=0}^{N-1} \widetilde{\alpha}(\sigma^n\omega) \in [c_1, c_2] \quad \textit{for all } \omega \in \Sigma.$$

Proof. For any integer $N \geq 1$, the cocycle identity of α yields

$$\begin{aligned} S_N\widetilde{\alpha}(\omega) &= \alpha(s_0^{-1}, w_*(\omega)) + \alpha(s_1^{-1}, s_0^{-1}w_*(\omega)) + \cdots + \alpha(s_{N-1}^{-1}, s_{N-2}^{-1}\cdots s_0^{-1}w_*(\omega)) \\ &= \alpha((s_0\cdots s_{N-1})^{-1}, w_*(\omega)), \end{aligned}$$

where $s_i = w(\omega_i)$ for $i = 0, \dots, N-1$. Letting $(\xi_-, \xi_+) = w_*(\omega)$, we take a C_δ-rough geodesic $\phi_{-,+}$ from ξ_- to ξ_+ with $\widehat{d}(\phi_{-,+}(0), o) \leq C_\delta$. Then we have

$$\alpha(\phi_{-,+}(r)^{-1}, \xi_-, \xi_+) = r \pm C_\delta, \quad \text{for } r \in \mathbb{R}.$$

Since for all large enough N, there exists an $r > 2C_\delta$ such that $\widehat{d}(\phi_{-,+}(r), s_1\cdots s_N) \leq C_\delta$, we have that $\alpha((s_0\cdots s_{N-1})^{-1}, w_*(\omega)) \geq \alpha(\phi_{-,+}(r)^{-1}, \xi_-, \xi_+) - 2C_\delta$ and thus there exist $N \geq 1$ and $c > 0$ such that $S_N\widetilde{\alpha}(\omega) \geq c$ for all $\omega \in \Sigma$. Letting

$$C := \sup\{|\alpha(x, w_*\omega)| : \omega \in \Sigma,\ x \in B_S(o, N)\} < \infty,$$

we obtain $S_N\widetilde{\alpha}(\omega) \leq C$ for all $\omega \in \Sigma$. □

4.3. *Coding the flow space.* Fix an integer $N \geq 1$ in Lemma 4.3. Let

$$\mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) := (\Sigma \times \mathbb{R})/\sim,$$

where

$$(\omega, t + S_N\widetilde{\alpha}(\omega)) \sim (\sigma^N\omega, t) \quad \text{for } (\omega, t) \in \Sigma \times \mathbb{R}.$$

We endow $\mathrm{Sus}(\Sigma, S_N\widetilde{\alpha})$ with the quotient topology from $\Sigma \times \mathbb{R}$. If we define the $\mathbb{R}$-action on $\mathrm{Sus}(\Sigma, S_N\widetilde{\alpha})$ by $\sigma_t[\omega, s] := [\omega, t + s]$ for $t \in \mathbb{R}$ and $[\omega, s] \in \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha})$, then $\mathrm{Sus}(\Sigma, S_N\widetilde{\alpha})$ defines a *suspension flow* over (Σ, σ^N).

Let us define $\widetilde{\Pi} : \Sigma \times \mathbb{R} \to \partial^2\Gamma \times \mathbb{R}$ by

$$\widetilde{\Pi}(\omega, t) := (w_*(\omega), t) \quad \text{for } (\omega, t) \in \Sigma \times \mathbb{R},$$

and the map by

$$\Pi : \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) \to \mathcal{F}_\alpha, \quad [\omega, t] \mapsto [w_*(\omega), t].$$

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We shall show that this map $\Pi : \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) \to \mathcal{F}_\alpha$ is well defined. For $(\omega, t) \in \Sigma \times \mathbb{R}$, we have

$$\widetilde{\Pi}(\omega, t + S_N\widetilde{\alpha}(\omega)) = (w_*(\omega), t + S_N\widetilde{\alpha}(\omega)) \quad \text{and} \quad \widetilde{\Pi}(\sigma^N\omega, t) = (x_N^{-1} \cdot w_*(\omega), t),$$

where $x_N := w(\omega_0) \cdots w(\omega_{N-1})$. By definition of the (Γ, α)-action on $\partial^2\Gamma \times \mathbb{R}$,

$$x_N^{-1} \cdot \widetilde{\Pi}(\omega, t + S_N\widetilde{\alpha}(\omega)) = (x_N^{-1} \cdot w_*(\omega), t + S_N\widetilde{\alpha}(\omega) - \alpha(x_N^{-1}, w_*(\omega))) = \widetilde{\Pi}(\sigma^N\omega, t),$$

where we have used $S_N\widetilde{\alpha}(\omega) = \alpha(x_N^{-1}, w_*(\omega))$ in the last equality. This shows that $\widetilde{\Pi}$ defines the map $\Pi : \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) \to \mathcal{F}_\alpha$.

PROPOSITION 4.4. *The map*

$$\Pi : \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) \to \mathcal{F}_\alpha, \quad [\omega, t] \mapsto [w_*(\omega), t]$$

is continuous and satisfies that $\Pi \circ \sigma_t = \Phi_t \circ \Pi$ *for* $t \in \mathbb{R}$. *Moreover,* Π *is a surjective map and there exists a constant* $C \geq 1$ *such that the cardinality of* $\Pi^{-1}(x)$ *is at most* C *for all* $x \in \mathcal{F}_\alpha$.

Proof. By definition $\Pi[\omega, t] = [\widetilde{\Pi}(\omega, t)]$ for $(\omega, t) \in \Sigma \times \mathbb{R}$, and since $\widetilde{\Pi}$ is continuous, Π is a continuous map. Furthermore, if we define an $\mathbb{R}$-action on $\Sigma \times \mathbb{R}$ by $\widetilde{\sigma}_t(\omega, s) := (\omega, s + t)$, then $\sigma_t[\omega, s] = [\widetilde{\sigma}_t(\omega, s)]$ and

$$\widetilde{\Pi} \circ \widetilde{\sigma}_t = \widetilde{\Phi}_t \circ \widetilde{\Pi} \quad \text{for } t \in \mathbb{R}.$$

This relation descends to $\Pi \circ \sigma_t = \Phi_t \circ \Pi$ for all $t \in \mathbb{R}$.

Let us show that the map $\Pi : \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) \to \mathcal{F}_\alpha$ is surjective. For given $(\xi_-, \xi_+) \in \partial^2\Gamma$, let $\phi = (\phi(n))_{n\in\mathbb{Z}}$ be a geodesic from ξ_- to ξ_+ in the Cayley graph $\mathrm{Cay}(\Gamma, S)$ where $\phi(n) \in \Gamma$ for $n \in \mathbb{Z}$. For $n \geq 1$, let $\xi_{-n} := \phi(-n)$ and $\xi_n := \phi(n)$, for which we have $\xi_{-n} \to \xi_-$ and $\xi_n \to \xi_+$ in $\Gamma \cup \partial\Gamma$ as $n \to \infty$. The strongly Markov automatic structure $(\mathcal{A}, w)$ gives a unique geodesic path ϕ_n from ξ_{-n} to ξ_n as the image of a directed path of length $2n + 1$ in $\mathcal{A}$ from the initial state s_*. We parametrize ϕ_n so that $\phi_n : [-n, n] \to \mathrm{Cay}(\Gamma, S)$ with $\xi_{-n} = \phi_n(-n)$ and $\xi_n = \phi_n(n)$ and extend ϕ_n on $\mathbb{Z}$ in such a way that ϕ_n and ϕ coincide on $(-\infty, -n] \cup [n, \infty)$. The δ-hyperbolicity of $\mathrm{Cay}(\Gamma, S)$ implies that $d_S(\phi(i), \phi_n(i)) \leq C_\delta$ for all $i \in [-n, n]$. Hence there exists a subsequence ϕ_{n_k} such that $\phi_{n_k}(i)$ converges to a $\gamma_i \in \Gamma$ for each $i \in \mathbb{Z}$ as $n_k \to \infty$. The path $(\gamma_i)_{i\in\mathbb{Z}}$ is a geodesic from ξ_- and ξ_+, and since any finite subpath $(\gamma_i)_{-N\leq i\leq N}$ is given by the image of a directed path in $\mathcal{A}$, there exists an $\omega \in \Sigma$ such that $w(\omega_i) = \gamma_i^{-1} \cdot \gamma_{i+1}$ for all $i \in \mathbb{Z}$. Then, we have

$$w_*(\omega) = (\gamma_0^{-1}\xi_-, \gamma_0^{-1}\xi_+).$$

This shows that for an arbitrary $(\xi_-, \xi_+, t) \in \partial^2\Gamma \times \mathbb{R}$, there exist an $\omega \in \Sigma$ and a $\gamma_0 \in \Gamma$ such that $\gamma_0 \cdot (w_*(\omega), t + \alpha(\gamma_0, \gamma_0^{-1}\xi_-, \gamma_0^{-1}\xi_+)) = (\xi_-, \xi_+, t)$. Therefore,

$$\Pi[\omega, t + \alpha(\gamma_0, \gamma_0^{-1}\xi_-, \gamma_0^{-1}\xi_+)] = [\xi_-, \xi_+, t],$$

and the map $\Pi : \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) \to \mathcal{F}_\alpha$ is surjective.

Let us show that the map Π is C-to-one for some constant $C \geq 1$. Since Π is surjective, for any $[\xi_-, \xi_+, s] \in \mathcal{F}_\alpha$, we have a $[\omega, s] \in \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha})$ such that $\Pi[\omega, s] = [\xi_-, \xi_+, s]$.

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Letting

$$Z := \{(\omega, t) \in \Sigma \times \mathbb{R} : 0 \le t < S_N\widetilde{\alpha}(\omega)\},$$

we identify Z with $\mathrm{Sus}(\Sigma, S_N\widetilde{\alpha})$. Fix a unique lift $(\omega, s) \in Z$ of $[\omega, s]$. We count the number of $(\omega', s') \in Z$ such that $\widetilde{\Pi}(\omega', s')$ and $\widetilde{\Pi}(\omega, s)$ are in the same (Γ, α)-orbit in $\partial^2\Gamma \times \mathbb{R}$. Note that $\widetilde{\Pi}(Z) = \{(w_*(\omega), t) \in \partial^2\Gamma \times \mathbb{R} : (\omega, t) \in Z\}$ is included in a compact set K in $\partial^2\Gamma \times \mathbb{R}$ since $0 < S_N\widetilde{\alpha}(\omega) \le C$ by Lemma 4.3 and $\langle \xi_- | \xi_+ \rangle_o \le C'$, where $w_*(\omega) = (\xi_-, \xi_+)$ for all $\omega \in \Sigma$. If there exists a $\gamma \in \Gamma$ such that $\gamma \cdot (w_*(\omega), s) = (w_*(\omega'), s')$, then the number of such γ is at most C_K, which is a constant depending only on K since the (Γ, α)-action on $\partial^2\Gamma \times \mathbb{R}$ is properly discontinuous by Lemma 3.2. For each such $\gamma \in \Gamma$, we have

$$(w_*(\omega'), s') = (\gamma \cdot \xi_-, \gamma \cdot \xi_+, s + \alpha(\gamma, \xi_-, \xi_+)),$$

where $w_*(\omega) = (\xi_-, \xi_+)$. There exists a constant $C \ge 1$ such that the number of ω' such that $w_*(\omega') = (\gamma \cdot \xi_-, \gamma \cdot \xi_+)$ is at most C by Lemma 4.2 (ii). Therefore, the number of $(\omega', s') \in Z$ such that $\Pi[\omega', s'] = \Pi[\omega, s]$ is at most $C_K \cdot C$. We conclude that $\Pi : \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) \to \mathcal{F}_\alpha$ is $(C_K \cdot C)$-to-one. □

4.4. *Coding the measures.* Let $\Pi : \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha}) \to \mathcal{F}_\alpha$ be the map in Proposition 4.4. For the simplicity of notation, fix N and we write $r := S_N\widetilde{\alpha}$ and $\mathrm{Sus}(\Sigma, r) := \mathrm{Sus}(\Sigma, S_N\widetilde{\alpha})$.

LEMMA 4.5. *For any $\{\Phi_t\}_{t\in\mathbb{R}}$-invariant Borel probability measure m on $\mathcal{F}_\alpha$, there exists a $\{\sigma_t\}_{t\in\mathbb{R}}$-invariant Borel probability measure $\widetilde{m}$ on* $\mathrm{Sus}(\Sigma, r)$ *such that*

$$\Pi_*\widetilde{m} = m.$$

Proof. Let m be any probability measure on $\mathcal{F}_\alpha$ such that $m \circ \Phi_{-t} = m$ for all $t \in \mathbb{R}$. If we have a continuous function $f \in C(\mathcal{F}_\alpha)$, then $f \circ \Pi \in C(\mathrm{Sus}(\Sigma, r))$, and the positive linear functional

$$L(f \circ \Pi) := \int_{\mathcal{F}_\alpha} f \, dm$$

extends to a linear functional $\widetilde{L}$ on $C(\mathrm{Sus}(\Sigma, r))$ such that $|\widetilde{L}(\widetilde{f})| \le \|\widetilde{f}\|_\infty$ for all $\widetilde{f} \in C(\mathrm{Sus}(\Sigma, r))$ by the Hahn–Banach theorem. Note that $\widetilde{L}(1) = 1$, and moreover $\widetilde{L}$ is positive. Indeed, suppose that there is an $\widetilde{f}$ on $\mathrm{Sus}(\Sigma, r)$ such that $\widetilde{f} \ge 0$ and $\widetilde{L}(\widetilde{f}) < 0$, then normalizing $\widetilde{f}$ so that $0 \le \widetilde{f} \le 1$ if necessary, we have

$$\widetilde{L}(1 - \widetilde{f}) = 1 - \widetilde{L}(\widetilde{f}) > 1;$$

since $\|1 - \widetilde{f}\|_\infty \le 1$, this is a contradiction. Hence by the Riesz representation theorem there exists a Borel probability measure $\widetilde{\mu}$ on $\mathrm{Sus}(\Sigma, r)$ such that

$$\int_{\mathrm{Sus}(\Sigma,r)} f \circ \Pi \, d\widetilde{\mu} = \int_{\mathcal{F}_\alpha} f \, dm \quad \text{for } f \in C(\mathcal{F}_\alpha).$$

For all $T > 0$, let $\widetilde{\mu}_T := (1/2T) \int_{-T}^{T} (\sigma_t)_*\widetilde{\mu} \, dt$. Then there exists a sequence $T_i \to \infty$ such that $\widetilde{\mu}_{T_i}$ weak-star converges to a probability measure $\widetilde{m}$, which is $\{\sigma_t\}_{t\in\mathbb{R}}$-invariant. Since $\Pi \circ \sigma_t = \Phi_t \circ \Pi$ for all $t \in \mathbb{R}$, we have $\Pi_*\widetilde{m} = m$ as required. □

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For any real values $a < b$, we denote by $\mathrm{Leb}_{[a,b)}$ the Lebesgue measure restricted on the interval $[a, b)$ in $\mathbb{R}$. For a probability measure λ on Σ, let $w_*\lambda := \lambda \circ w_*^{-1}$ be the pushforward of λ by w_* on $\partial^2\Gamma$.

