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A relationship between link polynomials

BY W. B. R. LICKORISH
Department of Pure Mathematics, University of Cambridge

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1. *Introduction*

The discovery by V. F. R. Jones of a Laurent polynomial invariant $V_L(t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ for every oriented link L in the 3-sphere prompted the finding of a more general 2-variable Laurent polynomial invariant $P_L(l, m) \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$, (see [4], [3] and [6]). The latter is defined uniquely for oriented links by the following formulae:

(i) $P_U = 1$ for the unknot U;

(ii) $lP_{L_+} + l^{-1}P_{L_-} + mP_{L_0} = 0$, where the L_j are identical except within a ball where



L_+ L_- L_0

Fig. 1

they are as in Fig. 1. The manner in which this polynomial specializes to become both the classical Alexander polynomial $\Delta_L(t)$ and the Jones polynomial is given by

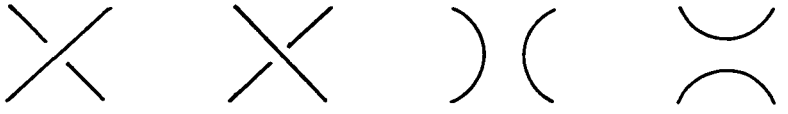
$$P_L(i,\ i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})) = \Delta_L(t)$$
$$P_L(it^{-1},\ -i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})) = V_L(t).$$

It follows almost at once that $\Delta_L(-1) = V_L(-1)$, the modulus of this value being the determinant of the link.

The formal ideas that produced P_L, and indeed the same style of proof, led to the discovery of a polynomial $Q_L(x) \in \mathbb{Z}[x^{\pm 1}]$ (called the *absolute* polynomial invariant), where L is now an unoriented link. The defining formulae for Q_L are

(i) $Q_U = 1$ where U is the unknot,

(ii) $Q_{L_+} + Q_{L_-} = x(Q_{L_0} + Q_{L_\infty})$ where the L_j are unoriented links, identical except within a ball where they are as in Fig. 2.



L_+ L_- L_0 L_∞

Fig. 2

Details are to be found in [2]. The P and Q polynomials are distinct in the sense that knots can be found that can be distinguished by one polynomial but not by the other.

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In [5], L. H. Kauffman defines yet another polynomial $F_L(a,x) \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$ for each oriented link L with the property that $F_L(1,x) = Q_{|L|}(x)$, where $|L|$ is L with its orientation ignored. A description of F will be given below; F is strictly stronger than Q and distinct from P. Of all these invariants P and F are the most general. As already stated, P specializes to V. The purpose of this note is to prove that F also specializes to V:

THEOREM. *For any oriented link L,*

$$V_L(t) = F_L(t^{-\frac{3}{4}}, -(t^{-\frac{1}{4}} + t^{\frac{1}{4}})).$$

2. *The Jones polynomial*

The Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ is, in the notation of the previous section, defined for oriented links by

(i) $V_U = 1$,

(ii) $t^{-1}V_{L_+} - tV_{L_-} + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{L_0} = 0$.

This invariant satisfies the Jones reversing result:

Reversing result. Suppose a component k of L has linking number λ with $(L-k)$ and that $\hat{L}$ is obtained from L by reversing the direction of k. Then

$$V_{\hat{L}} = t^{-3\lambda}V_L.$$

This is explained in [7] and from it (see [8], for example) can be deduced the following 'V_∞' formula devised by J. S. Birman [1].

'V_∞' *formula.* Let $c(L)$ denote the number of components of a link L. Let L_+, L_-, and L_0 be oriented links related as before. Let L_∞ be a link represented by one of the



(i) (ii)

Fig. 3

diagrams of Fig. 3, the choice being (i) if $c(L_+) < c(L_0)$ and (ii) if $c(L_+) > c(L_0)$. For each of these two possibilities the formula is

(i) $t^{-\frac{1}{2}}V_{L_+} - t^{\frac{1}{2}}V_{L_-} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{3\lambda}V_{L_\infty} = 0$, where λ is the linking number of the right-hand component of L_0 in Fig. 1 with the remainder of L_0;

(ii) $t^{-\frac{1}{2}}V_{L_+} - t^{\frac{1}{2}}V_{L_-} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})t^{3(\mu-\frac{1}{2})}V_{L_\infty} = 0$, where μ is the linking number of the bottom-right to top-left component of L_+ in Fig. 1 with the remainder of L_+.

