Commun. Comput. Phys.
doi: 10.4208/cicp.070814.190615a

Vol. **19**, No. 2, pp. 411-441
February 2016

The Mass-Preserving S-DDM Scheme for Two-Dimensional Parabolic Equations

Zhongguo Zhou[1] and Dong Liang[2,1,*]

[1] *School of Mathematics, Shandong University, Jinan, Shandong, 250100, China.*
[2] *Department of Mathematics and Statistics, York University, Toronto, Ontario, M3J 1P3, Canada.*

Received 7 August 2014; Accepted (in revised version) 19 June 2015

Abstract. In the paper, we develop and analyze a new mass-preserving splitting domain decomposition method over multiple sub-domains for solving parabolic equations. The domain is divided into non-overlapping multi-bock sub-domains. On the interfaces of sub-domains, the interface fluxes are computed by the semi-implicit (explicit) flux scheme. The solutions and fluxes in the interiors of sub-domains are computed by the splitting one-dimensional implicit solution-flux coupled scheme. The important feature is that the proposed scheme is mass conservative over multiple non-overlapping sub-domains. Analyzing the mass-preserving S-DDM scheme is difficult over non-overlapping multi-block sub-domains due to the combination of the splitting technique and the domain decomposition at each time step. We prove theoretically that our scheme satisfies conservation of mass over multi-block non-overlapping sub-domains and it is unconditionally stable. We further prove the convergence and obtain the error estimate in L^2-norm. Numerical experiments confirm theoretical results.

AMS subject classifications: 65M06, 65M12, 65M55, 76S05

Key words: Parabolic equations, S-DDM, mass-preserving, unconditional stability, convergence, multi-blocks, non-overlapping

1 Introduction

Parabolic equations are widely used in science and engineering, which describe water head in groundwater modelling, pressure in petroleum reservoir simulation, and diffusion phenomena in heat propagation, etc (see, for example, [1–3]). Due to the computational complexity and the large computational cost in long term and large scale simulations, there are strong interests in developing efficient domain decomposition methods for solving these kinds time-dependent PDEs in high dimensions.

*Corresponding author. *Email addresses:* zhouzhongguo@mail.sdu.edu.cn (Z. Zhou), dliang@mathstat.yorku.ca (D. Liang)

 ©2016 Global-Science Press

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Domain Decomposition Methods (DDMs) allow the reduction of the sizes of problems by dividing the large domain into smaller ones on which the PDEs can be solved by multiple computers in parallel, including the overlapping methods and the non-overlapping methods ([4,8,9,11,12,16,17], etc). Since non-overlapping methods have low computation and communication costs at each time step, the non-iterative explicit-implicit schemes on non-overlapping domain decompositions have been developed for solving large scale problems of parabolic types on massively parallel machine (see, [4–10, 13–15, 18, 19]). [4, 9] proposed the mixed/hybrid schemes, where the implicit schemes are used in sub-domains while the explicit schemes are used for the interface values on the interfaces of sub-domains. [6] further proposed the explicit-implicit methods for parabolic problems by either a multi-step explicit scheme or a high-order explicit scheme on the interfaces which relaxed the stability requirements. Further developments were done in [14, 19] by introducing the implicit correction step on the interfaces for improving the stability, where [14] analyzed theoretically the unconditional stability on the zigzag-shaped interfaces of sub-domains and obtained the convergence of the improved schemes in two dimensions. [18] studied the cell centered finite difference domain decomposition procedure for parabolic equations in one dimension. The fluxes at interface points were computed explicitly by nearby fluxes values at previous time. [15] proposed the parallel difference schemes for parabolic equations in two dimensions, where the predicted interface values on interfaces are employed some linear combination of the values on previous two time levels on the interfaces of sub-domains, which leads to a three level schemes. The methods in [4,6,18] can work efficiently for one dimensional problems and for stripe-divided sub-domains along one spatial variable for parabolic equations. More recently, over multiple block-divided sub-domains, [5,10] developed an efficient splitting domain decomposition methods (S-DDM) by combining the splitting technique with the non-overlapping decomposition for solving parabolic equations in two dimensions and compressible contamination fluid flows in porous media. While the local multi-level schemes with local time steps are used to compute the interface values on the interfaces of sub-domains, the interior values in sub-domains are solved by the splitting one-dimensional implicit schemes. But, these previous methods [4–6, 9, 10, 14, 15, 19] do not satisfy the important physical law of mass conservation on multiple sub-domains. It is an important and challenging task to develop and analyze the mass-preserving domain decomposition methods over multi-block sub-domains.

In this paper, over multiple block-divided sub-domains, we propose and analyze a new mass-preserving splitting domain decomposition method for solving two-dimensional parabolic equations by combining the non-overlapping domain decomposition, the splitting technique and the coupling of the solution and its fluxes on staggered meshes. In the method, the global domain is divided into multi-block non-overlapping sub-domains. At each time step level, we take two stages to compute the solutions and fluxes on sub-domains. Firstly, the intermediate interface fluxes of x-direction are computed by a semi-implicit (explicit) flux scheme. The intermediate solutions and fluxes in the interiors of sub-domains are computed by a x-directional splitting implicit solution-

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

flux coupled scheme. By substituting the intermediate fluxes, it leads to tridiagonal linear systems of intermediate solutions on sub-domains, which can be easily solved in parallel by Thomas' method. Secondly, the interface fluxes of y-direction are computed by a semi-implicit (explicit) flux scheme. The solutions and fluxes are computed by a y-directional splitting implicit solution-flux coupled scheme in the interiors of sub-domains.The important features are that the proposed scheme is mass conservative over multiple sub-domains and it is unconditionally stable. Meanwhile, the scheme has the advantage of easily extending to three or higher dimensional problems. We prove the scheme to satisfy conservation of mass over multi-block sub-domains and further analyze theoretically its unconditional stability and convergence. Analyzing the stability and convergence of the scheme over block-divided sub-domains is challenging due to the combination of the splitting technique and the domain decomposition. In order to overcome the difficulty, we derive out some auxiliary lemmas of estimating numerical fluxes in the interface areas and then prove the scheme unconditionally stable over multi-block sub-domains. Introducing intermediate variable values of solutions and applying the matrix theory for analyzing truncation errors, we prove the convergence of the scheme and obtain the error estimate in L^2-norm. Numerical experiments confirm theoretical results. The scheme keeps the excellent advantages of the non-overlapping multi-block domain decomposition and the splitting technique, preserves the important physical law of mass conservation, and is more suitable and powerful for parallel computing.

The paper is organized as follows. In Section 2, we introduce the parabolic equations and propose the mass-preserving S-DDM scheme. In Section 3, the mass conservation and unconditional stability of the scheme are proved. In Section 4, we prove convergence and error estimate of the scheme. Finally, numerical experiments are presented in Section 5.

2 The parabolic equations and the mass-preserving S-DDM scheme

Consider the two-dimensional parabolic equations with Neumann boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t}-\frac{\partial}{\partial x}(D_1\frac{\partial u}{\partial x})-\frac{\partial}{\partial y}(D_2\frac{\partial u}{\partial y})=f(x,y,t), & (x,y,t)\in\Omega\times(0,T], \\ \nabla u\cdot\vec{n}=0, & (x,y,t)\in\partial\Omega\times(0,T], \\ u(x,y,0)=u_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{2.1}$$

where $\Omega=(a,b)\times(c,d)$, $\partial\Omega$ is the boundary of Ω, and $\vec{n}$ is the normal vector along $\partial\Omega$. $T>0$ is the time period. $D_1>0$ and $D_2>0$ are the diffusion coefficients. $f(x,y,t)$ and $u_0(x,y)$ are the known smooth functions. Parabolic equations are important, which model reservoir pressure, groundwater water-head and heat temperature, etc. Many real applications are large scale and long term duration. They lead to developing the mass-preserving explicit/implicit domain decomposition methods over non-overlapping multi-block sub-domains.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

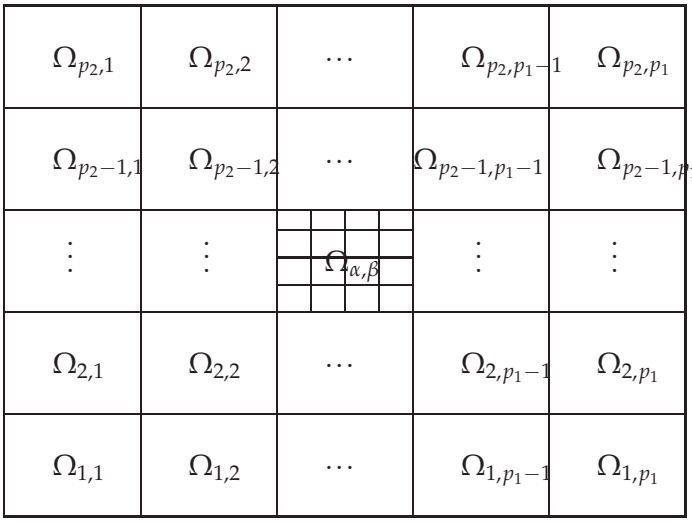


Figure 1: The domain Ω and its sub-domains $\Omega_{\alpha,\beta}$.

For designing our mass-preserving domain decomposition schemes, we first divide the domain Ω into multiple non-overlapping block-divided sub-domains $\Omega_{\alpha,\beta}$ (see, Fig. 1). In general, the number $p=p_1\times p_2$ is related to the problem and the number of processors in the computer platform, and the sub-domains may be of different widths. Let $\gamma^x_{\alpha,\beta}=\partial\Omega_{\alpha,\beta}\cap\partial\Omega_{\alpha+1,\beta}$, the interface of $\Omega_{\alpha,\beta}$ and $\Omega_{\alpha+1,\beta}$, and $\gamma^y_{\alpha,\beta}=\partial\Omega_{\alpha,\beta}\cap\partial\Omega_{\alpha,\beta+1}$, the interface of $\Omega_{\alpha,\beta}$ and $\Omega_{\alpha,\beta+1}$. Further, let $\Gamma^x_\alpha=\bigcup_{\beta=1}^{p_2}\gamma^x_{\alpha,\beta}$ and $\Gamma^y_\beta=\bigcup_{\alpha=1}^{p_1}\gamma^y_{\alpha,\beta}$. Each sub-domain $\Omega_{\alpha,\beta}$ is further divided by fine meshes.

In the construction, we approximate directly the solution u and fluxes $q^x=D_1\frac{\partial u}{\partial x}$ and $q^y=D_2\frac{\partial u}{\partial y}$ in a coupled system. For doing this, we introduce the staggered meshes for u,q^x and q^y. For simplicity, we describe the $p=2\times 2$ sub-domains of domain Ω and their staggered meshes (see, Fig. 2).