LEMMA 4.6. *For any σ^N-invariant Borel probability measure λ on Σ, and for any $T_0 > 0$, there exist a σ-invariant Borel probability measure $\widetilde{\lambda}$ on Σ and some $T \geq T_0$ such that the following inequalities hold on $\partial^2\Gamma \times \mathbb{R}$,*

$$\frac{1}{N} w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)} \leq w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)} \leq \frac{1}{N} \sum_{|\gamma|_S \leq N-1} \gamma.(w_*\lambda \otimes \mathrm{Leb}_{[-T,T)}).$$

Proof. Let $\widetilde{\lambda} := (1/N) \sum_{i=0}^{N-1} \sigma_*^i\lambda$. Then $\widetilde{\lambda}$ is a σ-invariant Borel probability measure on Σ. Since $\lambda \leq N \cdot \widetilde{\lambda}$, we have $w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)} \leq N w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)}$.

On the other hand, we note that for $\omega \in \Sigma$,

$$w_*(\sigma\omega) = (\xi_-(\sigma\omega), \xi_+(\sigma\omega)) = s_0^{-1}.(\xi_-(\omega), \xi_+(\omega)) = s_0^{-1}.w_*(\omega),$$

where $s_0 = w(\omega_0)$. For any Borel set A in $\partial^2\Gamma$ and for each $s \in S$, we have

$$\lambda(\sigma^{-1}w_*^{-1}A \cap \{w(\omega_0) = s\}) \leq \lambda(\{s^{-1}.w_*(\omega) \in A\}) = w_*\lambda(sA) = (s^{-1}).w_*\lambda(A),$$

and thus since $S = S^{-1}$, $w_*\sigma_*\lambda \leq \sum_{s\in S} s.w_*\lambda$. In fact, we have for all integer $i \geq 0$, $w_*\sigma_*^i\lambda \leq \sum_{|\gamma|_S=i} \gamma.w_*\lambda$ and

$$w_*\widetilde{\lambda} \leq \frac{1}{N} \sum_{|\gamma|_S \leq N-1} \gamma.w_*\lambda.$$

Let $T := T_0 + T'$, where

$$T' := \sup\{|\alpha(\gamma, \xi_-, \xi_+)| : |\gamma|_S \leq N-1,\ (\xi_-|\xi_+)_o \leq 4\delta\} < \infty.$$

For any compactly supported continuous function $f \geq 0$ on $\partial^2\Gamma \times \mathbb{R}$, we have

$$\begin{aligned}\int_{\partial^2\Gamma\times\mathbb{R}} f(\xi, t)\, d((\gamma.w_*\lambda) \otimes \mathrm{Leb}_{[0,T_0)}) &= \int_{\partial^2\Gamma\times\mathbb{R}} f(\gamma.\xi, t)\, d(w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)}) \\ &= \int_{\partial^2\Gamma\times\mathbb{R}} f \circ \gamma(\xi, t - \alpha(\gamma, \xi))\, d(w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)}) \\ &\leq \int_{\partial^2\Gamma\times\mathbb{R}} f \circ \gamma(\xi, t)\, d(w_*\lambda \otimes \mathrm{Leb}_{[-T,T)}),\end{aligned}$$

for all $\gamma \in \Gamma$ such that $|\gamma|_S \leq N-1$, where we have used the definition of $T = T_0 + T'$ in the last inequality. Therefore, we obtain

$$w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)} \leq \frac{1}{N} \sum_{|\gamma|_S \leq N-1} \gamma.(w_*\lambda \otimes \mathrm{Leb}_{[-T,T)}),$$

and thus we conclude the claim. □

Let us consider

$$Z := \{(\omega, t) \in \Sigma \times \mathbb{R} : 0 \leq t < r(\omega)\},$$

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and identify $\mathrm{Sus}(\Sigma, r)$ with Z and also the measures on them. Let $\mathcal{M}(\{\sigma_t\}_{t\in\mathbb{R}})$ be the set of flow $\{\sigma_t\}_{t\in\mathbb{R}}$-invariant Borel probability measures on $\mathrm{Sus}(\Sigma, r)$. There is a bijection between $\mathcal{M}(\sigma^N, \Sigma)$ and $\mathcal{M}(\{\sigma_t\}_{t\in\mathbb{R}})$,

$$\mathcal{M}(\sigma^N, \Sigma) \to \mathcal{M}(\{\sigma_t\}_{t\in\mathbb{R}}), \quad \lambda \mapsto \nu_\lambda := \frac{1}{\lambda \otimes \mathrm{Leb}(Z)} \lambda \otimes \mathrm{Leb}|_Z,$$

where Leb stands for the Lebesgue measure on $\mathbb{R}$. The reverse map is given by for $\nu \in \mathcal{M}(\{\sigma_t\}_{t\in\mathbb{R}})$ taking $\overline{\nu}$ on Σ by disintegration $\nu = \int_\Sigma \mathrm{Leb}_{[0,r(\omega))}\, d\overline{\nu}$ and assigning $\overline{\nu}(\Sigma)^{-1} \cdot \overline{\nu}$.

PROPOSITION 4.7. *There exist constants $T_0, c_1, c_2 > 0$ such that the following hold.*

(i) *For every flow $\{\Phi_t\}_{t\in\mathbb{R}}$-invariant Borel probability measure m on $\mathcal{F}_\alpha$, there exists a σ-invariant Borel probability measure $\widetilde{\lambda}$ on Σ satisfying that*

$$c_1 m \le \pi_*(w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)}) \le c_2 m,$$

where $\pi : \partial^2\Gamma \times \mathbb{R} \to \mathcal{F}_\alpha$ is the quotient map.

(ii) *For every Γ-invariant Radon measure Λ on $\partial^2\Gamma$, there exists a σ-invariant Borel probability measure $\widetilde{\lambda}$ on Σ satisfying that*

$$c_1 \Lambda \otimes dt \le \sum_{\gamma\in\Gamma} \gamma.(w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)}) \le c_2 \Lambda \otimes dt \quad \text{on } \partial^2\Gamma \times \mathbb{R}.$$

Note that for any compactly supported finite measure μ on $\partial^2\Gamma \times \mathbb{R}$, the sum $\sum_{\gamma\in\Gamma} \gamma.\mu$ is well defined since the (Γ, α)-action on $\partial^2\Gamma \times \mathbb{R}$ is properly discontinuous.

Proof. First we show (ii) by using (i). Let Λ be a Γ-invariant Radon measure on $\partial^2\Gamma$. Letting $\iota : \mathcal{F}_\alpha \to D$ be a Borel section, we have a $\{\Phi_t\}_{t\in\mathbb{R}}$-invariant finite measure m on $\mathcal{F}_\alpha$ such that

$$\Lambda \otimes dt = \sum_{\gamma\in\Gamma} \gamma.\iota_* m,$$

by Lemma 3.4 (and Remark 3.5). By (i), there exists a σ-invariant probability measure $\widetilde{\lambda}$ on Σ satisfying that $c_1 m \le \pi_*(w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)}) \le c_2 m$. Since we have that

$$\sum_{\gamma\in\Gamma} \gamma.\iota_*\pi_*(w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)}) = \sum_{\gamma\in\Gamma} \gamma.(w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)}),$$

we obtain (ii),

$$c_1 \Lambda \otimes dt \le \sum_{\gamma\in\Gamma} \gamma.(w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)}) \le c_2 \Lambda \otimes dt.$$

Next let us show (i). Let m be a $\{\Phi_t\}_{t\in\mathbb{R}}$-invariant probability measure on $\mathcal{F}_\alpha$. First we shall show that the claim holds for a σ^N-invariant probability measure λ on Σ for an integer $N \ge 1$. By Lemma 4.5, there exists a $\{\sigma_t\}_{t\in\mathbb{R}}$-invariant Borel probability measure $\widetilde{m}$ on $\mathrm{Sus}(\Sigma, r)$ such that $\Pi_*\widetilde{m} = m$, where $r = S_N\widetilde{\alpha}$ for some $N \ge 1$. Identifying $\mathrm{Sus}(\Sigma, r)$ with Z, we write the measure $\widetilde{m}$ by $(1/(\lambda \otimes \mathrm{Leb}(Z)))(\lambda \otimes \mathrm{Leb})|_Z$ where λ is a σ^N-invariant Borel probability measure on Σ. Formally, letting $\mathrm{pr} : \Sigma \times \mathbb{R} \to \mathrm{Sus}(\Sigma, r)$

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be the quotient map, we have

$$\widetilde{m} = \mathrm{pr}_* \frac{1}{\lambda \otimes \mathrm{Leb}(Z)} (\lambda \otimes \mathrm{Leb})|_Z.$$

Let $T_0 := \max_{\omega \in \Sigma} r(\omega)$ and $c_0 := \min_{\omega \in \Sigma} r(\omega)$, where we have $c_0 > 0$. Note that $c_0 \leq (\lambda \otimes \mathrm{Leb})(Z) \leq T_0$. By the definition of $\widetilde{\Pi}$, it holds that $\pi \circ \widetilde{\Pi} = \Pi \circ \mathrm{pr}$. Therefore, we have

$$m = \Pi_* \mathrm{pr}_* \frac{1}{\lambda \otimes \mathrm{Leb}(Z)} (\lambda \otimes \mathrm{Leb})|_Z = \pi_* \widetilde{\Pi}_* \frac{1}{\lambda \otimes \mathrm{Leb}(Z)} (\lambda \otimes \mathrm{Leb})|_Z.$$

By using disintegration with respect to $w_*\lambda$, we obtain

$$\widetilde{\Pi}_* (\lambda \otimes \mathrm{Leb})|_Z = \int_{\partial^2 \Gamma} l_\xi \, d(w_*\lambda)(\xi),$$

where $l_\xi = \sum_{w_*\omega = \xi} \mathrm{Leb}_{[0, r(\omega))}$ for $w_*\lambda$-almost every ξ in $\partial^2\Gamma$. Noting that the map $w_* : \Sigma \to \partial^2\Gamma$ is C-to-one for some $C \geq 1$ by Lemma 4.2 (ii), we have

$$\widetilde{\Pi}_* (\lambda \otimes \mathrm{Leb})|_Z \leq C \cdot w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)}.$$

This shows that

$$m \leq \frac{C}{c_0} \pi_* (w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)}). \tag{12}$$

We shall show the other estimate. Since $\widetilde{m}$ is $\{\sigma_t\}_{t \in \mathbb{R}}$-invariant, for any $T \geq 0$, $(T + c_0) \cdot \widetilde{m} = \int_{-c_0}^{T} (\sigma_t)_* \widetilde{m}$. Noting that $(\lambda \otimes \mathrm{Leb})|_Z \geq \lambda \otimes \mathrm{Leb}_{[0,c_0)}$, we have that for any $T \geq 0$,

$$\int_{-c_0}^{T} (\sigma_t)_* (\lambda \otimes \mathrm{Leb})|_Z \geq c_0 \cdot \lambda \otimes \mathrm{Leb}_{[0,T)} \quad \text{and} \quad \widetilde{m} \geq \frac{c_0}{T_0(T + c_0)} \mathrm{pr}_* (\lambda \otimes \mathrm{Leb}_{[0,T)}).$$

Hence by using the relation $\pi \circ \widetilde{\Pi} = \Pi \circ \mathrm{pr}$, we obtain for all $T \geq 0$,

$$\begin{aligned} m = \Pi_* \widetilde{m} &\geq \frac{c_0}{T_0(T + c_0)} \Pi_* \mathrm{pr}_* (\lambda \otimes \mathrm{Leb}_{[0,T)}) \\ &= \frac{c_0}{T_0(T + c_0)} \pi_* \widetilde{\Pi}_* (\lambda \otimes \mathrm{Leb}_{[0,T)}) \geq \frac{c_0}{T_0(T + c_0)} \pi_* (w_*\lambda \otimes \mathrm{Leb}_{[0,T)}). \end{aligned} \tag{13}$$

Finally, applying Lemma 4.6, we have a σ-invariant probability measure $\widetilde{\lambda}$ on Σ such that for some $T \geq T_0$,

$$\frac{1}{N} w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)} \leq w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)} \leq \frac{1}{N} \sum_{|\gamma|_S \leq N-1} \gamma.(w_*\lambda \otimes \mathrm{Leb}_{[-T,T)}). \tag{14}$$

By equation (12) and the first inequality of equation (14), we have $c_1 m \leq \pi_*(w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)})$. By equation (13), we have that for some positive constant $c > 0$,

$$3Tm = \int_{-2T}^{T} (\Phi_t)_* m \, dt \geq c \int_{-2T}^{T} (\Phi_t)_* \pi_* (w_*\lambda \otimes \mathrm{Leb}_{[0,T)}) \, dt \geq cT \pi_* (w_*\lambda \otimes \mathrm{Leb}_{[-T,T)}).$$

Since $\pi \circ \gamma = \pi$ for all $\gamma \in \Gamma$, by the second inequality of equation (14) we obtain

$$\pi_* (w_*\widetilde{\lambda} \otimes \mathrm{Leb}_{[0,T_0)}) \leq \frac{|B_S(o, N-1)|}{N} \pi_* (w_*\lambda \otimes \mathrm{Leb}_{[-T,T)}) \leq \frac{|B_S(o, N-1)|}{N} \cdot \frac{3}{c} m,$$