3. *The Kauffman polynomial*

An elegant definition of $F_L \in \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$ is given in [5]. From it can be deduced the following, here more convenient, characterization. Let L_+, L_-, L_0 and L_∞ be oriented links related as above, where, exactly as the the previous section, the definition of L_∞ depends on the relation between the numbers of components of L_+ and L_0. Then F is defined uniquely by $F_U = 1$ and the statement that for all such quadruples of oriented links,

(i) $aF_{L_+} + a^{-1}F_{L_-} = x(F_{L_0} + a^{-4\lambda}F_{L_\infty})$ if $c(L_+) < c(L_0)$,

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(ii) $aF_{L_+} + a^{-1}F_{L_-} = x(F_{L_0} + a^{-4\mu+2}F_{L_\infty})$ if $c(L_+) > c(L_0)$, where λ and μ are the linking numbers previously described.

This polynomial also has an easy formula for its behaviour under the operation of reversing the orientation of one component:

Reversing result. As in the last section, suppose that $\hat{L}$ is constructed from L by reversing the direction of a component of L which, prior to reversal, has linking number λ with the remainder of L. Then

$$F_{\hat{L}} = a^{4\lambda}F_L.$$

This reversing property nullifies the arbitrary nature of the choices made above for the orientations of L_∞.

4. *Proof of the Theorem*

Consider oriented links L_+, L_-, L_0, and L_∞ related as before with the above conventions for the orientations of L_∞.

Case (*i*). Suppose $c(L_+) < c(L_0)$. The defining formula for $V_L(t)$ and the 'V_∞' formula are

$$t^{-1}V_{L_+} - tV_{L_-} + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{L_0} = 0,$$
$$t^{-\frac{1}{2}}V_{L_+} - t^{\frac{1}{2}}V_{L_-} - (t^{-\frac{1}{2}} - t^{\frac{1}{2}})t^{3\lambda}V_{L_\infty} = 0.$$

Subtracting the second of these from the first gives

$$t^{-\frac{3}{2}}(t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{L_+} + t^{\frac{3}{2}}(t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{L_-} + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})(V_{L_0} + t^{3\lambda}V_{L_\infty}) = 0.$$

Of course, this simplifies to

$$t^{-\frac{3}{2}}V_{L_+} + t^{\frac{3}{2}}V_{L_-} + (t^{-\frac{1}{2}} + t^{\frac{1}{2}})(V_{L_0} + t^{3\lambda}V_{L_\infty}) = 0.$$

Case (*ii*). Suppose that $c(L_+) > c(L_0)$. The same reasoning gives

$$t^{-\frac{3}{2}}V_{L_+} + t^{\frac{3}{2}}V_{L_-} + (t^{-\frac{1}{2}} + t^{\frac{1}{2}})(V_{L_0} + t^{3(\mu-\frac{1}{2})}V_{L_\infty}) = 0.$$

Substitution of $a = t^{-\frac{3}{2}}$ and $x = -(t^{-\frac{1}{2}} + t^{\frac{1}{2}})$ in the formulae that characterize F_L yields exactly the same relationships in both of cases (i) and (ii). Thus the standard induction argument on the number of crossings of a link presentation and on the number of those crossings that must be reversed to produce an unlink shows that, for every oriented link L,

$$V_L(t) = F_L(t^{-\frac{3}{2}}, -(t^{-\frac{1}{2}} + t^{\frac{1}{2}})).$$

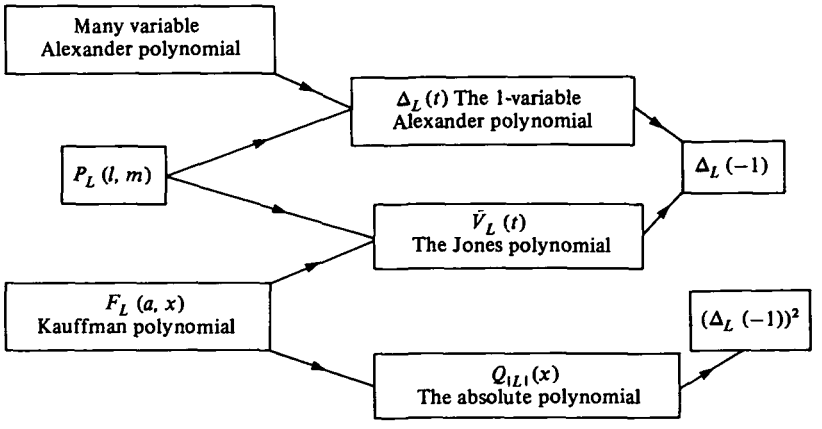


Fig. 4

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That completes the proof of the theorem. The interdependence of the invariants here discussed is shown schematically in Fig. 4, where an arrow indicates that one invariant produces another by means of a substitution of variables.

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