Let $\Delta x=\frac{b-a}{I}$, $\Delta y=\frac{d-c}{J}$. I,J, are the positive integers. Define

$$x_{i+\frac{1}{2}}=a+i\Delta x,\ \ i=0,1,\cdots,I,\quad x_i=a+\left(i-\frac{1}{2}\right)\Delta x,\ \ i=1,2,\cdots,I,$$
$$y_{j+\frac{1}{2}}=c+j\Delta y,\ \ j=0,1,\cdots,J,\quad y_j=c+\left(j-\frac{1}{2}\right)\Delta y,\ \ j=1,2,\cdots,J.$$

The lines Γ^x_1: $x=x_{i_1+\frac{1}{2}}=\frac{a+b}{2}$ Γ^y_1: $y=y_{j_1+\frac{1}{2}}=\frac{c+d}{2}$ are the interface lines of sub-domains $\Omega_{\alpha,\beta}$, $\alpha=1,2$; $\beta=1,2$. Nodes $\{(x_{i+\frac{1}{2}},y_j)\}$ and $\{(x_i,y_{j+\frac{1}{2}})\}$ are the nodes for the fluxes q^x and q^y and nodes $\{(x_i,y_j)\}$ are for the solution u.

Let $\Delta t=\frac{T}{N}$ and $t^n=n\Delta t$, $n=0,1,\cdots,N$. Denote $r_x=D_1\frac{\Delta t}{(\Delta x)^2}$ and $r_y=D_2\frac{\Delta t}{(\Delta y)^2}$. Let $w_{i,j},w_{i+\frac{1}{2},j},w_{i,j+\frac{1}{2}}$ be the values of $w(x,y)$ defined at nodes (x_i,y_j), $(x_{i+\frac{1}{2}},y_j)$ and $(x_i,y_{j+\frac{1}{2}})$.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

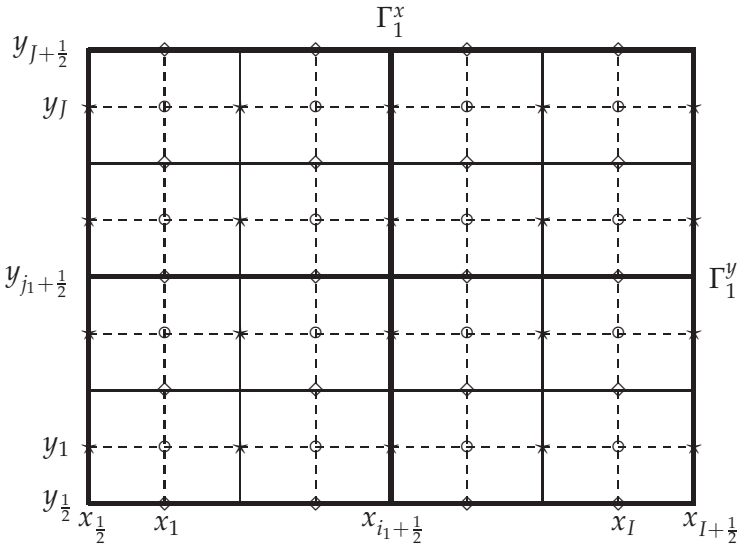


Figure 2: The sub-domains and their meshes: $\circ,\star,\diamond$ – the points of $(i,j),(i+\frac{1}{2},j),(i,j+\frac{1}{2})$.

Define the space center difference operators as

$$\delta_x w_{i+\frac{1}{2},j}=\frac{w_{i+1,j}-w_{i,j}}{\Delta x},\quad \delta_y w_{i,j+\frac{1}{2}}=\frac{w_{i,j+1}-w_{i,j}}{\Delta y},$$
$$\delta_x w_{i,j}=\frac{w_{i+\frac{1}{2},j}-w_{i-\frac{1}{2},j}}{\Delta x},\quad \delta_y w_{i,j}=\frac{w_{i,j+\frac{1}{2}}-w_{i,j-\frac{1}{2}}}{\Delta y}.$$

Define the time difference operators as

$$\hat{\partial}_t w^*=\frac{w^*-w^n}{\Delta t},\quad \check{\partial}_t w^{n+1}=\frac{w^{n+1}-w^*}{\Delta t},\quad \partial_t w^{n+1}=\frac{w^{n+1}-w^n}{\Delta t}.$$

Let $\{U_{i,j}^n\}$, $\{Q_{i+\frac{1}{2},j}^x\}$ and $\{Q_{i,j+\frac{1}{2}}^y\}$ be the numerical solution and the numerical fluxes. Let $f=f_1+f_2$ by using $f_1=\frac{1}{2}f, f_2=\frac{1}{2}f$, or $f_1=f, f_2=0$, or $f_1=0, f_2=f$ in computations.

Based on the notations, we now propose the mass-preserving splitting domain decomposition method over multi-block sub-domains as

Stage 1: Along x-direction:

(a) On the interface lines Γ_α^x, the intermediate fluxes $\{\tilde{Q}_{i_\alpha+\frac{1}{2},j}^{x,*}\}$ are computed by the semi-implicit (explicit) flux scheme

$$\tilde{Q}_{i_\alpha+\frac{1}{2},j}^{x,*}=Q_{i_\alpha+\frac{1}{2},j}^{x,n}+r_x(Q_{i_\alpha-\frac{1}{2},j}^{x,n}-2\tilde{Q}_{i_\alpha+\frac{1}{2},j}^{x,*}+Q_{i_\alpha+\frac{3}{2},j}^{x,n})+\Delta t D_1\delta_x f_{1\,i_\alpha+\frac{1}{2},j}^{n+1}. \tag{2.2}$$

(b) In the interiors of sub-domains $\Omega_{\alpha,\beta}$, the intermediate variables $\{U_{i,j}^*, Q_{i+\frac{1}{2},j}^{x,*}\}$ are

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

computed by the x-directional splitting implicit and solution-flux coupled scheme

$$\begin{cases} \frac{U^{*}_{i,j}-U^{n}_{i,j}}{\Delta t}=\delta_x Q^{x,*}_{i,j}+f_1{}^{n+1}_{i,j}, & (x_i,y_j)\in\Omega_{\alpha,\beta}, \\ Q^{x,*}_{i+\frac{1}{2},j}=D_1\delta_x U^{*}_{i+\frac{1}{2},j}, & (x_{i+\frac{1}{2}},y_j)\in\Omega_{\alpha,\beta}, \\ Q^{x,*}_{i_\alpha+\frac{1}{2},j}=\tilde{Q}^{x,*}_{i_\alpha+\frac{1}{2},j}, & (x_{i_\alpha+\frac{1}{2}},y_j)\in\Gamma^{x}_{\alpha}. \end{cases} \tag{2.3}$$

(c) Let

$$Q^{x,*}_{i_\alpha+\frac{1}{2},j}=D_1\delta_x U^{*}_{i_\alpha+\frac{1}{2},j}, \quad (x_{i_\alpha+\frac{1}{2}},y_j)\in\Gamma^{x}_{\alpha},$$

and

$$Q^{y,*}_{i,j+\frac{1}{2}}=D_2\delta_y U^{*}_{i,j+\frac{1}{2}}, \quad (x_i,y_{j+\frac{1}{2}})\in\Omega_{\alpha,\beta}.$$

Stage 2: Along y-direction:

(a) On the interface lines Γ^{y}_{β}, the interface fluxes $\{\tilde{Q}^{y,n+1}_{i,j_\beta+\frac{1}{2}}\}$ are computed by the semi-implicit (explicit) flux scheme

$$\tilde{Q}^{y,n+1}_{i,j_\beta+\frac{1}{2}}=Q^{y,*}_{i,j_\beta+\frac{1}{2}}+r_y(Q^{y,*}_{i,j_\beta-\frac{1}{2}}-2\tilde{Q}^{y,n+1}_{i,j_\beta+\frac{1}{2}}+Q^{y,*}_{i,j_\beta+\frac{3}{2}})+\Delta t D_2\delta_y f_2{}^{n+1}_{i,j_\beta+\frac{1}{2}}. \tag{2.4}$$

(b) In the interiors of sub-domains $\Omega_{\alpha,\beta}$, the numerical solutions and fluxes $\{U^{n+1}_{i,j},Q^{y,n+1}_{i,j+\frac{1}{2}}\}$ are solved by the y-directional splitting implicit and solution-flux coupled scheme

$$\begin{cases} \frac{U^{n+1}_{i,j}-U^{*}_{i,j}}{\Delta t}=\delta_y Q^{y,n+1}_{i,j}+f_2{}^{n+1}_{i,j}, & (x_i,y_j)\in\Omega_{\alpha,\beta}, \\ Q^{y,n+1}_{i,j+\frac{1}{2}}=D_2\delta_y U^{n+1}_{i,j+\frac{1}{2}}, & (x_i,y_{j+\frac{1}{2}})\in\Omega_{\alpha,\beta}, \\ Q^{y,n+1}_{i,j_\beta+\frac{1}{2}}=\tilde{Q}^{y,n+1}_{i,j_\beta+\frac{1}{2}}, & (x_i,y_{j_\beta+\frac{1}{2}})\in\Gamma^{y}_{\beta}. \end{cases} \tag{2.5}$$

(c) Let

$$Q^{y,n+1}_{i,j_\beta+\frac{1}{2}}=D_2\delta_y U^{n+1}_{i,j_\beta+\frac{1}{2}}, \quad (x_i,y_{j_\beta+\frac{1}{2}})\in\Gamma^{y}_{\beta},$$

and

$$Q^{x,n+1}_{i+\frac{1}{2},j}=D_1\delta_x U^{n+1}_{i+\frac{1}{2},j}, \quad (x_{i+\frac{1}{2}},y_j)\in\Omega_{\alpha,\beta}.$$

The boundary conditions are

$$\begin{aligned} &Q^{x,*}_{\frac{1}{2},j}=Q^{x,*}_{I+\frac{1}{2},j}=0, && (x_{\frac{1}{2}},y_j),(x_{I+\frac{1}{2}},y_j)\in\partial\Omega, \\ &Q^{y,n+1}_{i,\frac{1}{2}}=Q^{y,n+1}_{i,J+\frac{1}{2}}=0, && (x_i,y_{\frac{1}{2}}),(x_i,y_{J+\frac{1}{2}})\in\partial\Omega, \end{aligned} \tag{2.6}$$

and the initial values are given by

$$U^{0}_{i,j}=u_0(x_i,y_j), \quad (x_i,y_j)\in\Omega. \tag{2.7}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Remark 2.1. In the scheme (2.2)-(2.7), we normally use $f_1=\frac{1}{2}f, f_2=\frac{1}{2}f$, or $f_1=f, f_2=0$, or $f_1=0, f_2=f$. Combining the splitting technique with the coupling of solution and flux on staggered meshes, the scheme is developed over multi-block sub-domains, which is explicit-implicit and can be implemented easily. At Stage 1, the intermediate fluxes $\{\tilde{Q}^{x,*}_{i_\alpha+\frac{1}{2},j}\}$ on the interfaces are firstly computed by the semi-implicit flux scheme (2.2) explicitly. Substituting the intermediate fluxes $\{Q^{x,*}_{i+\frac{1}{2},j}\}$ into the first equation of (2.3), it leads to symmetric and tridiagonal algebraic systems of $\{U^*_{i,j}\}$ on sub-domains which can be easily solved by the Thomas' method. At Stage 2, $\{U^{n+1}_{i,j}\}$ and $\{Q^{y,n+1}_{i,j+\frac{1}{2}}\}$ can be solved on sub-domains in the same way as Stage 1.