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where $B_S(o, N-1)$ is the ball of radius $N-1$ centered at o in $\mathrm{Cay}(\Gamma, S)$. We conclude the claim. □

5. *Symbolic dynamics*

Recall that associated to the cocycle $\alpha : \Gamma \times \partial^2\Gamma \to \mathbb{R}$, we have defined the function

$$\widetilde{\alpha} : \Sigma \to \mathbb{R}, \quad \omega \mapsto \alpha(s_0^{-1}, w_*(\omega)),$$

where $s_0 := w(\omega_0)$ for $\omega = (\omega_i)_{i\in\mathbb{Z}}$. Similarly, we define

$$\widetilde{b} : \Sigma \to \mathbb{R}, \quad \omega \mapsto \widehat{b}_o(s_0, \xi_+(\omega)) \quad \text{and} \quad u : \Sigma \to \mathbb{R}, \quad \omega \mapsto \langle \xi_-(\omega)|\xi_+(\omega)\rangle_o.$$

LEMMA 5.1. *The functions $\widetilde{\alpha}$, $\widetilde{b}$, and u are Hölder continuous on (Σ, d_Σ), i.e., there exist constants $L \geq 0$ and $s > 0$ such that*

$$|\widetilde{\alpha}(\omega) - \widetilde{\alpha}(\omega')| \leq L\, d_\Sigma(\omega, \omega')^s \quad \textit{for all } \omega, \omega' \in \Sigma,$$

and the same estimates hold for $\widetilde{b}$ and u. Moreover, we have

$$\widetilde{\alpha}(\omega) = -\widetilde{b}(\omega) + u(\sigma\omega) - u(\omega) \quad \textit{for all } \omega \in \Sigma.$$

Proof. For any $n \geq 0$, let $\omega, \omega' \in \Sigma$ be any two sequences satisfying that $\omega \neq \omega'$ and $d_\Sigma(\omega, \omega') = e^{-n}$. Since the Busemann function $\widehat{b}_o(x, x_n) \to \widehat{b}_o(x, \xi)$ as $x_n \to \xi$ in $\Gamma \cup \partial\Gamma$ for each $x \in \Gamma$ and $\widehat{d}$ is strongly hyperbolic, by equation (4) in §2.4, there exist constant $L \geq 0$ and $s > 0$ such that

$$|\widehat{b}_o(s_0, \xi_+(\omega)) - \widehat{b}_o(s_0, \xi_+(\omega'))| \leq Le^{-s(n-1)},$$

where $s_0 = w(\omega_0)$, and thus $|\widetilde{b}(\omega) - \widetilde{b}(\omega')| \leq Le^s e^{-n} = Le^s\, d_\Sigma(\omega, \omega')$. Moreover, the strong hyperbolicity of $\widehat{d}$ implies that

$$|\widehat{d}(x, y) - \widehat{d}(x', y) - \widehat{d}(x, o) + \widehat{d}(x', o)| \leq Le^{-sn}$$

if two geodesic segments connecting x and y, and x' and o, respectively, have a common geodesic of length n in their C_δ-neighborhoods in the Cayley graph $\mathrm{Cay}(\Gamma, S)$. Adding $\widehat{d}(y, o)$ and $-\widehat{d}(y, o)$, we have that

$$|2\langle \xi_-(\omega')|\xi_+(\omega)\rangle_o - 2\langle \xi_-(\omega)|\xi_+(\omega)\rangle_o| \leq Le^{-sn}$$

and

$$|2\langle \xi_-(\omega')|\xi_+(\omega')\rangle_o - 2\langle \xi_-(\omega')|\xi_+(\omega)\rangle_o| \leq Le^{-sn}.$$

This shows that

$$|u(\omega) - u(\omega')| \leq Le^{-sn} = L\, d_\Sigma(\omega, \omega').$$

By the definition of α and the relation between $\widehat{b}_0$ and the Gromov product, we have

$$\widetilde{\alpha}(\omega) = \alpha(s_0^{-1}, \xi_-(\omega), \xi_+(\omega)) = \tfrac{1}{2}(\widehat{b}_o(s_0, \xi_-(\omega)) - \widehat{b}_o(s_0, \xi_+(\omega)))$$

and

$$\widehat{b}_o(s_0, \xi_-(\omega)) + \widehat{b}_o(s_0, \xi_+(\omega)) = 2\langle \xi_-(\omega)|\xi_+(\omega)\rangle_{s_0} - 2\langle \xi_-(\omega)|\xi_+(\omega)\rangle_o,$$

and thus $\widetilde{\alpha}(\omega) = -\widetilde{b}(\omega) + u(\sigma\omega) - u(\omega)$. Therefore, $\widetilde{\alpha}$ is also Hölder continuous. □

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5.1. *Thermodynamic formalism.* In this section, we work on the set of unilateral paths $\overline{\Sigma}^+$. It is better suited when we construct a shift-invariant measure on the set of bilateral paths Σ by using transfer operators. We introduce thermodynamic formalism on $\overline{\Sigma}^+$ following **[Gou14,** §3.3**]**.

Let $\sigma : \overline{\Sigma}^+ \to \overline{\Sigma}^+$ be the shift defined by deleting the first edge of paths. Define the metric in $\overline{\Sigma}^+$ by $d_{\overline{\Sigma}^+}(\omega, \omega') := e^{-n}$ where $n := \sup\{k \geq 0 : \omega_i = \omega'_i,\ 0 \leq i \leq k\}$. For every real-valued Hölder continuous function $\psi : \overline{\Sigma}^+ \to \mathbb{R}$ with respect to $d_{\overline{\Sigma}^+}$, we define the transfer operator $\mathcal{L}_\psi$ acting on the space of continuous functions f on $\overline{\Sigma}^+$ by

$$\mathcal{L}_\psi f(\omega) = \sum_{\sigma(\omega')=\omega} e^{\psi(\omega')} f(\omega'),$$

where the preimages of $\omega = \emptyset$ by the shift σ are paths of length 1. We will analyze the asymptotics of

$$\mathcal{L}_\psi^n 1(\emptyset) = \sum_{\omega' \text{ of length } n} e^{S_n\psi(\omega')}, \quad \text{where } S_n\psi(\omega) := \psi(\omega) + \psi(\sigma\omega) + \cdots + \psi(\sigma^{n-1}\omega).$$

We call a finite directed graph *recurrent* when any vertex is accessible from any other vertex by a directed path, and *topologically mixing* when it is recurrent and there exists N such that for all $n \geq N$ any two vertices are connected by a directed path of length n. If the underlying graph of $\mathcal{A}$ is topologically mixing, then the Ruelle–Perron–Frobenius theorem is applicable to describe the spectra of $\mathcal{L}_\psi$ ([**Bow08**, **PP90**, Theorem 1.7, Theorem 2.2] and see also **[Gou14,** Theorem 3.6**]**). If the graph $\mathcal{A}$ is only recurrent, then there is a period $p \geq 1$; every loop has the length of a multiple of p. In the case when $p > 1$, the set of vertices V of $\mathcal{A}$ is a disjoint union of subsets V_j for $j \in \mathbb{Z}/p\mathbb{Z}$ such that any edge of the origin in V_j has the terminus in V_{j+1}. We call such a decomposition $V = \bigsqcup_{j\in\mathbb{Z}/p\mathbb{Z}} V_j$ a *cyclic decomposition*. Let $\overline{\Sigma}_j^+$ be the set of path starting at a vertex in V_j and the empty path. Then $\sigma : \overline{\Sigma}_j^+ \to \overline{\Sigma}_{j+1}^+$ and, the restriction of σ^p on $\overline{\Sigma}_j^+$ is a topological mixing subshift of finite type for each $j \in \mathbb{Z}/p\mathbb{Z}$. In the general case when $\mathcal{A}$ is not necessarily recurrent, we decompose $\mathcal{A}$ into components. A *component* of $\mathcal{A}$ is a maximal subgraph, which is recurrent. For a general hyperbolic group, a strongly Markov automatic structure $\mathcal{A}$ can have several distinct components whose cardinality is greater than 1. (There are examples such as free groups and surface groups with the standard sets of generators admitting an $\mathcal{A}$ with a unique component, which is not a singleton.)

To each component $\mathcal{C}$, we define the transfer operator $\mathcal{L}_\mathcal{C}$ by restricting ψ on the set of paths staying in $\mathcal{C}$. Then $\mathcal{L}_\mathcal{C}$ has finitely many eigenvalues of maximal modulus $\mathcal{R}_\mathcal{C}$ and they are all simple and isolated. Let $\mathrm{Pr}_\mathcal{C}(\psi, \sigma) := \log \mathcal{R}_\mathcal{C}$. (This value will be called the *pressure*.) We define

$$\mathrm{Pr}(\psi, \sigma) := \max_\mathcal{C} \mathrm{Pr}_\mathcal{C}(\psi, \sigma),$$

where the maximum is taken over all components $\mathcal{C}$ of $\mathcal{A}$. We call a component $\mathcal{C}$ *maximal* if $\mathrm{Pr}(\psi, \sigma) = \mathrm{Pr}_\mathcal{C}(\psi, \sigma)$. A potential ψ is called *semisimple* if there is no directed path between any two different maximal components.

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Let $1_{[E_*]}$ be the indicator function on $\overline{\Sigma}^+$ taking the value 1 on the set of paths starting at s_* and 0 otherwise. Then we have

$$\mathcal{L}_\psi^n 1_{[E_*]}(\emptyset) = \sum e^{S_n\psi(\omega)},$$

where the summation runs over all paths ω of length n starting at s_*. The following lemma is used to compute $\mathrm{Pr}(\psi, \sigma)$ and decide whether the potential ψ is semisimple.

LEMMA 5.2. *Let $k \geq 1$ be an integer. If there exists a path from s_* in $\mathcal{A}$ passing through successively k different maximal components, then there exists a constant $C > 0$ such that for all integers $n \geq 1$, we have*

$$\mathcal{L}_\psi^n 1_{[E_*]}(\emptyset) \geq Cn^{k-1}e^{n\mathrm{Pr}(\psi,\sigma)}.$$

Let $L \geq 1$ be an integer. If there are L components in a finite directed graph of $\mathcal{A}$, then there exists a constant $C > 0$ such that for any integers $n \geq 1$,

$$\mathcal{L}_\psi^n 1_{[E_*]}(\emptyset) \leq Cn^{L}e^{n\mathrm{Pr}(\psi,\sigma)}.$$

Proof. The first claim is a special case of **[Gou14,** Lemma 3.7**]** and the proof of the second claim is in **[Tan17,** Lemma 4.7**]**. □

Although we do not use it directly in this paper, it is instructive to record the following theorem by Gouëzel in order to understand the situation when the potential ψ is semisimple. We denote by $\|\cdot\|_{\mathrm{H}}$ the Hölder norm with some fixed exponent (whose explicit value is not used).

THEOREM 5.3. (Theorem 3.8 in **[Gou14]**) *If ψ is a semisimple potential, and $\mathcal{C}_1, \ldots, \mathcal{C}_I$ are the maximal components with period p_i, where $\mathcal{C}_i = \bigsqcup_{j\in\mathbb{Z}/p_i\mathbb{Z}} \mathcal{C}_{i,j}$ is a cyclic decomposition for each $i = 1, \ldots, I$, then there exist Hölder continuous functions $h_{i,j}$ and measures $\lambda_{i,j}$ such that $\int_{\overline{\Sigma}^+} h_{i,j}\, d\lambda_{i,j} = 1$ and for every Hölder continuous function f, and for all $n \geq 0$,*

$$\left\| \mathcal{L}_\psi^n f - e^{n\mathrm{Pr}(\psi,\sigma)} \sum_{i=1}^{I} \sum_{j=0}^{p_i-1} \left(\int_{\overline{\Sigma}^+} f\, d\lambda_{i,(j-n \bmod p_i)} \right) h_{i,j} \right\|_{\mathrm{H}} \leq C\|f\|_{\mathrm{H}} \cdot e^{-n\varepsilon_0} e^{n\mathrm{Pr}(\psi,\sigma)},$$

where $C \geq 0$ and $\varepsilon_0 > 0$ are constants independent of n and f. For each $i = 1, \ldots, I$,

$$\mu_i := \frac{1}{p_i} \sum_{j=0}^{p_i-1} h_{i,j}\lambda_{i,j}$$

is a σ-invariant ergodic probability measure, and supported on Σ_i^+.

We extend the Busemann function $\widehat{b}_o$ (denoted by the same symbol) on the whole space $\Gamma \times (\Gamma \cup \partial\Gamma)$ by setting the same as the original Busemann function for $\widehat{d}$ on $\Gamma \times \partial\Gamma$ and

$$\widehat{b}_o(x, z) := \widehat{d}(x, z) - \widehat{d}(o, z) \quad \text{for } x, z \in \Gamma.$$

Further recalling the definition $w_* : \overline{\Sigma}^+ \to \Gamma \cup \partial\Gamma$, which assigns the terminus of the corresponding geodesic segments or rays (see, right after Definition 4.1), we define

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$\widetilde{b} : \overline{\Sigma}^+ \to \mathbb{R}$ by $\widetilde{b}(\omega) := \widehat{b}_o(w(\omega_0), w_*\omega)$. Note that $\widetilde{b}$ is Hölder continuous on $\overline{\Sigma}^+$ by Lemma 5.1. We denote the exponential volume growth rate $\widehat{v} := \mathrm{gr}(\widehat{d}\,)$ relative to $\widehat{d}$.

LEMMA 5.4. *If we define the potential by* $\psi := \widehat{v} \cdot \widetilde{b}$, *then* $\mathrm{Pr}(\psi, \sigma) = 0$.

Proof. For any integer $n \geq 1$ and for all path $\omega \in \overline{\Sigma}^+$ of length n, we have

$$S_k\psi(\omega) = \widehat{v} \cdot \widehat{b}_o(s_0 \cdots s_{k-1}, w_*\omega) \quad \text{for all } 1 \leq k \leq n,$$

where $s_k = w(\omega_k)$ for $k = 0, \ldots, n-1$, and thus $S_n\psi(\omega) = -\widehat{v} \cdot \widehat{d}(o, w_*\omega)$. Therefore, we have

$$\mathcal{L}_\psi^n 1_{[E_*]}(\emptyset) = \sum_{\omega:\ \text{length } n \text{ from } s_*} e^{S_n\psi(\omega)} = \sum_{x \in \mathbf{S}_n} e^{-\widehat{v}\cdot\widehat{d}(o,x)},$$

where $\mathbf{S}_n = \{x \in \Gamma : |x|_S = n\}$. Since the last term is bounded from above and from below by some constants independent of $n \geq 1$ by Lemma 2.10, Lemma 5.2 shows that $\mathrm{Pr}(\psi, \sigma) = 0$. □

Remark 5.5. In fact, the first part of Lemma 5.2 shows that the potential $\psi = \widehat{v} \cdot \widetilde{b}$ is semisimple, which we do not exploit in this paper.

The following lemma is specific to word metrics. Let $v_S := \mathrm{gr}(d_S)$ be the exponential volume growth rate with respect to d_S.

LEMMA 5.6. *Let* $\widetilde{b}_S(\omega) \equiv -1$ *for all* $\omega \in \overline{\Sigma}^+$ *and* $\psi_S := v_S \cdot \widetilde{b}_S$. *Then* $\mathrm{Pr}(\psi_S, \sigma) = 0$.

Proof. Note that $S_k\psi_S(\omega) = -v_S \cdot k$ for all $1 \leq k \leq n$ for all $\omega \in \overline{\Sigma}^+$ of length n. The proof follows as in Lemma 5.4. □

5.2. *Variational principle.* Let ψ be a Hölder continuous function on Σ where we assume that ψ depends only on coordinates of non-negative indices, i.e., $\psi(\omega) = \psi(\omega_0, \omega_1, \ldots)$. For each component $\mathcal{C}$ of $\mathcal{A}$, let $\Sigma_\mathcal{C}$ be the set of bilateral paths all the time staying in $\mathcal{C}$. Then it holds that

$$\mathrm{Pr}_\mathcal{C}(\psi, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{[\omega_0,\ldots,\omega_{n-1}]} \exp(S_{\omega_0,\ldots,\omega_{n-1}}\psi),$$

where $S_{\omega_0,\ldots,\omega_{n-1}}\psi := \sup\{S_n\psi(\omega) : \omega \in [\omega_0, \ldots, \omega_{n-1}]\}$ for cylinder sets $[\omega_0, \ldots, \omega_{n-1}]$ in $\Sigma_\mathcal{C}$.

For any subshift of finite type (Σ, σ), we denote by $\mathcal{M}(\sigma, \Sigma)$ the set of all σ-invariant probability measures on Σ. For any $\lambda \in \mathcal{M}(\sigma, \Sigma)$, let $h(\sigma, \lambda)$ be the measure theoretical entropy of $(\Sigma, \sigma, \lambda)$ (see §6.2 for the definition).

PROPOSITION 5.7. *Let* ψ *be a Hölder continuous potential on* Σ. *Then for each component* $\mathcal{C}$ *of* $\mathcal{A}$, *we have that*

$$\mathrm{Pr}_\mathcal{C}(\psi, \sigma) = \sup_{\lambda \in \mathcal{M}(\sigma, \Sigma_\mathcal{C})} \left\{ h(\lambda, \sigma) + \int_{\Sigma_\mathcal{C}} \psi \, d\lambda \right\}, \tag{15}$$

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and there exists a unique σ-invariant Borel probability measure $\mu_{\mathcal{C}}$ on $\Sigma_{\mathcal{C}}$ for which the supremum is attained. Moreover, there exist constants $c_1, c_2 > 0$ such that

$$\frac{\mu_{\mathcal{C}}[\omega_0, \ldots, \omega_{n-1}]}{\exp(-n\mathrm{Pr}_{\mathcal{C}}(\psi, \sigma) + S_n\psi(\omega))} \in [c_1, c_2], \tag{16}$$

for all $\omega \in [\omega_0, \ldots, \omega_{n-1}]$ and for all $n \geq 1$.

Proof. For each component $\mathcal{C}$, let $\mathcal{C} = \bigsqcup_{j\in\mathbb{Z}/p_{\mathcal{C}}\mathbb{Z}} \mathcal{C}_j$ be a cyclic decomposition, where $p_{\mathcal{C}}$ is the period of $\mathcal{C}$. Let $\Sigma_{\mathcal{C},j}$ be the space of bilateral paths $(\omega_i)_{i\in\mathbb{Z}}$ staying in $\mathcal{C}$ such that the edge ω_0 starts in $\mathcal{C}_j$. Then we have that $\Sigma_{\mathcal{C}} = \bigsqcup_{j=0}^{p_{\mathcal{C}}-1} \Sigma_{\mathcal{C},j}$ and $\sigma : \Sigma_{\mathcal{C},j} \to \Sigma_{\mathcal{C},j+1}$ for each $j \in \mathbb{Z}/p_{\mathcal{C}}\mathbb{Z}$. The (two-sided) shift space $(\Sigma_{\mathcal{C},j}, \sigma^{p_{\mathcal{C}}})$ is a topologically mixing subshift of finite type. Since ψ is Hölder continuous on Σ, $S_{p_{\mathcal{C}}}\psi$ is Hölder continuous on $\Sigma_{\mathcal{C},0}$ relative to the metric d_Σ restricted to $\Sigma_{\mathcal{C},0}$. Therefore, we have that

$$\mathrm{Pr}_{\mathcal{C}_0}(S_{p_{\mathcal{C}}}\psi, \sigma^{p_{\mathcal{C}}}) = \sup_{\lambda\in\mathcal{M}(\sigma^{p_{\mathcal{C}}}, \Sigma_{\mathcal{C},0})} \left\{ h(\lambda, \sigma^{p_{\mathcal{C}}}) + \int_{\Sigma_{\mathcal{C},0}} S_{p_{\mathcal{C}}}\psi \, d\lambda \right\}, \tag{17}$$

and the supremum in equation (17) is attained by a measure if and only if it is the unique Gibbs measure $\mu_{\mathcal{C},0}$ with potential $S_{p_{\mathcal{C}}}\psi$ **[Bow08,** p. 19, Theorem 1.22**]**. In this setting, the measure $\mu_{\mathcal{C},0}$ is a unique $\sigma^{p_{\mathcal{C}}}$-invariant probability measure $\mu_{\mathcal{C},0}$ on $\Sigma_{\mathcal{C},0}$ for which there exist constants $c_1, c_2 > 0$ such that for any integer $N \geq 1$ and any cylinder set $[\omega_0, \ldots, \omega_{Np_{\mathcal{C}}-1}]$ in $\Sigma_{\mathcal{C}}$ of a path $(\omega_0, \ldots, \omega_{Np_{\mathcal{C}}-1})$ starting in $\mathcal{C}_0$, one has

$$\frac{\mu_{\mathcal{C},0}[\omega_0, \ldots, \omega_{Np_{\mathcal{C}}-1}]}{\exp\left(-Np_{\mathcal{C}} \cdot \mathrm{Pr}_{\mathcal{C}}(\psi, \sigma) + S_{Np_{\mathcal{C}}}\psi(\omega)\right)} \in [c_1, c_2], \tag{18}$$

for any $\omega \in [\omega_0, \ldots, \omega_{Np_{\mathcal{C}}-1}]$.

Note that there is a one-to-one correspondence between $\mathcal{M}(\sigma, \Sigma_{\mathcal{C}})$ and $\mathcal{M}(\sigma^{p_{\mathcal{C}}}, \Sigma_{\mathcal{C},0})$. For $\lambda \in \mathcal{M}(\sigma, \Sigma_{\mathcal{C}})$, since $\lambda(\Sigma_{\mathcal{C},j}) = 1/p_{\mathcal{C}}$ for any $j \in \mathbb{Z}/p_{\mathcal{C}}\mathbb{Z}$, letting the restriction $\lambda_0 := (p_{\mathcal{C}}\lambda)|_{\Sigma_{\mathcal{C},0}}$ we obtain $\lambda_0 \in \mathcal{M}(\sigma^{p_{\mathcal{C}}}, \Sigma_{\mathcal{C},0})$. On the other hand, for $\lambda_0 \in \mathcal{M}(\sigma^{p_{\mathcal{C}}}, \Sigma_{\mathcal{C},0})$, letting $\lambda := (1/p_{\mathcal{C}}) \sum_{j=0}^{p_{\mathcal{C}}-1} \lambda_0 \circ \sigma^{-j}$, we obtain $\lambda \in \mathcal{M}(\sigma, \Sigma_{\mathcal{C}})$. These processes are mutually invertible. Moreover, we have $\mathrm{Pr}_{\mathcal{C}_0}(S_{p_{\mathcal{C}}}\psi, \sigma^{p_{\mathcal{C}}}) = p_{\mathcal{C}} \, \mathrm{Pr}_{\mathcal{C}}(\psi, \sigma)$,

$$h(\lambda_0, \sigma^{p_{\mathcal{C}}}) = p_{\mathcal{C}} \, h(\lambda, \sigma) \quad \text{and} \quad \int_{\Sigma_{\mathcal{C},0}} S_{p_{\mathcal{C}}}\psi \, d\lambda_0 = p_{\mathcal{C}} \int_{\Sigma_{\mathcal{C}}} \psi \, d\lambda.$$

Therefore, a probability measure $\lambda \in \mathcal{M}(\sigma, \Sigma_{\mathcal{C}})$ attains the supremum in equation (15) if and only if the corresponding probability measure $\lambda_0 \in \mathcal{M}(\sigma^{p_{\mathcal{C}}}, \Sigma_{\mathcal{C},0})$ attains the supremum equation (17). If we define $\mu_{\mathcal{C}} := (1/p_{\mathcal{C}}) \sum_{j=0}^{p_{\mathcal{C}}-1} \mu_{\mathcal{C},j}$, then $\mu_{\mathcal{C}}$ is the unique σ-invariant probability measure, which attains the supremum of equation (15). Moreover, $\mu_{\mathcal{C}}$ satisfies equation (16). This follows from equation (18) and the Hölder continuity of ψ. Indeed, for any $n \geq 1$, let N be the integer with $Np_{\mathcal{C}} \leq n < (N+1)p_{\mathcal{C}}$. Taking

$$[\omega_0, \ldots, \omega_{(N+1)p_{\mathcal{C}}-1}] \subset [\omega_0, \ldots, \omega_{n-1}] \subset [\omega_0, \ldots, \omega_{Np_{\mathcal{C}}-1}],$$

we have

$$\mu_{\mathcal{C}}[\omega_0, \ldots, \omega_{n-1}] \leq \frac{c_2}{p_{\mathcal{C}}} \cdot \exp(p_{\mathcal{C}}(|\mathrm{Pr}_{\mathcal{C}}(\psi, \sigma)| + \|\psi\|_\infty)) \cdot \exp(-n\mathrm{Pr}_{\mathcal{C}}(\psi, \sigma) + S_n\psi(\omega)),$$

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for any $\omega \in [\omega_0, \dots, \omega_{n-1}]$, and

$$\mu_{\mathcal{C}}[\omega_0, \dots, \omega_{n-1}] \geq \frac{c_1}{p_{\mathcal{C}}} \cdot \exp(-p_{\mathcal{C}}(|\mathrm{Pr}_{\mathcal{C}}(\psi, \sigma)| + \|\psi\|_\infty) - C') \cdot \exp(-n\mathrm{Pr}_{\mathcal{C}}(\psi, \sigma) + S_n\psi(\omega)),$$

for any $\omega \in [\omega_0, \dots, \omega_{n-1}]$, where $|S_{(N+1)p_{\mathcal{C}}}\psi(\omega') - S_n\psi(\omega)| \leq C' + p_{\mathcal{C}}\|\psi\|_\infty$ for any $\omega' \in [\omega_0, \dots, \omega_{(N+1)p_{\mathcal{C}}-1}]$ and any $\omega \in [\omega_0, \dots, \omega_{n-1}]$ for some constant $C' \geq 0$ independent of the cylinder sets and ω. □

5.3. *Coding Patterson–Sullivan measures.* Let us take a Hölder continuous potential $\psi := \widehat{v} \cdot \widetilde{b}$ on Σ, and let $\mathcal{C}_i$, $i = 1, \dots, I$ be the maximal components of ψ. For each $i = 1, \dots, I$, let μ_i be a unique σ-invariant probability measure on $\Sigma_{\mathcal{C}_i}$ satisfying equation (16) in Proposition 5.7. Let $\widetilde{\mu} := \sum_{i=1}^{I} \mu_i$ on Σ.

LEMMA 5.8. *Let $\mu_{\widehat{d}}$ be a Patterson–Sullivan measure for $\widehat{d}$ on $\partial\Gamma$. There exists a constant $C > 0$ such that*

$$w_*\widetilde{\mu} \leq C \cdot \mu_{\widehat{d}} \otimes \mu_{\widehat{d}} \quad \text{on } \partial^2\Gamma.$$

Moreover, for any $T > 0$, the measure

$$\sum_{\gamma \in \Gamma} \gamma.(w_*\widetilde{\mu} \otimes \mathrm{Leb}_{[0,T)})$$

is Radon and absolutely continuous with respect to $\mu_{\widehat{d}} \otimes \mu_{\widehat{d}} \otimes dt$ on $\partial^2\Gamma \times \mathbb{R}$.

Remark 5.9. Note that $w_*\widetilde{\mu}$ has the support in a compact subset of $\partial^2\Gamma$ while $\mu_{\widehat{d}} \otimes \mu_{\widehat{d}}$ has the full support $\partial^2\Gamma$. We do not have the reverse inequality in the claim.