Remark 2.2. Our mass-preserving S-DDM scheme (2.2)-(2.7) overcomes the limitation of the strip-divided sub-domains and the zigzag-shaped sub-domains. While it works efficiently over multiple block-divided non-overlapping sub-domains, our scheme preserves mass even though the explicit-implicit technique is used on interfaces and in multi-block sub-domains.

3 Conservation of mass and unconditional stability

In this section, we first prove our scheme to satisfy mass conservation and then prove its unconditional stability. The analysis of the stability for the scheme over non-overlapping multi-block sub-domains is difficult due to the combination of the splitting technique, the coupling of U, Q^x and Q^y on the staggered mesh on sub-domains and the semi-implicit (explicit) flux scheme used on interfaces of sub-domains. For analyzing its stability, we derive out Lemma 3.4 and Lemma 3.5 for $\hat{\partial}_t \parallel U^* \parallel^2$ and $\check{\partial}_t \parallel U^{n+1} \parallel^2$ of the solution and Lemma 3.6 and Lemma 3.7 for $\hat{\partial}_t \mid Q^{y,*} \mid^2_{\Gamma_{x,j+\frac{1}{2}}}$ and $\check{\partial}_t \mid Q^{x,n+1} \mid^2_{\Gamma_{i+\frac{1}{2},y}}$ of the fluxes on the interface lines and finally derive out the unconditionally stability of the proposed scheme.

Lemma 3.1. *Assume that* $Q^x_{\frac{1}{2},j}=Q^x_{I+\frac{1}{2},j}=0, j=1,2,\cdots,J$, *and* $Q^y_{i,\frac{1}{2}}=Q^y_{i,J+\frac{1}{2}}=0, i=1,2,\cdots,I$. *Then for the mesh functions* $\{Q^x_{i+\frac{1}{2},j}, Q^y_{i,j+\frac{1}{2}}\}$, *it holds that*

$$\sum_{j=1}^{J}\sum_{i=1}^{I}\delta_x Q^x_{i,j}=0, \quad \sum_{i=1}^{I}\sum_{j=1}^{J}\delta_y Q^y_{i,j}=0. \tag{3.1}$$

Theorem 3.1. (Conservation of mass) *The scheme* (2.2)-(2.7) *satisfies mass conservation over multi-block sub-domains, i.e.*

$$\sum_{i=1}^{I}\sum_{j=1}^{J}U^n_{i,j}\Delta x\Delta y=\sum_{i=1}^{I}\sum_{j=1}^{J}U^0_{i,j}\Delta x\Delta y+\sum_{l=0}^{n-1}\Big(\sum_{i=1}^{I}\sum_{j=1}^{J}f^{l+1}_{i,j}\Delta x\Delta y\Big)\Delta t, \quad for \ \ n\geq 1. \tag{3.2}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Proof. Multiplying the first equation of (2.3) with $\Delta x\Delta y$ and summing for i,j from 1 to I,J, and noting that $Q^{x,*}_{i_\alpha+\frac{1}{2},j}$ is given at the nodes on interfaces Γ^x_α, from the boundary condition (2.6) and Lemma 3.1, we can obtain that

$$\begin{aligned}\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{U^*_{i,j}-U^n_{i,j}}{\Delta t}\Delta x\Delta y&=\sum_{j=1}^{J}\sum_{i=1}^{I}\delta_x Q^{x,*}_{i,j}\Delta x\Delta y+\sum_{i=1}^{I}\sum_{j=1}^{J}f_1{}^{n+1}_{i,j}\Delta x\Delta y\\&=\sum_{i=1}^{I}\sum_{j=1}^{J}f_1{}^{n+1}_{i,j}\Delta x\Delta y.\end{aligned}\tag{3.3}$$

Similarly, multiplying the first equation of (2.5) with $\Delta x\Delta y$ and summing for i,j from 1 to I,J, we can have that

$$\begin{aligned}&\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{U^{n+1}_{i,j}-U^*_{i,j}}{\Delta t}\Delta x\Delta y\\&=\sum_{i=1}^{I}\sum_{j=1}^{J}\delta_y Q^{y,n+1}_{i,j}\Delta x\Delta y+\sum_{i=1}^{I}\sum_{j=1}^{J}f_2{}^{n+1}_{i,j}\Delta x\Delta y\\&=\sum_{i=1}^{I}\sum_{j=1}^{J}f_2{}^{n+1}_{i,j}\Delta x\Delta y.\end{aligned}\tag{3.4}$$

Combining (3.3) with (3.4), it yields that

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\frac{U^{n+1}_{i,j}-U^n_{i,j}}{\Delta t}\Delta x\Delta y=\sum_{i=1}^{I}\sum_{j=1}^{J}f^{n+1}_{i,j}\Delta x\Delta y.\tag{3.5}$$

Multiplying (3.5) with Δt and summing it from 0 to $n-1$, we finally get (3.2). □

Let $U=\{U_{i,j}\}$, $V=\{V_{i,j}\}$, $X=\{X_{i+\frac{1}{2},j}\}$, $Y=\{Y_{i+\frac{1}{2},j}\}$, $Z=\{Z_{i,j+\frac{1}{2}}\}$, and $W=\{W_{i,j+\frac{1}{2}}\}$ be discrete functions. Define the discrete inner products, and the corresponding discrete L^2-norms over Ω as follows,

$$\begin{aligned}&(U,V)=\textstyle\sum_{i=1}^{I}\sum_{j=1}^{J}U_{i,j}V_{i,j}\Delta x\Delta y,\quad \|U\|=(U,U)^{\frac{1}{2}},\\&(X,Y)=\textstyle\sum_{i=0}^{I-1}\sum_{j=1}^{J}X_{i+\frac{1}{2},j}Y_{i+\frac{1}{2},j}\Delta x\Delta y,\quad \|X\|=(X,X)^{\frac{1}{2}},\\&(Z,W)=\textstyle\sum_{i=1}^{I}\sum_{j=0}^{J-1}Z_{i,j+\frac{1}{2}}W_{i,j+\frac{1}{2}}\Delta x\Delta y,\quad \|Z\|=(Z,Z)^{\frac{1}{2}}.\end{aligned}\tag{3.6}$$

Let $\Gamma_{i_\alpha+\frac{1}{2},y}$, and $\Gamma_{x,j_\beta+\frac{1}{2}}$ be the interface lines, respectively. We define the corresponding inner products and norms on the interface lines.

$$\begin{aligned}&<X,Y>_{\Gamma_{i_\alpha+\frac{1}{2},y}}=\textstyle\sum_{j=1}^{J}X_{i_\alpha+\frac{1}{2},j}Y_{i+\frac{1}{2},j}\Delta y,\quad |X|_{\Gamma_{i_\alpha+\frac{1}{2},y}}=<X,X>^{\frac{1}{2}}_{\Gamma_{i_\alpha+\frac{1}{2},y}},\\&<Z,W>_{\Gamma_{x,j_\beta+\frac{1}{2}}}=\textstyle\sum_{i=1}^{I}Z_{i,j_\beta+\frac{1}{2}}W_{i,j_\beta+\frac{1}{2}}\Delta x,\quad |Z|_{\Gamma_{x,j_\beta+\frac{1}{2}}}=<Z,W>^{\frac{1}{2}}_{\Gamma_{x,j_\beta+\frac{1}{2}}}.\end{aligned}\tag{3.7}$$

We first give some preliminary lemmas.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Lemma 3.2. *Let* $U=\{U_{i,j}\}$ *and* $V=\{V_{i,j}\}$. *Then*

$$\left(\frac{U-V}{\Delta t},U\right)=\frac{1}{2\Delta t}(\|U\|^2-\|V\|^2)+\frac{1}{2\Delta t}\|U-V\|^2. \tag{3.8}$$

Lemma 3.3. *Let* $Q^x_{i+\frac{1}{2},j}=D_1\delta_x U_{i+\frac{1}{2},j}$ *and* $Q^y_{i,j+\frac{1}{2}}=D_2\delta_y U_{i,j+\frac{1}{2}}$. *If* $Q^x_{\frac{1}{2},j}=D_1\delta_x U_{\frac{1}{2},j}=0$, $Q^x_{I+\frac{1}{2},j}=D_1\delta_x U_{I+\frac{1}{2},j}=0$, $Q^y_{i,\frac{1}{2}}=D_2\delta_y U_{i,\frac{1}{2}}=0$ *and* $Q^y_{i,J+\frac{1}{2}}=D_2\delta_y U_{i,J+\frac{1}{2}}=0$, *then we have that*

$$(\delta_x Q^x,U)=-\frac{1}{D_1}\|Q^x\|^2,\quad (\delta_y Q^y,U)=-\frac{1}{D_2}\|Q^y\|^2, \tag{3.9}$$

and

$$D_2\delta_y Q^x_{i+\frac{1}{2},j+\frac{1}{2}}=D_1\delta_x Q^y_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{3.10}$$

Proof. Note that

$$\begin{aligned}(\delta_x Q^x,U)&=\sum_{i=1,j=1}^{I,J}\delta_x Q^x_{i,j}U_{i,j}\Delta x\Delta y=\sum_{j=1}^{J}\sum_{i=1}^{I}\frac{(Q^x_{i+\frac{1}{2},j}-Q^x_{i-\frac{1}{2},j})U_{i,j}}{\Delta x}\Delta x\Delta y\\&=\sum_{j=1}^{J}\sum_{i=1}^{I}\frac{Q^x_{i+\frac{1}{2},j}U_{i,j}}{\Delta x}\Delta x\Delta y-\sum_{j=1}^{J}\sum_{i=0}^{I-1}\frac{Q^x_{i+\frac{1}{2},j}U_{i+1,j}}{\Delta x}\Delta x\Delta y\\&=\sum_{j=1}^{J}\left(-\sum_{i=1}^{I-1}Q^x_{i+\frac{1}{2},j}\delta_x U_{i+\frac{1}{2},j}-\frac{Q^x_{\frac{1}{2},j}U_{1,j}}{\Delta x}+\frac{Q^x_{I+\frac{1}{2},j}U_{I,j}}{\Delta x}\right)\Delta x\Delta y.\end{aligned} \tag{3.11}$$

Using the definition $Q^x_{i+\frac{1}{2},j}=D_1\delta_x U_{i+\frac{1}{2},j}$ and the conditions $Q^x_{\frac{1}{2},j}=Q^x_{I+\frac{1}{2},j}=0$, we can get that $(\delta_x Q^x,U)=-\frac{1}{D_1}\|Q^x\|^2$. Similarly, we have that $(\delta_y Q^y,U)=-\frac{1}{D_2}\|Q^y\|^2$.