Proof. Fixing a large enough $R > 0$, we define shadows $O_o(x, R)$ on $\mathrm{Cay}(\Gamma, S)$. For any fixed $x, y \in \Gamma$, let us consider any paths $\omega \in \Sigma$ satisfying that for some $m, n \geq 0$,

$$\xi_{-m}(\omega) \in B_S(x, R) \quad \text{and} \quad \xi_n(\omega) \in B_S(y, R). \tag{19}$$

Note that if some ω in $[\omega_{-m}, \dots, \omega_n]$ satisfies equation (19) with the indices $m, n \geq 0$, then in fact every ω in the same cylinder set satisfies equation (19), and thus it is not ambiguous to say that a cylinder set $[\omega_{-m}, \dots, \omega_n]$ satisfies equation (19). For every ω in $[\omega_{-m}, \dots, \omega_{n-1}]$ with equation (19), we have

$$\widehat{b}_o(s_0 \cdots s_{n-1}, w_*\omega) = -\widehat{d}(o, s_0 \cdots s_{n-1}) \pm C_\delta$$

since $\widehat{d}$ is rough geodesic (see equation (5) in §2.4), and thus

$$S_n\psi(\omega) = \widehat{v} \cdot \widehat{b}_o(s_0 \cdots s_{n-1}, w_*\omega) = -\widehat{v} \cdot \widehat{d}(o, s_0 \cdots s_{n-1}) \pm C_\delta = -\widehat{v} \cdot \widehat{d}(o, y) \pm C_{R,\delta},$$

where in the last estimates we have used the triangle inequality, and

$$S_m\psi(\sigma^{-m}\omega) = -\widehat{v} \cdot \widehat{d}(o, s_{-m} \cdots s_{-1}) \pm C_\delta = -\widehat{v} \cdot \widehat{d}(x, o) \pm C_{R,\delta},$$

for $s_i = w(\omega_i)$ for $i = -m, \dots, n-1$. Since by Proposition 2.8,

$$\mu_{\widehat{d}}(O_o(x, R)) = \exp(-\widehat{v} \cdot \widehat{d}(o, x) \pm C),$$

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we obtain for all $\omega \in [\omega_{-m}, \dots, \omega_{n-1}]$ with equation (19),

$$\mu_{\widehat{d}}(O_o(x, R)) = \exp(S_m\psi(\sigma^{-m}\omega) \pm C'_{R,\delta}) \quad \text{and} \quad \mu_{\widehat{d}}(O_o(y, R)) = \exp(S_n\psi(\omega) \pm C'_{R,\delta}).$$

Note that for any $x, y \in \Gamma$, we have

$$\begin{aligned} w_*\widetilde{\mu}(O_o(x, R) \times O_o(y, R)) &= \widetilde{\mu}(\omega \in \Sigma : w_*(\omega) \in O_o(x, R) \times O_o(y, R)) \\ &\le \sum_{[\omega_{-m},\dots,\omega_{n-1}]} \widetilde{\mu}[\omega_{-m}, \dots, \omega_{n-1}], \end{aligned}$$

where the summation runs over all cylinder sets $[\omega_{-m}, \dots, \omega_{n-1}]$ satisfying equation (19) (where m, n also vary). Since the measures μ_i are σ-invariant and satisfy equation (16) in Proposition 5.7 and $\Pr(\psi, \sigma) = 0$ by Lemma 5.4, we have that for all $\omega \in [\omega_{-m}, \dots, \omega_{n-1}]$ with equation (19),

$$\begin{aligned} \widetilde{\mu}[\omega_{-m}, \dots, \omega_{n-1}] \le Ce^{S_{n+m}\psi(\sigma^{-m}\omega)} &= Ce^{S_m\psi(\sigma^{-m}\omega)} \cdot e^{S_n\psi(\omega)} \\ &\le C'\mu_{\widehat{d}}(O_o(x, R)) \cdot \mu_{\widehat{d}}(O_o(y, R)). \end{aligned}$$

Note that for each pair $x, y \in \Gamma$, the number of all cylinder sets $[\omega_{-m}, \dots, \omega_{n-1}]$ with equation (19) is at most $|B_S(o, R)|^2$. Hence for any $x, y \in \Gamma$, we obtain

$$w_*\widetilde{\mu}(O_o(x, R) \times O_o(y, R)) \le C'|B_S(o, R)|^2 \cdot \mu_{\widehat{d}}(O_o(x, R)) \cdot \mu_{\widehat{d}}(O_o(y, R)).$$

(Note that the left hand side may be 0 for some x and y.) Comparing shadows with balls (Lemma 2.3), we have constants $C, C' > 0$ such that for any $(\xi, \eta) \in \partial^2\Gamma$ and for any $r, s > 0$,

$$w_*\widetilde{\mu}(B_\rho(\xi, r) \times B_\rho(\eta, s)) \le C \cdot \mu_{\widehat{d}} \otimes \mu_{\widehat{d}}(B_\rho(\xi, C'r) \times B_\rho(\eta, C's)).$$

In the following discussion, we use a metric bi-Lipschitz to ρ^ε for some $\varepsilon \in (0, 1)$ in $\partial\Gamma$ (§2.1.3) and denote it by the same symbol ρ. Let us consider $\rho_\times((\xi_1, \eta_1), (\xi_2, \eta_2)) = \max\{\rho(\xi_1, \xi_2), \rho(\eta_1, \eta_2)\}$ in $\partial^2\Gamma$. For any Borel set A in $\partial^2\Gamma$ and for any $\varepsilon > 0$, we take a compact set K and an open set U in $\partial^2\Gamma$ such that both $w_*\widetilde{\mu}$ and $\mu_{\widehat{d}} \otimes \mu_{\widehat{d}}$ have their measures of $U\backslash K$ less than ε. Since K is compact, there exists an $\varepsilon_0 > 0$ such that every ball (relative to $\rho_\times$) of radius less than ε_0 centered in K is included in U. Let us take a countable family of balls $B_{\rho_\times}(x_i, r_i)$ of radius $r_i \le \varepsilon_0$ covering K such that $B_{\rho_\times}(x_i, r_i/2)$ are disjoint. Then we have that

$$w_*\widetilde{\mu}(K) \le \sum_i w_*\widetilde{\mu}(B_{\rho_\times}(x_i, r_i)) \le C \sum_i \mu_{\widehat{d}} \otimes \mu_{\widehat{d}}(B_{\rho_\times}(x_i, C'r_i)).$$

Noting that $\mu_{\widehat{d}} \otimes \mu_{\widehat{d}}$ is doubling relative to $\rho_\times$ by Lemma 2.9 (whose proof is adapted to the metric ρ) and $B_\rho(x_i, r_i/2)$ are disjoint, we obtain a constant $C' > 0$ such that the last term is at most

$$C' \sum_i \mu_{\widehat{d}} \otimes \mu_{\widehat{d}}(B_{\rho_\times}(x_i, r_i/2)) = C'\mu_{\widehat{d}} \otimes \mu_{\widehat{d}}\left(\bigcup_i B_{\rho_\times}(x_i, r_i/2) \right) \le C'\mu_{\widehat{d}} \otimes \mu_{\widehat{d}}(U).$$

Letting $\varepsilon \to 0$, we have $w_*\widetilde{\mu}(A) \le C'\mu_{\widehat{d}} \otimes \mu_{\widehat{d}}(A)$ for any Borel set A in $\partial^2\Gamma$. Since the constant C' is independent of A, we conclude the first claim.

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Let us show the second claim. Note that for any $T > 0$ and for every $\gamma \in \Gamma$, the measure $\gamma.(w_*\widetilde{\mu} \otimes \mathrm{Leb}_{[0,T)})$ is absolutely continuous with respect to $\mu_{\widehat{d}} \otimes \mu_{\widehat{d}} \otimes dt$, and the finite measure $w_*\widetilde{\mu} \otimes \mathrm{Leb}_{[0,T)}$ is supported on a compact set in $\partial^2\Gamma \times \mathbb{R}$ and the (Γ, α)-action of Γ on $\partial^2\Gamma \times \mathbb{R}$ is properly discontinuous. Therefore, for any $T > 0$ the measure $\sum_{\gamma\in\Gamma} \gamma.(w_*\widetilde{\mu} \otimes \mathrm{Leb}_{[0,T)})$ is Radon and absolutely continuous with respect to $\mu_{\widehat{d}} \otimes \mu_{\widehat{d}} \otimes dt$. □

For the potential $\psi_S = v_S \cdot \widetilde{b}_S$ as in Lemma 5.6, letting $\mathcal{C}_i$, $i = 1, \ldots, I$ be the corresponding maximal components of ψ_S, we have a unique σ-invariant Borel probability measure μ_i on $\Sigma_{\mathcal{C}_i}$ in Proposition 5.7. (In fact, this specific case allows us to describe each μ_i as the invariant measure for a Markov chain on $\mathcal{C}_i$ [**CF10,** §4.2].) Let $\widetilde{\mu}_S := \sum_{i=1}^{I} \mu_i$ on Σ. The following lemma is proved as in Lemma 5.8; we omit the proof.

LEMMA 5.10. *Let μ_S be a Patterson–Sullivan measure associated with a word metric d_S on $\partial\Gamma$. There exists a constant $C > 0$ such that*

$$w_*\widetilde{\mu}_S \le C \cdot \mu_S \otimes \mu_S \quad \text{on } \partial^2\Gamma.$$

Moreover, for any $T > 0$, the measure $\sum_{\gamma\in\Gamma} \gamma.(w_*\mu_S \otimes \mathrm{Leb}_{[0,T)})$ is Radon and absolutely continuous with respect to $\mu_S \otimes \mu_S \otimes dt$ on $\partial^2\Gamma \times \mathbb{R}$.

6. *Dimension and entropy*

6.1. *Dimension.* For each $(\xi_-, \xi_+) \in \partial^2\Gamma$, let ξ be a geodesic in the Cayley graph $\mathrm{Cay}(\Gamma, S)$ such that $\xi_{-n} \to \xi_-$ and $\xi_n \to \xi_+$ as $n \to \infty$ in $\Gamma \cup \partial\Gamma$, respectively. For any hyperbolic metric d in $\mathcal{D}_\Gamma$, we define

$$\chi(\xi_-, \xi_+; d) := \liminf_{n\to\infty} \frac{1}{2n} d(\xi_{-n}, \xi_n).$$

Note that $\chi(\xi_-, \xi_+)$ is independent of the choice of a geodesic ξ since any two geodesics with the same pair of extreme points are within a bounded distance in $\mathrm{Cay}(\Gamma, S)$ and d is quasi-isometric to a word metric. For any $d \in \mathcal{D}_\Gamma$, there exist constants $L_1, L_2 > 0$ such that

$$L_1 \le \chi(\xi_-, \xi_+; d) \le L_2 \quad \text{for all } (\xi_-, \xi_+) \in \partial^2\Gamma.$$

LEMMA 6.1. *Let $d \in \mathcal{D}_\Gamma$, and let Λ be a Γ-invariant Radon measure on $\partial^2\Gamma$. For Λ-almost every $(\xi_-, \xi_+) \in \partial^2\Gamma$, and for any geodesic ξ in $\mathrm{Cay}(\Gamma, S)$ with extreme points (ξ_-, ξ_+), the limit exists*

$$\chi(\xi_-, \xi_+; d) = \lim_{n\to\infty} \frac{1}{2n} d(\xi_{-n}, \xi_n). \tag{20}$$

Moreover, if Λ is ergodic, then $\chi(\xi_-, \xi_+; d)$ is constant Λ-almost everywhere.

Proof. For $(\xi_-, \xi_+) \in \partial^2\Gamma$, if the limit exists along a geodesic ξ in equation (20), then in fact the limit exists along any geodesics with the same pairs of extreme points. Let us take a geodesic ϕ_{ξ_-,ξ_+} in $\mathrm{Cay}(\Gamma, S)$ such that the map $(\xi_-, \xi_+) \mapsto \phi_{\xi_-,\xi_+}$ is measurable. (Choosing a lexicographic ordering in the space of geodesics endowed with the topology of pointwise convergence, we assign the minimum ϕ_{ξ_-,ξ_+} in this ordering for each (ξ_-, ξ_+);

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then this map is Borel measurable.) Then, note that the set

$$B := \left\{ (\xi_-, \xi_+) \in \partial^2\Gamma : \chi(\xi_-, \xi_+; d) = \lim_{n\to\infty} \frac{1}{2n} d(\xi_{-n}, \xi_n) \right\}$$

is Borel measurable and Γ-invariant in $\partial^2\Gamma$. Proposition 4.7 (ii) implies that for any Γ-invariant Radon measure Λ on $\partial^2\Gamma$, there exists a σ-invariant probability measure λ on Σ such that

$$c_1 \Lambda \otimes dt \le \sum_{\gamma\in\Gamma} \gamma.(w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)}) \le c_2 \Lambda \otimes dt \quad \text{on } \partial^2\Gamma \times \mathbb{R} \tag{21}$$

for some constants $c_1, c_2, T_0 > 0$. The Kingman subadditive ergodic theorem implies that there exist the limits

$$\lim_{n\to\infty} \frac{1}{n} d(\xi_0(\omega), \xi_n(\omega)) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} d(\xi_{-n}(\omega), \xi_0(\omega)) \quad \text{for } \lambda\text{-almost every } \omega \in \Sigma.$$

Since $d \in \mathcal{D}_\Gamma$ is quasi-isometric to a word metric, there exists a constant $C_{d,S} \ge 0$ such that for all $\omega \in \Sigma$ and for all $n \ge 1$,

$$0 \le d(\xi_{-n}(\omega), o) + d(o, \xi_n(\omega)) - d(\xi_{-n}(\omega), \xi_n(\omega)) = 2(\xi_{-n}(\omega)|\xi_n(\omega))_o \le C_{d,S}.$$

Therefore, the limit exists

$$\chi(\xi_-(\omega), \xi_+(\omega); d) = \lim_{n\to\infty} \frac{1}{2n} d\,(\xi_{-n}(\omega), \xi_n(\omega)) \quad \text{for } \lambda\text{-almost every } \omega \in \Sigma,$$

and thus $\lambda(w_*^{-1}B) = 1$. Since B is Γ-invariant, if we apply the set $(\partial^2\Gamma \backslash B) \times [0, 1]$ to the first inequality in equation (21), then we obtain $\Lambda(\partial^2\Gamma \backslash B) = 0$. Hence equation (20) holds for Λ-almost every $(\xi_-, \xi_+) \in \partial^2\Gamma$. Moreover, since $\chi(\xi_-, \xi_+; d)$ is Γ-invariant on $\partial^2\Gamma$, if Λ is ergodic with respect to the Γ-action, then $\chi(\xi_-, \xi_+; d)$ has to be a constant function Λ-almost everywhere on $\partial^2\Gamma$. □