From $D_2\delta_y Q^x_{i+\frac{1}{2},j+\frac{1}{2}}=D_1\delta_y\delta_x U_{i+\frac{1}{2},j+\frac{1}{2}}$, and $\delta_y\delta_x U_{i+\frac{1}{2},j+\frac{1}{2}}=\delta_x\delta_y U_{i+\frac{1}{2},j+\frac{1}{2}}$, we have that

$$D_2\delta_y(D_1\delta_x U_{i+\frac{1}{2},j})=D_1\delta_x(D_2\delta_y U_{i,j+\frac{1}{2}})=D_1\delta_x Q^y_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{3.12}$$

This ends the proof. □

Without loss of generality, we will give the proof of stability of the scheme (2.2)-(2.7) over 2×2 sub-domains. The interface indexes are $i=i_1$ and $j=j_1$. The first equation of Eq. (2.3) and the first equation of (2.5) can be rewritten into the equivalent forms

$$\begin{cases}\hat{\partial}_t U^*_{i,j}=\delta_x Q^{x,*}_{i,j}+f1^{n+1}_{i,j}, & i=1,2,\cdots,i_1-1,i_1+2,\cdots,I,\\ \hat{\partial}_t U^*_{i_1,j}=\delta_x Q^{x,*}_{i_1,j}+f1^{n+1}_{i_1,j}-\frac{Q^{x,*}_{i_1+\frac{1}{2},j}-\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}}{\Delta x}, & i=i_1,\\ \hat{\partial}_t U^*_{i_1+1,j}=\delta_x Q^{x,*}_{i_1+1,j}+f1^{n+1}_{i_1+1,j}+\frac{Q^{x,*}_{i_1+\frac{1}{2},j}-\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}}{\Delta x}, & i=i_1+1,\end{cases} \tag{3.13}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

and

$$\begin{cases} \check{\partial}_t U_{i,j}^{n+1} = \delta_y Q_{i,j}^{y,n+1} + f_{2i,j}^{n+1}, & j=1,2,\cdots,j_1-1,j_1+2,\cdots,J, \\ \check{\partial}_t U_{i,j_1}^{n+1} = \delta_y Q_{i,j_1}^{y,n+1} + f_{2i,j_1}^{n+1} - \frac{Q_{i,j_1+\frac{1}{2}}^{y,n+1} - \tilde{Q}_{i,j_1+\frac{1}{2}}^{y,n+1}}{\Delta y}, & j=j_1, \\ \check{\partial}_t U_{i,j_1+1}^{n+1} = \delta_y Q_{i,j_1+1}^{y,n+1} + f_{2i,j_1+1}^{n+1} + \frac{Q_{i,j_1+\frac{1}{2}}^{y,n+1} - \tilde{Q}_{i,j_1+\frac{1}{2}}^{y,n+1}}{\Delta y}, & j=j_1+1. \end{cases} \tag{3.14}$$

Lemma 3.4. *Let $\{U_{i,j}^*\}$ be the solution of Stage 1 of the scheme. We have that*

$$\begin{aligned} &(1-\Delta t)\hat{\partial}_t \| U^* \|^2 + \frac{(\Delta t)^2}{\Delta x}(\hat{\partial}_t | Q^{x,*} |^2_{\Gamma_{i_1-\frac{1}{2},y}} + \hat{\partial}_t | Q^{x,*} |^2_{\Gamma_{i_1+\frac{3}{2},y}}) \\ &\le \| U^n \|^2 + M(\| {f_1}^{n+1} \|^2 + \| {f_1}^{n+1} \|^2_\Gamma), \end{aligned} \tag{3.15}$$

where $M>0$ is a positive constant and

$$\| f_1^{n+1} \|^2_\Gamma = \frac{(\Delta t)^2}{\Delta x}(| {f_1}^{n+1} |^2_{\Gamma_{i_1,y}} + | {f_1}^{n+1} |^2_{\Gamma_{i_1+1,y}}). \tag{3.16}$$

Proof. Multiplying both sides of (3.13) by $U_{i,j}^* \Delta x \Delta y$, respectively, and summing them up for i,j from 1 to I,J, we obtain that

$$(\hat{\partial}_t U^*, U^*) = (\delta_x Q^{x,*}, U^*) + ({f_1}^{n+1}, U^*) + R, \tag{3.17}$$

where

$$\begin{aligned} R &= \sum_{j=1}^{J} \frac{Q_{i_1+\frac{1}{2},j}^{x,*} - \tilde{Q}_{i_1+\frac{1}{2},j}^{x,*}}{\Delta x}(U_{i_1+1,j}^* - U_{i_1,j}^*)\Delta x \Delta y \\ &= \frac{\Delta x}{D_1} < Q^{x,*} - \tilde{Q}^{x,*}, Q^{x,*} >_{\Gamma_{i_1+\frac{1}{2},y}} \\ &= \frac{\Delta x}{D_1}\Big((| Q^{x,*} |^2_{\Gamma_{i_1+\frac{1}{2},y}} - | \tilde{Q}^{x,*} |^2_{\Gamma_{i_1+\frac{1}{2},y}}) - < Q^{x,*} - \tilde{Q}^{x,*}, \tilde{Q}^{x,*} >_{\Gamma_{i_1+\frac{1}{2},y}} \Big). \end{aligned} \tag{3.18}$$

Subtracting the second equation of (3.13) from the third equation of (3.13), we get that at interface $x = x_{i_1+\frac{1}{2}}$

$$\frac{Q_{i_1+\frac{1}{2},j}^{x,*} - Q_{i_1+\frac{1}{2},j}^{x,n}}{\Delta t} = D_1 \frac{Q_{i_1+\frac{3}{2},j}^{x,*} - 2\tilde{Q}_{i_1+\frac{1}{2},j}^{x,*} + Q_{i_1-\frac{1}{2},j}^{x,*}}{\Delta x^2} + D_1 \delta_x {f_1}_{i_1+\frac{1}{2},j}^{n+1}. \tag{3.19}$$

Using (2.2) and (3.19), we have that

$$Q_{i_1+\frac{1}{2},j}^{x,*} - \tilde{Q}_{i_1+\frac{1}{2},j}^{x,*} = r_x(Q_{i_1-\frac{1}{2},j}^{x,*} + Q_{i_1+\frac{3}{2},j}^{x,*}) - r_x(Q_{i_1-\frac{1}{2},j}^{x,n} + Q_{i_1+\frac{3}{2},j}^{x,n}). \tag{3.20}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

So, we get that

$$-<Q^{x,*}-\tilde{Q}^{x,*},\tilde{Q}^{x,*}>_{\Gamma_{i_1+\frac{1}{2},y}}=-r_x\sum_{j=1}^{J}(Q^{x,*}_{i_1-\frac{1}{2},j}+Q^{x,*}_{i_1+\frac{3}{2},j})\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}\Delta y$$
$$+r_x\sum_{j=1}^{J}(Q^{x,n}_{i_1-\frac{1}{2},j}+Q^{x,n}_{i_1+\frac{3}{2},j})\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}\Delta y. \tag{3.21}$$

Applying the equality $2ab=a^2+b^2-(a-b)^2$ and the second equation of Eq. (3.13), we have that for the right side term of (3.21)

$$\begin{aligned}-r_x\sum_{j=1}^{J}Q^{x,*}_{i_1-\frac{1}{2},j}\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}\Delta y&=\frac{r_x}{2}\sum_{j=1}^{J}(\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}-Q^{x,*}_{i_1-\frac{1}{2},j})^2\Delta y-\frac{r_x}{2}(|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|\tilde{Q}^{x,*}|^2_{\Gamma_{i_1+\frac{1}{2},y}})\\&=\frac{D_1\Delta t}{2}\sum_{j=1}^{J}(\partial_t U^*_{i_1,j}-{f_1}^{n+1}_{i_1,j})^2\Delta y-\frac{r_x}{2}(|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|\tilde{Q}^{x,*}|^2_{\Gamma_{i_1+\frac{1}{2},y}})\\&=\frac{D_1\Delta t}{2}(|\hat{\partial}_t U^*|^2_{\Gamma_{i_1,y}}-|{f_1}^{n+1}|^2_{\Gamma_{i_1,y}})-\frac{r_x}{2}(|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|\tilde{Q}^{x,*}|^2_{\Gamma_{i_1+\frac{1}{2},y}})\\&\quad-D_1\frac{\Delta t}{\Delta x}\sum_{j=1}^{J}(\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}-Q^{x,*}_{i_1-\frac{1}{2},j}){f_1}^{n+1}_{i_1,j}\Delta y.\end{aligned} \tag{3.22}$$

Similarly, applying the third equation of Eq. (3.13), we have that

$$\begin{aligned}&-r_x\sum_{j=1}^{J}Q^{x,*}_{i_1+\frac{3}{2},j}\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}\Delta y\\&=\frac{D_1\Delta t}{2}(|\hat{\partial}_t U^*|^2_{\Gamma_{i_1+1,y}}-|{f_1}^{n+1}|^2_{\Gamma_{i_1+1,y}})-\frac{r_x}{2}(|Q^{x,*}|^2_{\Gamma_{i_1+\frac{3}{2},y}}+|\tilde{Q}^{x,*}|^2_{\Gamma_{i_1+\frac{1}{2},y}})\\&\quad-D_1\frac{\Delta t}{\Delta x}\sum_{j=1}^{J}(Q^{x,*}_{i_1+\frac{3}{2},j}-\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}){f_1}^{n+1}_{i_1,j}\Delta y.\end{aligned} \tag{3.23}$$