LEMMA 6.2. *If we define the function $\widetilde{b}$ on Σ for the metric $\widehat{d}$, then for every σ-invariant Borel probability measure λ on Σ, we have*

$$\int_\Sigma \widetilde{b}\, d\lambda = -\int_\Sigma \chi(\xi_-(\omega), \xi_+(\omega); \widehat{d}\,)\, d\lambda.$$

Proof. Since $\widehat{d}$ is hyperbolic and quasi-isometric to a word metric, for any $\omega \in \Sigma$ and any $n \ge 1$, we have for some constants $L, C \ge 0$,

$$\langle \xi_{-n}(\omega)|\xi_n(\omega)\rangle_o \le L(\xi_{-n}(\omega)|\xi_n(\omega))_o + C,$$

(cf. **[BS00,** Proposition 5.5(1)**]**) and thus

$$S_n\widetilde{b}(\sigma^{-n}\omega) + S_n\widetilde{b}(\omega) = -\widehat{d}(o, \xi_{-n}(\omega)) - \widehat{d}(o, \xi_n(\omega)) \pm C = -\widehat{d}(\xi_{-n}(\omega), \xi_n(\omega)) \pm C'.$$

Therefore, it holds that

$$\lim_{n\to\infty} \frac{1}{2n}(S_n\widetilde{b}(\sigma^{-n}\omega) + S_n\widetilde{b}(\omega)) = -\chi(\xi_-(\omega), \xi_+(\omega); \widehat{d}\,) \quad \text{for } \lambda\text{-almost every } \omega \in \Sigma.$$

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Since $\widetilde{b}$ is bounded measurable on Σ, the Lebesgue dominated convergence theorem yields

$$\int_\Sigma \widetilde{b}\, d\lambda = -\int_\Sigma \chi(\xi_-(\omega), \xi_+(\omega); \widehat{d}\,)\, d\lambda,$$

as required. □

6.2. *Entropy.* Let X be a compact topological space, and $f : X \to X$ be a continuous map. We denote by $\mathcal{M}(f, X)$ the space of all f-invariant Borel probability measures ν on X. For any $\nu \in \mathcal{M}(f, X)$, for any finite Borel partition $\mathcal{P} = \{P_1, \dots, P_k\}$ of X, i.e., each P_i is a Borel set and $X = \bigsqcup_{i=1}^k P_i$, let us define the entropy of $\mathcal{P}$ by

$$H(\nu, \mathcal{P}) := -\sum_{i=1}^k \nu(P_i) \log \nu(P_i).$$

We define the entropy of (X, ν, f) relative to $\mathcal{P}$ by

$$h(f, \nu, \mathcal{P}) := \lim_{n\to\infty} \frac{1}{n} H\Big(\nu, \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}\Big) = \inf_{n\geq 1} \frac{1}{n} H\Big(\nu, \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}\Big),$$

where $\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P} := \mathcal{P} \vee f^{-1}\mathcal{P} \vee \cdots \vee f^{-(n-1)}\mathcal{P}$ is the partition consisting of all sets of the form

$$P_{i_1} \cap f^{-1}P_{i_2} \cap \cdots \cap f^{-(n-1)}P_{i_n}, \quad i_1, \dots, i_n \in \{1, \dots, k\},$$

and the limit exists by the subadditivity of the function $n \mapsto H(\nu, \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P})$. The *measure theoretical entropy* of f for (X, ν) is defined by

$$h(f, \nu) := \sup_{\mathcal{P}} h(f, \nu, \mathcal{P}),$$

where the supremum is taken over all finite Borel partitions $\mathcal{P}$. For the suspension flow $\{\sigma_t\}_{t\in\mathbb{R}}$ on $\mathrm{Sus}(\Sigma, r)$, we consider the time one map σ_1 and consider $h(\sigma_1, \nu)$ for a flow invariant probability measure ν on $\mathrm{Sus}(\Sigma, r)$.

For any Radon measure Λ on $\partial^2\Gamma$, let

$$D_\rho(\xi_-, \xi_+; \Lambda) := \liminf_{r\to 0} \frac{\log \Lambda(B_\rho(\xi_-, r) \times B_\rho(\xi_+, r))}{\log r} \quad \text{for } (\xi_-, \xi_+) \in \partial^2\Gamma.$$

LEMMA 6.3. *Let $d \in \mathcal{D}_\Gamma$ and ρ be the associated gauge in $\partial\Gamma$. Let Λ be a Γ-invariant Radon measure on $\partial^2\Gamma$, and λ be a σ-invariant Borel probability measure on Σ such that for a constant $C > 0$, we have*

$$w_*\lambda \leq C \cdot \Lambda \quad \text{on } \partial^2\Gamma.$$

If $D_\rho(\xi_-, \xi_+; \Lambda) \geq D$ for Λ-almost every (ξ_-, ξ_+) in $\partial^2\Gamma$, then we have that

$$h(\sigma, \lambda) + \frac{D}{2}\int_\Sigma \widetilde{b}\, d\lambda \geq 0.$$

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Proof. We identify $\mathrm{Sus}(\Sigma, r)$ with $Z = \{(\omega, t) \in \Sigma \times \mathbb{R} : 0 \le t < r(\omega)\}$. Letting W_N be the set of paths $(\omega_0, \dots, \omega_{N-1})$ of length N in $\mathcal{A}$, we decompose Z into

$$\{(\omega, t) \in Z : \omega \in [\omega_0, \dots, \omega_{N-1}],\ 0 \le t < r(\omega)\}$$

for each cylinder set $[\omega_0, \dots, \omega_{N-1}]$ corresponding to the path in W_N and define $\mathcal{P}$ the resulting partition of Z. Thereby in $\mathrm{Sus}(\Sigma, r)$ we obtain the partition, which we denote by the same symbol $\mathcal{P}$. Let us fix $0 < \varepsilon_0 < \min_{\omega\in\Sigma} r(\omega)$, and $\sigma_* := \sigma_{\varepsilon_0}$ for the flow $\{\sigma_t\}_{t\in\mathbb{R}}$ on $\mathrm{Sus}(\Sigma, r)$. We consider $\sigma_*^n\mathcal{P} \vee \cdots \vee \sigma_*^{-n}\mathcal{P}$.

For any $[\omega, t] \in \mathrm{Sus}(\Sigma, r)$ and for any integer $n \ge 1$, let us denote by $\mathcal{P}_{[-n,n]}([\omega, t])$ the set in $\sigma_*^n\mathcal{P} \vee \cdots \vee \sigma_*^{-n}\mathcal{P}$ containing $[\omega, t]$. Fix the corresponding point $(\omega, t) \in Z$. Then for $w_*(\omega) = (\xi_-(\omega), \xi_+(\omega)) \in \partial^2\Gamma$, take a C-rough geodesic ϕ in (Γ, d) with $d(\phi(0), o) \le C$ such that $\phi(-n) \to \xi_-(\omega)$ and $\phi(n) \to \xi_+(\omega)$ as $n \to \infty$, respectively (Lemma 2.2). Note that ϕ and $\xi(\omega)$ are within a bounded distance, where $\xi(\omega)$ is a geodesic in $\mathrm{Cay}(\Gamma, S)$ since d and d_S are quasi-isometric. Fixing a large enough $R > 0$, we have that

$$[\omega', t'] \in \mathcal{P}_{[-n,n]}([\omega, t]) \implies w_*(\omega') \in O_o(\phi(t - \varepsilon_0 n), R) \times O_o(\phi(t + \varepsilon_0 n), R).$$

This follows since $\xi(\omega)$ and $\xi(\omega')$ coincides at the coordinates $i = -Nm', \dots, Nm$, where $m := \max\{k \ge 0 : S_k r(\omega) \le t + \varepsilon_0 n\}$ and $m' := \min\{k \ge 0 : S_k r(\sigma^{-kN}\omega) \ge -t + \varepsilon_0 n\}$.

By the assumption we have $w_*\lambda \le C \cdot \Lambda$. Let $\nu_\lambda := (1/(\lambda \otimes \mathrm{Leb}(Z)))(\lambda \otimes \mathrm{Leb})|_Z$. Then we have that for $T_0 := \sup_{\omega\in\Sigma} r(\omega)$,

$$\begin{aligned}\nu_\lambda(\mathcal{P}_{[-n,n]}([\omega, t])) &\le \nu_\lambda(w_*(\omega') \in O_o(\phi(t - \varepsilon_0 n), R) \times O_o(\phi(t + \varepsilon_0 n), R), |t - t'| \le T_0)\\ &\le C \cdot \Lambda(O_o(\phi(t - \varepsilon_0 n), R) \times O_o(\phi(t + \varepsilon_0 n), R)) \cdot (2T_0).\end{aligned}$$

Comparing shadows with balls, we have that for all $n \ge 1$,

$$O_o(\phi(t - \varepsilon_0 n), R) \times O_o(\phi(t + \varepsilon_0 n), R) \subset B_\rho(\xi_-(\omega), C_R e^{-\varepsilon_0 n}) \times B_\rho(\xi_+(\omega), C_R e^{-\varepsilon_0 n}),$$

we obtain

$$\liminf_{n\to\infty} -\frac{1}{n} \log \nu_\lambda(\mathcal{P}_{[-n,n]}([\omega, t])) \ge \varepsilon_0 \cdot D_\rho(\xi_-(\omega), \xi_+(\omega); \Lambda).$$

Noting that λ is $\{\sigma_t\}_{t\in\mathbb{R}}$-invariant, by the Fatou lemma we obtain

$$\begin{aligned}h(\sigma_*, \nu_\lambda, \mathcal{P}) &= \lim_{n\to\infty} \frac{1}{2n} H\Big(\nu_\lambda, \bigvee_{i=0}^{2n+1} \sigma_*^{-i}\mathcal{P}\Big)\\ &= \lim_{n\to\infty} \frac{1}{2n} H\Big(\nu_\lambda, \bigvee_{i=-n}^{n} \sigma_*^{-i}\mathcal{P}\Big)\\ &= \liminf_{n\to\infty} -\frac{1}{2n} \int_{\mathrm{Sus}(\Sigma,r)} \log \nu_\lambda\big(\mathcal{P}_{[-n,n]}([\omega, t])\big)\, d\nu_\lambda\\ &\ge \frac{\varepsilon_0}{2} \int_{\mathrm{Sus}(\Sigma,r)} D_\rho(\xi_-(\omega), \xi_+(\omega); \Lambda)\, d\nu_\lambda.\end{aligned}$$

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We have that $h(\sigma_*, \nu_\lambda) \geq h(\sigma_*, \nu_\lambda, \mathcal{D})$ by definition of the measure theoretic entropy. Since $w_*\lambda \leq C \cdot \Lambda$, if $D_\rho(\xi_-, \xi_+; \Lambda) \geq D$ for Λ-almost every (ξ_-, ξ_+), then $D_\rho(\xi_-, \xi_+; \Lambda) \geq D$ for $w_*\lambda$-almost every (ξ_-, ξ_+) in $\partial^2\Gamma$, and thus we have $h(\sigma_*, \nu_\lambda) \geq \varepsilon_0 D/2$.

Then we apply the Abramov formula and its consequence (e.g., **[FH18,** Theorem 4.1.4 and Corollary 4.1.10**]**)

$$h(\sigma^N, \lambda) = h(\sigma_1, \nu_\lambda) \cdot \int_\Sigma r \, d\lambda \quad \text{and} \quad h(\sigma_{\varepsilon_0}, \nu_\lambda) = \varepsilon_0 \cdot h(\sigma_1, \nu_\lambda) \quad \text{for } \lambda \in \mathcal{M}(\sigma^N, \Sigma).$$

Note that λ is σ-invariant on Σ; we have $h(\sigma^N, \lambda) = N \cdot h(\sigma, \lambda)$, and $\int_\Sigma r \, d\lambda = N \int_\Sigma \widetilde{\alpha} \, d\lambda$. Moreover, by Lemma 5.1, we have that $\widetilde{\alpha} = -\widetilde{b} + u \circ \sigma - u$ and thus $\int_\Sigma \widetilde{\alpha} \, d\lambda = -\int_\Sigma \widetilde{b} \, d\lambda$. Therefore, we obtain

$$h(\sigma, \lambda) \geq -\frac{D}{2} \cdot \int_\Sigma \widetilde{b} \, d\lambda,$$

as required. □

6.3. *Invariant Radon measures of maximal dimensions.*

LEMMA 6.4. *Let λ be a σ-invariant Borel probability measure on Σ. For any component $\mathcal{C}$ of the underlying graph in the automatic structure $\mathcal{A}$, letting $\Sigma_\mathcal{C}$ be the set of bilateral paths all the time staying in $\mathcal{C}$, we have that*

$$\lambda\left(\Sigma \setminus \bigsqcup_\mathcal{C} \Sigma_\mathcal{C}\right) = 0.$$

Proof. Let us denote by $\Sigma_{\to,\mathcal{C}}$ the set of bilateral paths eventually staying in $\mathcal{C}$ (and never leaving $\mathcal{C}$). We note that

$$\Sigma = \bigsqcup_\mathcal{C} \Sigma_{\to,\mathcal{C}},$$

since there is no loop in the components graph (which is a directed graph obtained by identifying each component $\mathcal{C}$ in $\mathcal{A}$ with a point). The set $\Sigma_{\to,\mathcal{C}}$ is σ-invariant and $\Sigma_{\to,\mathcal{C}}$ are disjoint for different $\mathcal{C}$. For any integer K, if we define

$$\Sigma_{K,\mathcal{C}} := \{\omega \in \Sigma : \omega_i \text{ is an edge in } \mathcal{C} \text{ for all } i \geq K\},$$

then we have $\Sigma_{\to,\mathcal{C}} = \bigcup_{K=-\infty}^{\infty} \Sigma_{K,\mathcal{C}}$ and $\Sigma_\mathcal{C} = \bigcap_{K=-\infty}^{\infty} \Sigma_{K,\mathcal{C}}$. Note that $\Sigma_{K,\mathcal{C}} \subset \Sigma_{K+1,\mathcal{C}} \subset \sigma^{-1}\Sigma_{K,\mathcal{C}}$, and $\Sigma_{\to,\mathcal{C}} \setminus \Sigma_\mathcal{C} = \bigsqcup_{K=-\infty}^{\infty} (\Sigma_{K+1,\mathcal{C}} \setminus \Sigma_{K,\mathcal{C}})$. Hence for any σ-invariant probability measure λ on $\Sigma_{\to,\mathcal{C}}$, we have that for each $K \in \mathbb{Z}$,

$$\lambda(\Sigma_{K+1,\mathcal{C}} \setminus \Sigma_{K,\mathcal{C}}) \leq \lambda(\sigma^{-1}\Sigma_{K,\mathcal{C}}) - \lambda(\Sigma_{K,\mathcal{C}}) = 0,$$

and thus $\lambda(\Sigma_{\to,\mathcal{C}} \setminus \Sigma_\mathcal{C}) = 0$ for each $\mathcal{C}$. Therefore, every σ-invariant probability measure λ on $\Sigma = \bigsqcup_\mathcal{C} \Sigma_{\to,\mathcal{C}}$ is supported on $\bigsqcup_\mathcal{C} \Sigma_\mathcal{C}$. □