Substituting (3.22) and (3.23) into Eq. (3.21), we get that

$$\begin{aligned}&-<Q^{x,*}-\tilde{Q}^{x,*},\tilde{Q}^{x,*}>_{\Gamma_{i_1+\frac{1}{2},y}}\\&=\frac{D_1\Delta t}{2}(|\hat{\partial}_t U^*|^2_{\Gamma_{i_1,y}}+|\hat{\partial}_t U^*|^2_{\Gamma_{i_1+1,y}}-|{f_1}^{n+1}|^2_{\Gamma_{i_1,y}}-|{f_1}^{n+1}|^2_{\Gamma_{i_1+1,y}})\\&\quad-\frac{r_x}{2}(|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|Q^{x,*}|^2_{\Gamma_{i_1+\frac{3}{2},y}}+2|\tilde{Q}^{x,*}|^2_{\Gamma_{i_1+\frac{1}{2},y}})\\&\quad+r_x\sum_{j=1}^{J}(Q^{x,n}_{i_1-\frac{1}{2},j}+Q^{x,n}_{i_1+\frac{3}{2},j})\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}\Delta y\\&\quad+\frac{D_1\Delta t}{\Delta x}\sum_{j=1}^{J}\Big({f_1}^{n+1}_{i_1,j}(Q^{x,*}_{i_1-\frac{1}{2},j}-\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j})-{f_1}^{n+1}_{i_1+1,j}(Q^{x,*}_{i_1+\frac{3}{2},j}-\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j})\Big)\Delta y.\end{aligned} \tag{3.24}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

For the third and fourth terms of (3.24), we use Holder-inequality and ϵ-inequality, respectively, to get

$$\begin{aligned}
&r_x\sum_{j=1}^{J}(Q^{x,n}_{i_1-\frac{1}{2},j}+Q^{x,n}_{i_1+\frac{3}{2},j})\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j}\Delta y\\
&\leq\frac{r_x}{2}(|Q^{x,n}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|Q^{x,n}|^2_{\Gamma_{i_1+\frac{3}{2},y}}+2|\tilde{Q}^{x,*}|^2_{\Gamma_{i_1+\frac{1}{2},y}}),
\end{aligned}\tag{3.25}$$

$$\begin{aligned}
&\sum_{j=1}^{J}\frac{D_1\Delta t}{\Delta x}\Big(f_1{}^{n+1}_{i_1,j}(Q^{x,*}_{i_1-\frac{1}{2},j}-\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j})-f_1{}^{n+1}_{i_1+1,j}(Q^{x,*}_{i_1+\frac{3}{2},j}-\tilde{Q}^{x,*}_{i_1+\frac{1}{2},j})\Big)\Delta y\\
&\leq M\frac{(\Delta t)^2}{(\Delta x)^2}(|f_1{}^{n+1}|^2_{\Gamma_{i_1,y}}+|f_1{}^{n+1}|^2_{\Gamma_{i_1+1,y}})+\epsilon(|Q^{x,*}|^2_{\Gamma_{i_1+\frac{3}{2},y}}+|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|\tilde{Q}^{x,*}|^2_{\Gamma_{i_1+\frac{1}{2},y}}).
\end{aligned}\tag{3.26}$$

Substituting (3.25) and (3.26) into (3.24), and further using (3.18), we obtain that

$$\begin{aligned}
R\leq&\frac{\Delta t\Delta x}{2}(|\hat{\partial}_t U^*|^2_{\Gamma_{i_1,y}}+|\hat{\partial}_t U^*|^2_{\Gamma_{i_1+1,y}})+\frac{\Delta x}{D_1}|Q^{x,*}|^2_{\Gamma_{i_1+\frac{1}{2},y}}\\
&-\frac{\Delta t}{2\Delta x}\Big((|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|Q^{x,*}|^2_{\Gamma_{i_1+\frac{3}{2},y}})-(|Q^{x,n}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|Q^{x,n}|^2_{\Gamma_{i_1+\frac{3}{2},y}})\Big)\\
&+\frac{\epsilon\Delta x}{D_1}(|Q^{x,*}|^2_{\Gamma_{i_1+\frac{3}{2},y}}+|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}})+M\frac{(\Delta t)^2}{\Delta x}(|f_1{}^{n+1}|^2_{\Gamma_{i_1,y}}+|f_1{}^{n+1}|^2_{\Gamma_{i_1+1,y}}).
\end{aligned}\tag{3.27}$$

By Lemma 3.2 and Lemma 3.3, we have that

$$(\hat{\partial}_t U^*,U^*)=\frac{1}{2\Delta t}(\|U^*\|^2-\|U^n\|^2)+\frac{\Delta t}{2}\|\hat{\partial}_t U^*\|^2,\tag{3.28}$$

$$(\delta_x Q^{x,*},U^*)=-\frac{1}{D_1}\|Q^{x,*}\|^2.\tag{3.29}$$

Substituting (3.27)-(3.29) into (3.17), we obtain that

$$\begin{aligned}
&\frac{1}{2}\hat{\partial}_t\|U^*\|^2+\frac{\Delta t}{2\Delta x}\Big(|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|Q^{x,*}|^2_{\Gamma_{i_1+\frac{3}{2},y}}-(|Q^{x,n}|^2_{\Gamma_{i_1-\frac{1}{2},y}}+|Q^{x,n}|^2_{\Gamma_{i_1+\frac{3}{2},y}})\Big)\\
&\leq\frac{1}{2}\|U^*\|^2+M\Big(\|f_1{}^{n+1}\|^2+\frac{(\Delta t)^2}{\Delta x}(|f_1{}^{n+1}|^2_{\Gamma_{i_1,y}}+|f_1{}^{n+1}|^2_{\Gamma_{i_1+1,y}})\Big).
\end{aligned}\tag{3.30}$$

This ends the proof. □

Similarly, for Stage 2 of the scheme, we have Lemma 3.5.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Lemma 3.5. *Let* $\{U_{i,j}^{n+1}\}$ *be the solution of Stage 2 of the scheme, we have that*

$$\begin{aligned}&\check{\partial}_t \| U^{n+1} \|^2 + \frac{(\Delta t)^2}{\Delta y}(\check{\partial}_t | Q^{y,n+1} |^2_{\Gamma_{x,j_1-\frac{1}{2}}} + \check{\partial}_t | Q^{y,n+1} |^2_{\Gamma_{x,j_1+\frac{3}{2}}})\\ &\leq \| U^{n+1} \|^2 + M(\| f_2{}^{n+1} \|^2 + \| f_2{}^{n+1} \|^2_{\Gamma}),\end{aligned} \tag{3.31}$$

where $M>0$ *is a positive constant and*

$$\| f_2^{n+1} \|^2_{\Gamma} = \frac{(\Delta t)^2}{\Delta y}(| f_2{}^{n+1} |^2_{\Gamma_{x,j_1}} + | f_2{}^{n+1} |^2_{\Gamma_{x,j_1+1}}). \tag{3.32}$$

As the scheme (2.2)-(2.7) involves the numerical fluxes $Q^{x,*}, Q^{y,*}$ on interfaces. In order to study the stability of the scheme, we need to analyze the fluxes $Q^{x,*}$ and $Q^{y,*}$ in terms of $\hat{\partial}_t |Q^{x,*}|^2_{\Gamma_{i+\frac{1}{2},y}}$ and $\check{\partial}_t |Q^{y,n+1}|^2_{\Gamma_{x,j+\frac{1}{2}}}$ on the interface lines. Applying the operator $D_2\delta_y$ to (3.13) and (3.14), we get the equations of $Q^{y,*}_{i,j_1+\frac{3}{2}}$ and $Q^{y,*}_{i,j_1-\frac{1}{2}}$ as follows,

$$\begin{cases}\hat{\partial}_t Q^{y,*}_{i,j_1+\frac{3}{2}} = D_2\delta_y\delta_x Q^{x,*}_{i,j_1+\frac{3}{2}} + D_2\delta_y f1^{n+1}_{i,j_1+\frac{3}{2}}, & i=1,2,\cdots,i_1-1,i_1+2,\cdots,I,\\ \hat{\partial}_t Q^{y,*}_{i_1,j_1+\frac{3}{2}} = D_2\delta_y\delta_x Q^{x,*}_{i_1,j_1+\frac{3}{2}} + D_2\delta_y f1^{n+1}_{i_1,j_1+\frac{3}{2}} & \\ \qquad -D_2\frac{\delta_y Q^{x,*}_{i_1+\frac{1}{2},j_1+\frac{3}{2}} - \delta_y \tilde{Q}^{x,*}_{i_1+\frac{1}{2},j_1+\frac{3}{2}}}{\Delta x}, & i=i_1,\\ \hat{\partial}_t Q^{y,*}_{i_1+1,j_1+\frac{3}{2}} = D_2\delta_y\delta_x Q^{x,*}_{i_1+1,j_1+\frac{3}{2}} + D_2\delta_y f1^{n+1}_{i_1+1,j_1+\frac{3}{2}} & \\ \qquad +D_2\frac{\delta_y Q^{x,*}_{i_1+\frac{1}{2},j_1+\frac{3}{2}} - \delta_y \tilde{Q}^{x,*}_{i_1+\frac{1}{2},j_1+\frac{3}{2}}}{\Delta x}, & i=i_1+1,\end{cases} \tag{3.33}$$

with the boundary conditions $\delta_x Q^{y,*}_{\frac{1}{2},j_1+\frac{3}{2}} = \delta_x Q^{y,*}_{I+\frac{1}{2},j_1+\frac{3}{2}} = 0$, and

$$\begin{cases}\hat{\partial}_t Q^{y,*}_{i,j_1-\frac{1}{2}} = D_2\delta_y\delta_x Q^{x,*}_{i,j_1-\frac{1}{2}} + D_2\delta_y f1^{n+1}_{i,j_1-\frac{1}{2}}, & i=1,2,\cdots,i_1-1,i_1+2,\cdots,I,\\ \hat{\partial}_t Q^{y,*}_{i_1,j_1-\frac{1}{2}} = D_2\delta_y\delta_x Q^{x,*}_{i_1,j_1-\frac{1}{2}} + D_2\delta_y f1^{n+1}_{i_1,j_1-\frac{1}{2}} & \\ \qquad -D_2\frac{\delta_y Q^{x,*}_{i_1+\frac{1}{2},j_1-\frac{1}{2}} - \delta_y \tilde{Q}^{x,*}_{i_1+\frac{1}{2},j_1-\frac{1}{2}}}{\Delta x}, & i=i_1,\\ \hat{\partial}_t Q^{y,*}_{i_1+1,j_1-\frac{1}{2}} = D_2\delta_y\delta_x Q^{x,*}_{i_1+1,j_1-\frac{1}{2}} + D_2\delta_y f1^{n+1}_{i_1+1,j_1-\frac{1}{2}} & \\ \qquad +D_2\frac{\delta_y Q^{x,*}_{i_1+\frac{1}{2},j_1-\frac{1}{2}} - \delta_y \tilde{Q}^{x,*}_{i_1+\frac{1}{2},j_1-\frac{1}{2}}}{\Delta x}, & i=i_1+1,\end{cases} \tag{3.34}$$

with the boundary conditions $\delta_x Q^{y,*}_{\frac{1}{2},j_1-\frac{1}{2}} = \delta_x Q^{y,*}_{I+\frac{1}{2},j_1-\frac{1}{2}} = 0$. Using the equality (3.9),