THEOREM 6.5. *Let Γ be a non-elementary hyperbolic group and consider a strongly hyperbolic metric $\widehat{d} \in \mathcal{D}_\Gamma$ in Γ and the associated gauge $\widehat{\rho}$ in $\partial\Gamma$. For any Γ-invariant Radon measure Λ on $\partial^2\Gamma$, if $\underline{\dim}_H(\Lambda, \widehat{\rho}_\times) = \dim_H(\partial^2\Gamma, \widehat{\rho}_\times)$, then Λ is a constant multiple of $\Lambda_{\widehat{d}}$.*

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Proof. Let us denote by $\widehat{v} := \mathrm{gr}(\widehat{d}\,)$ the exponential volume growth rate relative to $\widehat{d}$. For any Γ-invariant Radom measure Λ on $\partial^2\Gamma$, there exists a σ-invariant probability measure λ on Σ such that

$$c_1\Lambda \otimes dt \le \sum_{\gamma\in\Gamma} \gamma.(w_*\lambda \otimes \mathrm{Leb}_{[0,T_0)}) \le c_2\Lambda \otimes dt,$$

for some constants $c_1, c_2, T_0 > 0$ by Proposition 4.7 (ii). Lemma 6.4 implies that λ is supported on $\bigsqcup_{\mathcal{C}} \Sigma_{\mathcal{C}}$. Letting $\psi = \widehat{v}\cdot\widetilde{b}$ and $\lambda = \sum_{\mathcal{C}} a_{\mathcal{C}}\cdot\lambda_{\mathcal{C}}$, where $a_{\mathcal{C}} \ge 0$ with $\sum_{\mathcal{C}} a_{\mathcal{C}} = 1$, and $\lambda_{\mathcal{C}}$ is a σ-invariant probability measure supported on $\Sigma_{\mathcal{C}}$ for each component $\mathcal{C}$, we have

$$h(\sigma,\lambda) + \int_\Sigma \psi\, d\lambda = \sum_{\mathcal{C}} a_{\mathcal{C}}\left(h(\sigma,\lambda_{\mathcal{C}}) + \int_\Sigma \psi\, d\lambda_{\mathcal{C}}\right).$$

The assumption $\underline{\dim}_H(\Lambda, \widehat{\rho}_\times) = 2\widehat{v}$ implies that $D_{\widehat{\rho}}(\xi_-,\xi_+;\Lambda) \ge 2\widehat{v}$ for Λ-almost every (ξ_-,ξ_+) in $\partial^2\Gamma$. Hence Lemma 6.3 yields

$$h(\sigma,\lambda) + \int_\Sigma \psi\, d\lambda = h(\sigma,\lambda) + \widehat{v}\int_\Sigma \widetilde{b}\, d\lambda \ge 0.$$

The variational principle (Proposition 5.7) implies that for each $\mathcal{C}$,

$$\mathrm{Pr}_{\mathcal{C}}(\psi,\sigma) \ge h(\sigma,\lambda_{\mathcal{C}}) + \int_\Sigma \psi\, d\lambda_{\mathcal{C}}, \tag{22}$$

and this implies that

$$\mathrm{Pr}(\psi,\sigma) = \max_{\mathcal{C}} \mathrm{Pr}_{\mathcal{C}}(\psi,\sigma) \ge \sum_{\mathcal{C}} a_{\mathcal{C}}\cdot\mathrm{Pr}_{\mathcal{C}}(\psi,\sigma) \ge h(\sigma,\lambda) + \int_\Sigma \psi\, d\lambda \ge 0.$$

For the potential $\psi = \widehat{v}\cdot\widetilde{b}$, we have $\mathrm{Pr}(\psi,\sigma) = \max_{\mathcal{C}} \mathrm{Pr}_{\mathcal{C}}(\psi,\sigma) = 0$ by Lemma 5.4. Therefore, we obtain

$$\sum_{\mathcal{C}} a_{\mathcal{C}}\cdot\mathrm{Pr}_{\mathcal{C}}(\psi,\sigma) = 0 \quad\text{and}\quad \mathrm{Pr}_{\mathcal{C}}(\psi,\sigma) \le 0 \quad \text{for all } \mathcal{C}.$$

If $\mathcal{C}$ is not maximal for ψ, then $\mathrm{Pr}_{\mathcal{C}}(\psi,\sigma) < 0$ and thus $a_{\mathcal{C}} = 0$ for all component $\mathcal{C}$, which is not maximal for ψ. Hence λ is supported on $\bigsqcup_{i\in I} \Sigma_{\mathcal{C}_i}$ where $\mathcal{C}_i$ for $i \in I$ are maximal components for ψ. If $a_{\mathcal{C}_i} > 0$ for some $i \in I$, then for such an $i \in I$, we have that

$$\mathrm{Pr}_{\mathcal{C}_i}(\psi,\sigma) = h(\sigma,\lambda_{\mathcal{C}_i}) + \int_\Sigma \psi\, d\lambda_{\mathcal{C}_i}.$$

Since for each $\mathcal{C}_i$, there exists a unique σ-invariant probability measure μ_i, which attains the equality in equation (22) (Proposition 5.7), $\lambda_{\mathcal{C}_i} = \mu_i$. This shows that $\lambda = \sum_{\mathcal{C}} a_{\mathcal{C}}\cdot\lambda_{\mathcal{C}}$ is absolutely continuous with respect to $\widetilde{\mu} = \sum_{i\in I}\mu_i$, and thus $w_*\lambda$ is absolutely continuous with respect to $w_*\widetilde{\mu}$. By Lemma 5.8, we know that $w_*\widetilde{\mu}$ is absolutely continuous with respect to $\Lambda_{\widehat{d}}$. Therefore, $\Lambda\otimes dt$ is absolutely continuous with respect to $\Lambda_{\widehat{d}}\otimes dt$, which shows that Λ is absolutely continuous with respect to $\Lambda_{\widehat{d}}$. Note that $d\Lambda/d\Lambda_{\widehat{d}}$ is locally integrable since Λ and $\Lambda_{\widehat{d}}$ are Radon and Γ-invariant. This shows that $d\Lambda/d\Lambda_{\widehat{d}} = c$ for a constant c almost everywhere relative to $\Lambda_{\widehat{d}}$ since $\Lambda_{\widehat{d}}$ is ergodic with

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respect to the Γ-action on $\partial^2\Gamma$ by Corollary 3.7. Therefore, we obtain $\Lambda = c \cdot \Lambda_{\widehat{d}}$ for some $c > 0$ as desired. □

7. *Mean distortion for word metrics*

7.1. *Word metrics.* Let S be a finite set of generators S with $S = S^{-1}$. In this section, we focus on a word metric d_S and denote by ρ_S the associated gauge in $\partial\Gamma$. Let us consider the two-sided shift space (Σ, σ) based on an automatic structure $(\mathcal{A}, w, S)$ with respect to S.

LEMMA 7.1. *Let Λ be a Γ-invariant Radon measure on $\partial^2\Gamma$, and λ be a σ-invariant Borel probability measure on Σ such that for a constant $C > 0$, we have*

$$w_*\lambda \le C \cdot \Lambda \quad \text{on } \partial^2\Gamma.$$

If $D_{\rho_S}(\xi_-, \xi_+; \Lambda) \ge D$ for Λ-almost every (ξ_-, ξ_+) in $\partial^2\Gamma$, then we have that

$$h(\sigma, \lambda) \ge \frac{D}{2}.$$

Proof. The proof runs similarly as in Lemma 6.3. For any integer $n \ge 1$, let $[\omega_{-n}, \dots, \omega_n]$ be any cylinder set of Σ. Fix a large enough $R > 0$ and we define shadows $O_o(x, R)$ on $\mathrm{Cay}(\Gamma, S)$. If $\omega \in [\omega_{-n}, \dots, \omega_n]$, then

$$w_*(\omega) \in O_o(\xi_{-n}(\omega), R) \times O_o(\xi_n(\omega), R).$$

Therefore, for any $[\omega_{-n}, \dots, \omega_n]$, we have that

$$\begin{aligned}\lambda[\omega_{-n}, \dots, \omega_n] &\le \lambda \circ w_*^{-1}(O_o(\xi_{-n}(\omega), R) \times O_o(\xi_n(\omega), R)) \\ &= (w_*\lambda)(O_o(\xi_{-n}(\omega), R) \times O_o(\xi_n(\omega), R)).\end{aligned}$$

By the assumption that $w_*\lambda \le C \cdot \Lambda$, the last term is at most

$$C \cdot \Lambda(O_o(\xi_{-n}(\omega), R) \times O_o(\xi_n(\omega), R)).$$

Comparing shadows with balls (Lemma 2.3), we have for any $n \ge 1$ and for any $\omega \in \Sigma$,

$$O_o(\xi_{-n}(\omega), R) \subset B_{\rho_S}(\xi_-(\omega), C_R e^{-n}) \quad \text{and} \quad O_o(\xi_n(\omega), R) \subset B_{\rho_S}(\xi_+(\omega), C_R e^{-n}),$$

for some constant $C_R > 0$ depending only on R, and for any $\omega \in \Sigma$,

$$\begin{aligned}\liminf_{n\to\infty} -\frac{1}{2n} \log \lambda[\omega_{-n}, \dots, \omega_n] &\ge \liminf_{n\to\infty} -\frac{1}{2n} \log \Lambda(B_{\rho_S}(\xi_-(\omega), C_R e^{-n}) \\ &\quad \times B_{\rho_S}(\xi_+(\omega), C_R e^{-n})) \\ &= D_{\rho_S}(\xi_-(\omega), \xi_+(\omega)).\end{aligned}$$

Let $\mathcal{P} := \{\mathcal{P}_e\}_{e\in E}$ where $\mathcal{P}_e := \{\omega \in \Sigma : \omega_0 = e\}$ is the partition of Σ according to the set of alphabets E. By the definition of measure theoretic entropy, since λ is σ-invariant, we have

$$\begin{aligned}h(\sigma, \lambda, \mathcal{P}) &= \lim_{n\to\infty} \frac{1}{2n} H\left(\sigma, \lambda, \bigvee_{i=0}^{2n} \sigma^{-i}\mathcal{P}\right) \\ &= \lim_{n\to\infty} -\frac{1}{2n} \sum_{[\omega_{-n},\dots,\omega_n]} \lambda[\omega_{-n}, \dots, \omega_n] \log \lambda[\omega_{-n}, \dots, \omega_n],\end{aligned}$$

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and by the Fatou lemma,

$$\liminf_{n\to\infty} -\frac{1}{2n} \sum_{[\omega_{-n},\dots,\omega_n]} \lambda[\omega_{-n},\dots,\omega_n] \log[\omega_{-n},\dots,\omega_n]$$
$$\geq \int_\Sigma \liminf_{n\to\infty} -\frac{1}{2n} \log \lambda[\omega_{-n},\dots,\omega_n] d\lambda.$$

Hence we obtain

$$h(\sigma,\lambda) \geq \int_\Sigma D_{\rho_S}(\xi_-(\omega),\xi_+(\omega))\, d\lambda \geq \frac{D}{2}$$

if $D_{\rho_S}(\xi_-,\xi_+) \geq D/2$ for Λ-almost every (ξ_-,ξ_+) in $\partial^2\Gamma$. □

Let $\mathrm{gr}(S) := \mathrm{gr}(d_S)$ be the exponential volume growth rate with respect to d_S and Λ_S be the corresponding Bowen–Margulis current for d_S. Note that $\dim_H(\partial^2\Gamma, \rho_{S,\times}) = 2\mathrm{gr}(S)$.

THEOREM 7.2. *Let Γ be a non-elementary hyperbolic group, d_S be a word metric on Γ and ρ_S be the associated gauge in $\partial\Gamma$. For any Γ-invariant Radon measure Λ on $\partial^2\Gamma$, if $\underline{\dim}_H(\Lambda, \rho_{S,\times}) = 2\mathrm{gr}(S)$, then Λ is a constant multiple of Λ_S.*

Proof. The proof is analogous to Theorem 6.5 and we indicate the place specific to word metrics. Let $v_S := \mathrm{gr}(S)$. For any Γ-invariant Radom measure Λ on $\partial^2\Gamma$, we take a σ-invariant probability measure λ on Σ in Proposition 4.7 and λ is supported on $\bigsqcup_{\mathcal{C}} \Sigma_{\mathcal{C}}$ by Lemma 6.4. Let $\psi_S := v_S \cdot \widetilde{b}_S$ where $\widetilde{b}_S \equiv -1$. The assumption $\underline{\dim}_H(\Lambda, \rho_{S,\times}) = 2v_S$ implies that $D_{\rho_S}(\xi_-,\xi_+;\Lambda) \geq 2v_S$ for Λ-almost every (ξ_-,ξ_+) in $\partial^2\Gamma$. Hence Lemma 7.1 implies that $h(\sigma,\lambda) \geq v_S$, and thus

$$h(\sigma,\lambda) + \int_\Sigma \psi_S\, d\lambda \geq 0.$$

For the potential $\psi_S = v_S \cdot \widetilde{b}_S$, we have $\Pr(\psi_S,\sigma) = \max_{\mathcal{C}} \Pr_{\mathcal{C}}(\psi_S,\sigma) = 0$ by Lemma 5.6. We use the measure $\widetilde{\mu}_S$ on Σ in Lemma 5.10. The rest follows similarly as in Theorem 6.5. □

LEMMA 7.3. *Let S and $S^\star$ be finite symmetric sets of generators in Γ. There exists a constant $\tau^\star$ such that for μ_S-almost every point ξ in $\partial\Gamma$ and for any geodesic ray ξ_n in* $\mathrm{Cay}(\Gamma, S)$ *converging to ξ, we have*

$$\lim_{n\to\infty} \frac{1}{n} d_{S^\star}(o,\xi_n) = \tau^\star,$$

where μ_S is a Patterson–Sullivan measure for d_S.

Proof. Let Λ_S be the Bowen–Margulis current on $\partial^2\Gamma$ associated with d_S. Then we have that for Λ_S-almost every $(\xi_-,\xi_+) \in \partial^2\Gamma$ and for any geodesic ξ in $\mathrm{Cay}(\Gamma, S)$ with extreme points (ξ_-,ξ_+), the limit exists

$$\chi^+(\xi_-,\xi_+;d_{S^\star}) := \lim_{n\to\infty} \frac{1}{n} d_{S^\star}(o,\xi_n).$$

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Indeed, if this convergence holds for some geodesic ray toward ξ, then in fact it holds for any geodesic ray toward ξ since any two geodesic rays converging to the same extreme point are eventually within bounded distance, and this shows that the set where this limit exits is Γ-invariant; the rest follows as in Lemma 6.1. Since Λ_S is ergodic with respect to Γ-action on $\partial^2\Gamma$ by Corollary 3.7, this $\chi^+(\xi_-, \xi_+; d_{S^\star})$ is a constant function Λ_S-almost everywhere on $\partial^2\Gamma$. Note that Λ_S is equivalent to $\mu_S \otimes \mu_S$ and the limit does not depend on the choice of geodesic ξ converging to ξ_+. Letting the constant $\tau^\star = \chi^+(\xi_-, \xi_+; d_S)$, we obtain the claim. □

7.2. *Mean distortion.* Recall that for any pair of finite symmetric sets of generators S and $S^\star$ in Γ, we have defined

$$\tau(S^\star/S) := \liminf_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathrm{Unif}_{S,n}}|x_n|_{S^\star},$$

where x_n has the uniform distribution $\mathrm{Unif}_{S,n}$ on $\mathbf{S}_n = \{x \in \Gamma : |x|_S = n\}$. We shall show that the liminf is actually the limit, and the weak law of large number holds.

THEOREM 7.4. *Let Γ be a non-elementary hyperbolic group. For any pair of finite symmetric sets of generators S and $S^\star$ in Γ, we have that*

$$\tau(S^\star/S) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_{\mathrm{Unif}_{S,n}}|x_n|_{S^\star},$$

and for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{Unif}_{S,n}\left\{x \in \mathbf{S}_n : \frac{|d_{S_*}(o, x) - n\tau(S^\star/S)|}{n} > \varepsilon\right\} = 0.$$

Proof. By Lemma 7.3, there exists a constant $\tau^\star$ such that for μ_S-almost every ξ in $\partial\Gamma$ and for any geodesic ray ξ_n converging to ξ in $\mathrm{Cay}(\Gamma, S)$, we have $d_{S^\star}(o, \xi_n)/n \to \tau^\star$ as $n \to \infty$. For any $\varepsilon > 0$, let

$$A_n := \left\{x \in \mathbf{S}_n : \frac{|d_{S_*}(o, x) - n\tau^\star|}{n} > \varepsilon\right\}.$$

Fix a large enough $R > 0$, and define shadows $O_o(x, R)$ in $\mathrm{Cay}(\Gamma, S)$. We have that $c_1 \le \mu_S(O_o(x, R)) \cdot |\mathbf{S}_n| \le c_2$ for all $x \in \mathbf{S}_n$ and for all $n \ge 0$ by Proposition 2.8 and Lemma 2.10 where we apply to d_S. Note that shadows $O_o(x, R)$ for $|x|_S = n$ cover the boundary $\partial\Gamma$, and each $\xi \in \partial\Gamma$ is included at most $|B_S(o, 4R)|$ shadows $O_o(x, R)$ with $|x|_S = n$. Therefore, we have that

$$\frac{|A_n|}{|\mathbf{S}_n|} \le C \sum_{x\in A_n} \mu_S(O_o(x, R)) \le C'\mu_S\left(\bigcup_{x\in A_n} O_o(x, R)\right). \tag{23}$$

If $\xi \in \bigcup_{x\in A_n} O_o(x, R)$, then $\xi_n \in B_S(x, 2R)$ for some $x \in A_n$, and thus $|d_{S^*}(o, \xi_n) - n\tau^\star| \ge \varepsilon n - 2LR$ where L is a Lipschitz constant: $d_{S^\star} \le L d_S$. Hence Lemma 7.3 implies that the last term in equation (23) tends to 0 as $n \to \infty$, and we obtain $\mathrm{Unif}_{S,n}(A_n) \to 0$ as $n \to \infty$. This shows that for any $\varepsilon > 0$, for all large enough n,

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we have $(\tau^\star - \varepsilon)n \le |x_n|_{S^\star} \le (\tau^\star + \varepsilon)n$ with probability at least $1 - \varepsilon$, and thus

$$(1-\varepsilon)(\tau^\star - \varepsilon)n \le \mathbb{E}_{\mathrm{Unif}_{S,n}} |x_n|_{S^\star} \le (\tau^\star + \varepsilon)n + L\varepsilon n.$$

Therefore, we obtain

$$\frac{1}{n}\mathbb{E}_{\mathrm{Unif}_{S,n}} |x_n|_{S^\star} \to \tau^\star \quad \text{as } n \to \infty,$$

and thus $\tau^\star = \tau(S^\star/S)$. We conclude the claim. □

Remark 7.5. Gekhtman, Taylor, and Tiozzo have discussed a more general setting where a hyperbolic group acts on a (possibly locally noncompact) hyperbolic metric space. Actually, Lemma 7.3 is a special case of **[GTT18,** Theorem 1.4**]**, and (a part of) Theorem 7.4 is included in **[GTT18,** Theorems 1.7 and 7.3**]**.

LEMMA 7.6. *Fix a word metric $d_{S^\star}$ in Γ. Let μ_S be a Patterson–Sullivan measure for d_S. Then*

$$\dim_H(\mu_S, \rho_{S^\star}) = \frac{\mathrm{gr}(S)}{\tau(S^\star/S)},$$

where $\rho_{S^\star}$ is the gauge in $\partial\Gamma$ relative to $d_{S^\star}$.

Proof. For any ξ in $\partial\Gamma$, let us take a geodesic ray $\xi_n^\star$ from o converging to ξ in $\mathrm{Cay}(\Gamma, S^\star)$. Then $\xi^\star$ is a (L, C)-quasi-geodesic ray in $\mathrm{Cay}(\Gamma, S)$, and there exists a geodesic ray ξ_n from o toward ξ in $\mathrm{Cay}(\Gamma, S)$ such that $\xi_n^\star$ and ξ_n are within bounded Hausdorff distance in $\mathrm{Cay}(\Gamma, S)$. Namely, there exists $D \ge 0$ such that for any n we have $d_S(\xi_n^\star, \xi_{k_n}) \le D$ for some k_n. Lemma 7.3 implies that for μ_S-almost every ξ in $\partial\Gamma$, we have that

$$\tau(S^*/S) \le \liminf_{n\to\infty} \frac{d_{S^\star}(o, \xi_{k_n})}{d_S(o, \xi_{k_n})} \le \limsup_{n\to\infty} \frac{d_{S^\star}(o, \xi_{k_n})}{d_S(o, \xi_{k_n})} \le \tau(S^\star/S),$$

and thus since $d_S(\xi_n^\star, \xi_{k_n}) \le D$ and $d_{S^\star}(\xi_n^\star, \xi_{k_n}) \le L \cdot D$ for all $n \ge 1$,

$$\lim_{n\to\infty} \frac{d_{S^\star}(o, \xi_n^\star)}{d_S(o, \xi_n^\star)} = \tau(S^\star/S) \quad \text{for } \mu_S\text{-almost every } \xi.$$

Now we have $d_{S^\star}(o, \xi_n^\star) = n$ and $\mu_S(O_o(x, R))$ is comparable to $\exp(-\mathrm{gr}(S)|x|_S)$ for all $x \in \Gamma$ by Proposition 2.8, it holds that

$$\begin{aligned} \lim_{n\to\infty} -\frac{1}{n} \log \mu_S(O_o(\xi_n^\star, R)) &= \lim_{n\to\infty} \mathrm{gr}(S) \frac{d_S(o, \xi_n^\star)}{d_{S^\star}(o, \xi_n^\star)} \\ &= \frac{\mathrm{gr}(S)}{\tau(S^\star/S)} \quad \text{for } \mu_S\text{-almost every } \xi. \end{aligned} \tag{24}$$

Comparing shadows with balls by Lemma 2.3 and by the Frostman-type lemma (Lemma 2.5), we obtain $\dim_H(\mu_S, \rho_{S^\star}) = \mathrm{gr}(S)/\tau(S^\star/S)$. □

THEOREM 7.7. *Let Γ be a non-elementary hyperbolic group. For any pair of finite symmetric sets of generators S and $S^\star$ in Γ, it holds that*

$$\tau(S^\star/S) \ge \frac{\mathrm{gr}(S)}{\mathrm{gr}(S^\star)},$$

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and $\tau(S^\star/S) = \mathrm{gr}(S)/\mathrm{gr}(S^\star)$ *if and only if* $d_{S^\star}$ *and* d_S *are roughly similar, i.e., there exist constants* $\tau > 0$ *and* $D \geq 0$ *such that*

$$|d_{S^\star}(x, y) - \tau d_S(x, y)| \leq D \quad \text{for all } x, y \in \Gamma.$$

Proof. Since $\dim_H(\mu_S, \rho_{S^\star}) \leq \mathrm{gr}(S^\star)$, Lemma 7.6 shows that we have that $\tau(S^\star/S) \geq \mathrm{gr}(S)/\mathrm{gr}(S^\star)$. If there exist $\tau > 0$ and $D \geq 0$ such that $|d_{S^\star}(x, y) - \tau d_S(x, y)| \leq D$ for all $x, y \in \Gamma$, then $\tau(S^*/S) = \tau$ and $\mathrm{gr}(S) = \tau \cdot \mathrm{gr}(S^\star)$ and thus the equality $\tau(S^\star/S) = \mathrm{gr}(S)/\mathrm{gr}(S^\star)$ holds.

Let us assume that $\tau(S^\star/S) = \mathrm{gr}(S)/\mathrm{gr}(S^\star)$. Since Λ_S and $\mu_S \otimes \mu_S$ are equivalent on $\partial^2\Gamma$ and by (24) in Lemma 7.6, the Frostman-type lemma (Lemma 2.5) implies that

$$\dim_H(\Lambda_S, \rho_{S^\star \times}) = \frac{2\mathrm{gr}(S)}{\tau(S^\star/S)} = 2\mathrm{gr}(S^\star).$$

Therefore, Theorem 7.2 implies that $\Lambda_S = c \cdot \Lambda_{S^\star}$ for some constant $c > 0$. This shows that μ_S and $\mu_{S^\star}$ are mutually absolutely continuous and thus $\mu_S = f\mu_{S^\star}$ for a density function $f \geq 0$ on $\partial\Gamma$. We shall show that f is uniformly bounded from above and away from 0. Since $\Lambda_S = c \cdot \Lambda_{S^\star}$, by the definition of Λ_S and $\Lambda_{S^\star}$, we have that the ratio between

$$f(\xi)f(\eta) \quad \text{and} \quad \frac{e^{2\mathrm{gr}(S^\star)(\xi|\eta)^\star_o}}{e^{2\mathrm{gr}(S)(\xi|\eta)_o}}$$

are uniformly bounded from above and away from 0 for all $\xi \neq \eta$ where $(\cdot|\cdot)_o$ and $(\cdot|\cdot)^\star_o$ are Gromov products with respect to d_S and $d_{S^\star}$, respectively. If the density f is not bounded on $B_{\rho_S}(\xi, \varepsilon)$, then fixing $\eta \neq \xi$ with $f(\eta) > 0$, we have that $f(\xi')$ is arbitrary large for $\xi' \in B_{\rho_S}(\xi, \varepsilon)$ for all small enough $\varepsilon > 0$ while $(\xi'|\eta)^\star_o$ and $(\xi'|\eta)_o$ are bounded for $\xi' \in B_{\rho_S}(\xi, \varepsilon)$; this is absurd. Hence f has to be uniformly bounded from above and the same argument shows that f is uniformly bounded away from 0. Therefore, μ_S and $\mu_{S^\star}$ are comparable, and for a fixed $R > 0$ the ratio between $\mu_S(O_o(x, R))$ and $\mu_{S^\star}(O_o(x, R))$ are uniformly bounded from above and away from 0. By Proposition 2.8, we have that

$$\mathrm{gr}(S^\star)|x|_{S^\star} = \mathrm{gr}(S)|x|_S + O(1) \quad \text{for all } x \in \Gamma.$$

This shows that $d_{S^\star}$ and d_S are roughly similar, as required. □

Acknowledgements. The author would like to thank Professor Vadim Kaimanovich for sharing the questions and his insight, which leads this work as well as valuable comments on an earlier version of this paper, Professors Jérémie Brieussel, Thibault Godin, Jayadev Athreya, Hidetoshi Masai, Takefumi Kondo, Hiroyasu Izeki and Koji Fujiwara for helpful discussions on this topic, Professors Rémi Coulon and Ilya Gekhtman for informing their works, and Professor Marc Bourdon for sending him his PhD thesis and valuable comments on the preprint. He also would like to thank the organizers of the workshop 'Equilibrium states for dynamical systems arising from geometry' held at American Institute of Mathematics, where he had beneficial discussions with participants, and last but not least, an anonymous referee for careful reading of the paper and useful comments.

https://doi.org/10.1017/etds.2020.101 Published online by Cambridge University Press

The author is supported by JSPS Grant-in-Aid for Young Scientists (B) JP17K14178 and JST, ACT-X Grant Number JPMJAX190J, Japan.

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