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Eqs. (3.33) and (3.34) can be rewritten that

$$\begin{cases} \hat{\partial}_t Q^{y,*}_{i,j_1+\frac{3}{2}} = \delta_x(D_1\delta_x Q^{y,*}_{i,j_1+\frac{3}{2}}) + D_2\delta_y f1^{n+1}_{i,j_1+\frac{3}{2}}, & i=1,2,\cdots,i_1-1,i_1+2,\cdots,I, \\ \hat{\partial}_t Q^{y,*}_{i_1,j_1+\frac{3}{2}} = \delta_x(D_1\delta_x Q^{y,*}_{i_1,j_1+\frac{3}{2}}) + D_2\delta_y f1^{n+1}_{i_1,j_1+\frac{3}{2}} \\ \qquad - \frac{D_1\delta_x Q^{y,*}_{i_1+\frac{1}{2},j_1+\frac{3}{2}} - D_1\delta_x \tilde{Q}^{y,*}_{i_1+\frac{1}{2},j_1+\frac{3}{2}}}{\Delta x}, & i=i_1, \\ \hat{\partial}_t Q^{y,*}_{i_1+1,j_1+\frac{3}{2}} = \delta_x(D_1\delta_x Q^{y,*}_{i_1+1,j_1+\frac{3}{2}}) + D_2\delta_y f1^{n+1}_{i_1+1,j_1+\frac{3}{2}} \\ \qquad + \frac{D_1\delta_x Q^{y,*}_{i_1+\frac{1}{2},j_1+\frac{3}{2}} - D_1\delta_x \tilde{Q}^{y,*}_{i_1+\frac{1}{2},j_1+\frac{3}{2}}}{\Delta x}, & i=i_1+1, \end{cases} \tag{3.35}$$

and

$$\begin{cases} \hat{\partial}_t Q^{y,*}_{i,j_1-\frac{1}{2}} = \delta_x(D_1\delta_x Q^{y,*}_{i,j_1-\frac{1}{2}}) + D_2\delta_y f1^{n+1}_{i,j_1-\frac{1}{2}}, & i=1,2,\cdots,i_1-1,i_1+2,\cdots,I, \\ \hat{\partial}_t Q^{y,*}_{i_1,j_1-\frac{1}{2}} = \delta_x(D_1\delta_x Q^{y,*}_{i_1,j_1-\frac{1}{2}}) + D_2\delta_y f1^{n+1}_{i_1,j_1-\frac{1}{2}} \\ \qquad - \frac{D_1\delta_x Q^{y,*}_{i_1+\frac{1}{2},j_1-\frac{1}{2}} - D_1\delta_x \tilde{Q}^{y,*}_{i_1+\frac{1}{2},j_1-\frac{1}{2}}}{\Delta x}, & i=i_1, \\ \hat{\partial}_t Q^{y,*}_{i_1+1,j_1-\frac{1}{2}} = \delta_x(D_1\delta_x Q^{y,*}_{i_1+1,j_1-\frac{1}{2}}) + D_2\delta_y f1^{n+1}_{i_1+1,j_1-\frac{1}{2}} \\ \qquad + \frac{D_1\delta_x Q^{y,*}_{i_1+\frac{1}{2},j_1-\frac{1}{2}} - D_1\delta_y \tilde{Q}^{y,*}_{i_1+\frac{1}{2},j_1-\frac{1}{2}}}{\Delta x}, & i=i_1+1. \end{cases} \tag{3.36}$$

In the similar way of the proof of Lemma 3.4, from (3.35) and (3.36) we obtain the following Lemma 3.6.

Lemma 3.6. *On the interface lines* $\Gamma_{x,j_1-\frac{1}{2}}$ *and* $\Gamma_{x,j_1+\frac{3}{2}}$, *respectively, we have that*

$$\begin{aligned} &(1-\Delta t)\hat{\partial}_t \,|\, Q^{y,*} \,|^2_{\Gamma_{x,j_1+\frac{3}{2}}} + \frac{(\Delta t)^2}{\Delta x}\left(\hat{\partial}_t(D_1\delta_x Q^{y,*}_{i_1-\frac{1}{2},j_1+\frac{3}{2}})^2 + \hat{\partial}_t(D_1\delta_x Q^{y,*}_{i_1+\frac{3}{2},j_1+\frac{3}{2}})^2\right) \\ &\le |\, Q^{y,n} \,|^2_{\Gamma_{x,j_1+\frac{3}{2}}} + M(|\, \delta_y f1^{n+1} \,|^2_{\Gamma_{x,j_1+\frac{3}{2}}} + |\, \delta_y f1^{n+1} \,|^2_{I\Gamma_{x,j_1+\frac{3}{2}}}) \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} &(1-\Delta t)\hat{\partial}_t \,|\, Q^{y,*} \,|^2_{\Gamma_{x,j_1-\frac{1}{2}}} + \frac{(\Delta t)^2}{\Delta x}\left(\hat{\partial}_t(D_1\delta_x Q^{y,*}_{i_1-\frac{1}{2},j_1-\frac{1}{2}})^2 + \hat{\partial}_t(D_1\delta_x Q^{y,*}_{i_1+\frac{3}{2},j_1-\frac{1}{2}})^2\right) \\ &\le |\, Q^{y,n} \,|^2_{\Gamma_{x,j_1-\frac{1}{2}}} + M(|\, \delta_y f1^{n+1} \,|^2_{\Gamma_{x,j_1-\frac{1}{2}}} + |\, \delta_y f1^{n+1} \,|^2_{I\Gamma_{x,j_1-\frac{1}{2}}}), \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} |\, \delta_y f1^{n+1} \,|^2_{I\Gamma_{x,j_1+\frac{3}{2}}} &= \frac{(\Delta t)^2}{\Delta x}(|\, \delta_y f1^{n+1}_{i_1,j_1+\frac{3}{2}} \,|^2 + |\, \delta_y f1^{n+1}_{i_1+1,j_1+\frac{3}{2}} \,|^2), \\ |\, \delta_y f1^{n+1} \,|^2_{I\Gamma_{x,j_1-\frac{1}{2}}} &= \frac{(\Delta t)^2}{\Delta x}(|\, \delta_y f1^{n+1}_{i_1,j_1-\frac{1}{2}} \,|^2 + |\, \delta_y f1^{n+1}_{i_1+1,j_1-\frac{1}{2}} \,|^2). \end{aligned} \tag{3.39}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Similarly, taking the operator $D_1\delta_x$ to Eqs. (3.14), we have the equations of $Q^{x,n+1}_{i_1+\frac{3}{2},j}$ and $Q^{x,n+1}_{i_1-\frac{1}{2},j}$ as follows,

$$\begin{cases} \check{\partial}_t Q^{x,n+1}_{i_1+\frac{3}{2},j} = \delta_y(D_2\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},j}) + D_1\delta_x f_2{}^{n+1}_{i_1+\frac{3}{2},j}, & j=1,2,\cdots,j_1-1,j_1+2,\cdots,J,\\ \check{\partial}_t Q^{x,n+1}_{i_1+\frac{3}{2},j_1} = \delta_y(D_2\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},j_1}) + D_1\delta_x f_2{}^{n+1}_{i_1+\frac{3}{2},j_1} & \\ \quad - \frac{D_2\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},j_1+\frac{1}{2}} - D_2\delta_y \tilde{Q}^{x,n+1}_{i_1+\frac{3}{2},j_1+\frac{1}{2}}}{\Delta y}, & j=j_1,\\ \check{\partial}_t Q^{x,n+1}_{i_1+\frac{3}{2},j_1+1} = \delta_y(\delta_y D_2 Q^{x,n+1}_{i_1+\frac{3}{2},j_1+1}) + D_1\delta_x f_2{}^{n+1}_{i_1+\frac{3}{2},j_1+1} & \\ \quad + \frac{D_2\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},j_1+\frac{1}{2}} - D_2\delta_y \tilde{Q}^{x,n+1}_{i_1+\frac{3}{2},j_1+\frac{1}{2}}}{\Delta y}, & j=j_1+1, \end{cases} \tag{3.40}$$

with the boundary conditions $\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},\frac{1}{2}} = \delta_y Q^{x,n+1}_{i_1+\frac{3}{2},J+\frac{1}{2}} = 0$, and

$$\begin{cases} \check{\partial}_t Q^{x,n+1}_{i_1-\frac{1}{2},j} = \delta_y(\delta_y D_2 Q^{x,n+1}_{i_1-\frac{1}{2},j}) + D_1\delta_x f_2{}^{n+1}_{i_1-\frac{1}{2},j}, & j=1,2,\cdots,j_1-1,j_1+2,\cdots,J,\\ \check{\partial}_t Q^{x,n+1}_{i_1-\frac{1}{2},j_1} = \delta_y(\delta_y D_2 Q^{x,n+1}_{i_1-\frac{1}{2},j_1}) + D_1\delta_x f_2{}^{n+1}_{i_1-\frac{1}{2},j_1} & \\ \quad - \frac{D_2\delta_y Q^{x,n+1}_{i_1-\frac{1}{2},j_1+\frac{1}{2}} - D_2\delta_y \tilde{Q}^{x,n+1}_{i_1-\frac{1}{2},j_1+\frac{1}{2}}}{\Delta y}, & j=j_1,\\ \check{\partial}_t Q^{x,n+1}_{i_1-\frac{1}{2},j_1+1} = \delta_y(\delta_y D_2 Q^{x,n+1}_{i_1-\frac{1}{2},j_1+1}) + D_1\delta_x f_2{}^{n+1}_{i_1-\frac{1}{2},j_1+1} & \\ \quad + \frac{D_2\delta_y Q^{x,n+1}_{i_1-\frac{1}{2},j_1+\frac{1}{2}} - D_2\delta_y \tilde{Q}^{x,n+1}_{i_1-\frac{1}{2},j_1+\frac{1}{2}}}{\Delta y}, & j=j_1+1, \end{cases} \tag{3.41}$$

with the boundary conditions $\delta_y Q^{x,n+1}_{i_1-\frac{1}{2},\frac{1}{2}} = \delta_y Q^{x,n+1}_{i_1-\frac{1}{2},J+\frac{1}{2}} = 0$.

Further, we can get from (3.40) and (3.41) the following lemma.

Lemma 3.7. *On the interface lines $\Gamma_{i_1+\frac{3}{2},y}$ and $\Gamma_{i_1-\frac{1}{2},y}$, respectively, we have that*

$$\begin{aligned} &\check{\partial}_t |\, Q^{x,n+1}|^2_{\Gamma_{i_1+\frac{3}{2},y}} + \frac{(\Delta t)^2}{\Delta y}\left(\check{\partial}_t (D_2\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},j_1-\frac{1}{2}})^2 + \check{\partial}_t (D_2\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},j_1+\frac{3}{2}})^2\right)\\ \le &|\, Q^{x,n+1}|^2_{\Gamma_{i_1+\frac{3}{2},y}} + M(|\,\delta_x f_2{}^{n+1}|^2_{\Gamma_{i_1+\frac{3}{2},y}} + |\,\delta_x f_2{}^{n+1}|^2_{I\Gamma_{i_1+\frac{3}{2},y}}) \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} &\check{\partial}_t |\, Q^{x,n+1}|^2_{\Gamma_{i_1-\frac{1}{2},y}} + \frac{(\Delta t)^2}{\Delta y}\left(\check{\partial}_t (D_2\delta_y Q^{x,n+1}_{i_1-\frac{1}{2},j_1-\frac{1}{2}})^2 + \check{\partial}_t (D_2\delta_y Q^{x,n+1}_{i_1-\frac{1}{2},j_1+\frac{3}{2}})^2\right)\\ \le &|\, Q^{x,n+1}|^2_{\Gamma_{i_1-\frac{1}{2},y}} + M(|\,\delta_x f_2{}^{n+1}|^2_{\Gamma_{i_1-\frac{1}{2},y}} + |\,\delta_x f_2{}^{n+1}|^2_{I\Gamma_{i_1-\frac{1}{2},y}}), \end{aligned} \tag{3.43}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

where

$$|\delta_x f_2^{n+1}|^2_{I\Gamma_{i_1+\frac{3}{2},y}} = \frac{(\Delta t)^2}{\Delta y}(|\delta_x f_2^{n+1}_{i_1+\frac{3}{2},j_1}|^2 + |\delta_x f_2^{n+1}_{i_1+\frac{3}{2},j_1+1}|^2),$$
$$|\delta_x f_2^{n+1}|^2_{I\Gamma_{i_1-\frac{1}{2},y}} = \frac{(\Delta t)^2}{\Delta y}(|\delta_x f_2^{n+1}_{i_1-\frac{1}{2},j_1}|^2 + |\delta_x f_2^{n+1}_{i_1-\frac{1}{2},j_1+1}|^2). \tag{3.44}$$

Define

$$\begin{aligned}|||\{U,Q\}|||^2 = \|U\|^2 &+ \frac{(\Delta t)^2}{\Delta x}(|Q^x|^2_{\Gamma_{i_1-\frac{1}{2},y}} + |Q^x|^2_{\Gamma_{i_1+\frac{3}{2},y}}) + \frac{(\Delta t)^2}{\Delta y}(|Q^y|^2_{\Gamma_{x,j_1-\frac{1}{2}}} + |Q^y|^2_{\Gamma_{x,j_1+\frac{3}{2}}}) \\ &+ \frac{(\Delta t)^4}{\Delta x\Delta y}\Big[(D_2\delta_y Q^x_{i_1-\frac{1}{2},j_1-\frac{1}{2}})^2 + (D_2\delta_y Q^x_{i_1+\frac{3}{2},j_1-\frac{1}{2}})^2 \\ &\quad + (D_2\delta_y Q^x_{i_1-\frac{1}{2},j_1+\frac{3}{2}})^2 + (D_2\delta_y Q^x_{i_1+\frac{3}{2},j_1+\frac{3}{2}})^2\Big].\end{aligned} \tag{3.45}$$

Finally, we can have the following stability theorem.

Theorem 3.2. (Stability) *The scheme* (2.2)-(2.7) *is unconditionally stable over multi-block subdomains, i.e.,*

$$|||\{U^{n+1},Q^{n+1}\}|||^2 \le M\Big(|||\{U^0,Q^0\}|||^2 + \sum_{l=1}^{n+1}\Delta t(\|f_1^l\|^2 + \|f_2^l\|^2 + |||f^l|||^2_\Gamma)\Big), \tag{3.46}$$

where $M>0$ *is a positive constant independent of* $\Delta t,\Delta x,\Delta y$, *and*

$$\begin{aligned}|||f^l|||^2_\Gamma = &\frac{(\Delta t)^2}{\Delta x}(|{f_1}^l|^2_{\Gamma_{i_1},y} + |{f_1}^l|^2_{\Gamma_{i_1+1},y}) + \frac{(\Delta t)^2}{\Delta y}(|{f_2}^l|^2_{\Gamma_{x,j_1}} + |{f_2}^l|^2_{\Gamma_{x,j_1+1}}) \\ &+ \frac{(\Delta t)^2}{\Delta y}(|\delta_y {f_1}^l|^2_{\Gamma_{x,j_1-\frac{1}{2}}} + |\delta_y {f_1}^l|^2_{\Gamma_{x,j_1+\frac{3}{2}}}) + \frac{(\Delta t)^2}{\Delta x}(|\delta_x {f_2}^l|^2_{\Gamma_{i_1-\frac{1}{2},y}} + |\delta_x {f_2}^l|^2_{\Gamma_{i_1+\frac{3}{2},y}}) \\ &+ \frac{(\Delta t)^4}{\Delta x\Delta y}(|\delta_y {f_1}^l_{i_1,j_1-\frac{1}{2}}|^2 + |\delta_y {f_1}^l_{i_1+1,j_1-\frac{1}{2}}|^2 + |\delta_y {f_1}^l_{i_1,j_1+\frac{3}{2}}|^2 + |\delta_y {f_1}^l_{i_1+1,j_1+\frac{3}{2}}|^2) \\ &+ \frac{(\Delta t)^4}{\Delta x\Delta y}(|\delta_x {f_2}^l_{i_1-\frac{1}{2},j_1}|^2 + |\delta_x {f_2}^l_{i_1-\frac{1}{2},j_1+1}|^2 + |\delta_x {f_2}^l_{i_1+\frac{3}{2},j_1}|^2 + |\delta_x {f_2}^l_{i_1+\frac{3}{2},j_1+1}|^2).\end{aligned} \tag{3.47}$$

Proof. Multiplying both sides of (3.31) with $(1-\Delta t)$ and adding together with (3.15), we obtain that

$$\begin{aligned}&\partial_t\|U^{n+1}\|^2 + \frac{(\Delta t)^2}{\Delta x}(\hat\partial_t|Q^{x,*}|^2_{\Gamma_{i_1-\frac{1}{2},y}} + \hat\partial_t|Q^{x,*}|^2_{\Gamma_{i_1+\frac{3}{2},y}}) \\ &\quad + (1-\Delta t)\frac{(\Delta t)^2}{\Delta y}(\check\partial_t|Q^{y,n+1}|^2_{\Gamma_{x,j_1-\frac{1}{2}}} + \check\partial_t|Q^{y,n+1}|^2_{\Gamma_{x,j_1+\frac{3}{2}}}) \\ &\le (2-\Delta t)\|U^{n+1}\|^2 + M(\|{f_1}^{n+1}\|^2 + \|{f_2}^{n+1}\|^2 + \|{f_1}^{n+1}\|^2_\Gamma + \|{f_2}^{n+1}\|^2_\Gamma).\end{aligned} \tag{3.48}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Multiplying (3.37), (3.38) with $\frac{(\Delta t)^2}{\Delta y}$, respectively, multiplying (3.42), (3.43) with $\frac{(\Delta t)^2}{\Delta x}$, respectively, and adding with (3.48), applying the equality (3.9), we can obtain that

$$\begin{aligned}
&\partial_t \| U^{n+1} \|^2 + \frac{(\Delta t)^2}{\Delta x}(\partial_t |Q^{x,n+1}|^2_{\Gamma_{i_1+\frac{3}{2},y}} + \partial_t |Q^{x,n+1}|^2_{\Gamma_{i_1-\frac{1}{2},y}}) \\
&\quad + \frac{(\Delta t)^2}{\Delta y}(\partial_t |Q^{y,n+1}|^2_{\Gamma_{x,j_1+\frac{3}{2}}} + \partial_t |Q^{y,n+1}|^2_{\Gamma_{x,j_1-\frac{1}{2}}}) \\
&\quad + \frac{(\Delta t)^4}{\Delta x \Delta y}\Big[\partial_t (D_2\delta_y Q^{x,n+1}_{i_1-\frac{1}{2},j_1-\frac{1}{2}})^2 + \partial_t (D_2\delta_y Q^{x,n+1}_{i_1-\frac{1}{2},j_1+\frac{3}{2}})^2 \\
&\quad + \partial_t (D_2\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},j_1-\frac{1}{2}})^2 + \partial_t (D_2\delta_y Q^{x,n+1}_{i_1+\frac{3}{2},j_1+\frac{3}{2}})^2\Big] \\
\le &(2-\Delta t)\| U^{n+1} \|^2 + \frac{(\Delta t)^2}{\Delta x}(|Q^{x,n+1}|^2_{\Gamma_{i_1+\frac{3}{2},y}} + |Q^{x,n+1}|^2_{\Gamma_{i_1-\frac{1}{2},y}}) \\
&\quad + \frac{(\Delta t)^2}{\Delta y}(|Q^{y,n+1}|^2_{\Gamma_{x,j_1+\frac{3}{2}}} + |Q^{y,n+1}|^2_{\Gamma_{x,j_1-\frac{1}{2}}}) + M(\| f_1^{n+1} \|^2 + \| f_2^{n+1} \|^2 + \| |f^{n+1}| \|^2_\Gamma). \quad (3.49)
\end{aligned}$$

From (3.49), we have that

$$\partial_t \| |\{U^{n+1}, Q^{n+1}\}| \|^2 \le 2 \| |\{U^{n+1}, Q^{n+1}\}| \|^2 + M(\| f_1^{n+1} \|^2 + \| f_2^{n+1} \|^2 + \| |f^{n+1}| \|^2_\Gamma). \quad (3.50)$$

By Gronwall's lemma, (3.50) leads to (3.46). This ends the proof. □

Remark 3.1. In the scheme, we choose $f_1 = \frac{1}{2}f, f_2 = \frac{1}{2}f$, or $f_1 = f, f_2 = 0$, or $f_1 = 0, f_2 = f$. Our scheme (2.2)-(2.7) is unconditionally stable. The unconditional stability is proved in the sense that the numerical solutions of the scheme are continuously dependent on the initial value $u_0(x,y)$ and the right side term $f(x,y,t)$ without any restrictions of the mesh ratios (see, the description on p.1586, and Lemma 6.1, Lemma 6.2 and Theorem 6.3 on zigzag-shaped sub-domains in [14]).

The new important feature of the scheme (2.2)-(2.7) is that the scheme is proved to preserve mass over multi-block sub-domains.

4 Convergence

In this section, we will analyze the convergence and error estimate of the scheme in L^2-norm, without loss of generality, over 2×2 sub-domains with interface indexes $i = i_1$ and $j = j_1$. Let $u^n_{i,j}$ denote the value of the exact solution u at the point (x_i, y_j, t^n) and the fluxes $q^{x,n}_{i+\frac{1}{2},j} = D_1\delta_x u^n_{i+\frac{1}{2},j}$ and $q^{y,n}_{i,j+\frac{1}{2},j} = D_2\delta_y u^n_{i,j+\frac{1}{2}}$.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Define $u^*_{i,j}$,$\tilde{q}^{x,*}_{i_1+\frac{1}{2},j}$,$q^{x,*}_{i+\frac{1}{2},j}$,$q^{y,*}_{i,j+\frac{1}{2}}$ and $\tilde{q}^{y,n+1}_{i,j_1+\frac{1}{2}}$ to satisfy the following equations.

$$\begin{cases} u^*_{i,j}=u^n_{i,j}+\Delta t\delta_x q^{x,*}_{i,j}+\Delta t f^{n+1}_{i,j}, & i=1,\cdots,i_1-1,i_1+2,\cdots,I,\\ u^*_{i_1,j}=u^n_{i_1,j}+\Delta t\frac{\tilde{q}^{x,*}_{i_1+\frac{1}{2},j}-q^{x,*}_{i_1-\frac{1}{2},j}}{\Delta x}+\Delta t f^{n+1}_{i_1,j}, & i=i_1,\\ u^*_{i_1+1,j}=u^n_{i_1+1,j}+\Delta t\frac{q^{x,*}_{i_1+\frac{3}{2},j}-\tilde{q}^{x,*}_{i_1+\frac{1}{2},j}}{\Delta x}+\Delta t f^{n+1}_{i_1+1,j}, & i=i_1+1, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \tilde{q}^{x,*}_{i_1+\frac{1}{2},j}=q^{x,n}_{i_1+\frac{1}{2},j}+\Delta t D_1\delta_x^2 q^{x,n}_{i_1+\frac{1}{2},j}+\Delta t D_1\delta_x f^n_{i_1+\frac{1}{2},j},\\ q^{x,*}_{i+\frac{1}{2},j}=D_1\delta_x u^*_{i+\frac{1}{2},j}, \quad i=1,2,\cdots,I-1,\ j=1,2,\cdots,J,\\ q^{y,*}_{i,j+\frac{1}{2}}=D_2\delta_y u^*_{i,j+\frac{1}{2}}, \quad i=1,2,\cdots,I,\ j=1,2,\cdots,J-1,\\ \tilde{q}^{y,n+1}_{i,j_1+\frac{1}{2}}=q^{y,*}_{i,j_1+\frac{1}{2}}+\frac{\Delta t}{1+2r_y}D_2\delta_y^2 q^{y,*}_{i,j_1+\frac{1}{2}}+\frac{2r_y\Delta t}{1+2r_y}D_2\delta_y^2 q^{y,n}_{i,j_1+\frac{1}{2}}. \end{cases} \tag{4.2}$$

From the definition of $u^*_{i,j}$, we can obtain the truncation errors of the first equation of Eqs. (2.3) and (2.5) of the scheme as

$$\begin{cases} \hat{\partial}_t u^*_{i,j}=\delta_x q^{x,*}_{i,j}+f^{n+1}_{i,j}+\theta^{n+1}_{i,j}, & i=1,\cdots,i_1-1,i_1+1,\cdots,I,\\ \hat{\partial}_t u^*_{i_1,j}=\frac{\tilde{q}^{x,*}_{i_1+\frac{1}{2},j}-q^{x,*}_{i_1-\frac{1}{2},j}}{\Delta x}+f^{n+1}_{i_1,j}+\theta^{n+1}_{i_1,j}, & i=i_1,\\ \hat{\partial}_t u^*_{i_1+1,j}=\frac{q^{x,*}_{i_1+\frac{3}{2},j}-\tilde{q}^{x,*}_{i_1+\frac{1}{2},j}}{\Delta x}+f^{n+1}_{i_1+1,j}+\theta^{n+1}_{i_1+1,j}, & i=i_1+1, \end{cases} \tag{4.3}$$

and

$$\begin{cases} \check{\partial}_t u^{n+1}_{i,j}=\delta_y q^{y,n+1}_{i,j}+\eta^{n+1}_{i,j}, & j=1,2,\cdots,j_1-1,j_1+2,\cdots,J,\\ \check{\partial}_t u^{n+1}_{i,j_1}=\frac{\tilde{q}^{y,n+1}_{i,j_1+\frac{1}{2}}-q^{y,n+1}_{i,j_1-\frac{1}{2}}}{\Delta y}+\eta^{n+1}_{i,j_1}, & j=j_1,\\ \check{\partial}_t u^{n+1}_{i,j_1+1}=\frac{q^{y,n+1}_{i,j_1+\frac{3}{2}}-\tilde{q}^{y,n+1}_{i,j_1+\frac{1}{2}}}{\Delta y}+\eta^{n+1}_{i,j_1+1}, & j=j_1+1. \end{cases} \tag{4.4}$$

In order to analyze the truncation error $\eta^{n+1}_{i,j}$, we define the new variables $\psi^{n+1}_{i,j}$ for $\eta^{n+1}_{i,j}$ such that

$$\begin{cases} \psi^{n+1}_{i,j}=(1-D_1\Delta t\delta_x^2)\eta^{n+1}_{i,j}, & i=1,2,\cdots,i_1-1,i_1+2,\cdots,I,\\ \psi^{n+1}_{i_1,j}=\eta^{n+1}_{i_1,j}+D_1\frac{\Delta t}{\Delta x}\delta_x\eta^{n+1}_{i_1-\frac{1}{2},j}, & i=i_1,\\ \psi^{n+1}_{i_1+1,j}=\eta^{n+1}_{i_1+1,j}-D_1\frac{\Delta t}{\Delta x}\delta_x\eta^{n+1}_{i_1+\frac{3}{2},j}, & i=i_1+1. \end{cases} \tag{4.5}$$

Now, we first estimate $\theta^{n+1}_{i,j}$ and $\psi^{n+1}_{i,j}$ by the following lemma.

Lemma 4.1. *Assume that the exact solution u satisfies the following regularity condition:*

$$u\in C^0([0,T];C^5(\Omega))\cap C^1([0,T];C^3(\Omega))\cap C^2([0,T];C^1(\Omega)). \tag{4.6}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

We have that

$$\theta_{i,j}^{n+1}=0,\quad \forall i,j, \tag{4.7}$$

and

$$\|\psi^{n+1}\|\leq M\left(\Delta t+(\Delta x)^2+(\Delta y)^2+\frac{(\Delta t)^2}{\Delta x}+\frac{(\Delta t)^2}{\Delta y}\right), \tag{4.8}$$

where $M>0$ is a constant independent of $\Delta t,\Delta x$ and Δy.

Proof. By the definition for $u_{i,j}^*$, we can easily get that $\theta_{i,j}^{n+1}=0$. In the following, we will give the detailed estimates of $\psi_{i,j}^{n+1}$.

For all the strict interior nodes, $i\notin\{i_1,i_1+1\},j\notin\{j_1,j_1+1\}$, from the first equation of (4.4) and (4.5), we can have that

$$\begin{aligned}\psi_{i,j}^{n+1}&=(1-D_1\Delta t\delta_x^2)\left(\frac{u_{i,j}^{n+1}-u_{i,j}^*}{\Delta t}-\delta_y q_{i,j}^{y,n+1}\right)\\&=\frac{u_{i,j}^{n+1}-(1-D_1\Delta t\delta_x^2)u_{i,j}^*}{\Delta t}-\delta_x q_{i,j}^{y,n+1}-\delta_y q_{i,j}^{y,n+1}+D_1\Delta t\delta_x^2\delta_y q_{i,j}^{y,n+1}.\end{aligned} \tag{4.9}$$

Eliminating $u_{i,j}^*$, by using the first equation of (4.1) and using Taylor formula, we can obtain that

$$\begin{aligned}\psi_{i,j}^{n+1}&=\frac{u_{i,j}^{n+1}-u_{i,j}^n}{\Delta t}-\delta_x q_{i,j}^{x,n+1}-\delta_y q_{i,j}^{y,n+1}-f_{i,j}^{n+1}+D_1\Delta t\delta_x^2\delta_y q_{i,j}^{y,n+1}\\&=\mathcal{O}(\Delta t+(\Delta x)^2+(\Delta y)^2).\end{aligned} \tag{4.10}$$

For the near interface nodes, $j=j_1,i\notin\{i_1,i_1+1\}$, on the near interface line Γ_{x,j_1}, from the first equation of (4.4) and (4.5), it holds that

$$\psi_{i,j_1}^{n+1}=(1-D_1\Delta t\delta_x^2)\left(\frac{u_{i,j_1}^{n+1}-u_{i,j_1}^*}{\Delta t}-\delta_y q_{i,j_1}^{y,n+1}\right)-(1-D_1\Delta t\delta_x^2)\frac{\tilde{q}_{i,j_1+\frac{1}{2}}^{y,n+1}-q_{i,j_1+\frac{1}{2}}^{y,n+1}}{\Delta y}. \tag{4.11}$$

For the second term of (4.11), using the definition of $\tilde{q}_{i,j_1+\frac{1}{2}}^{y,n+1}$ of (4.2), the first equation of (4.1), Eq. (2.1), and the Taylor formula, we can get that

$$\begin{aligned}&\frac{1}{\Delta y}(1-D_1\Delta t\delta_x^2)(\tilde{q}_{i,j_1+\frac{1}{2}}^{y,n+1}-q_{i,j_1+\frac{1}{2}}^{y,n+1})\\&=\frac{1}{\Delta y}(1-D_1\Delta t\delta_x^2)\left(q_{i,j_1+\frac{1}{2}}^{y,*}+\frac{1}{1+2r_y}D_2\Delta t\delta_y^2 q_{i,j_1+\frac{1}{2}}^{y,*}+\frac{2r_y}{1+2r_y}D_2\Delta t\delta_y^2 q_{i,j_1+\frac{1}{2}}^{y,n}-q_{i,j_1+\frac{1}{2}}^{y,n+1}\right)\end{aligned}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

$$
\begin{aligned}
&=\frac{1}{\Delta y}\Big[q^{y,n}_{i,j_1+\frac{1}{2}}+D_2\Delta t\delta_y f^{n+1}_{i,j_1+\frac{1}{2}}+\frac{1}{1+2r_y}D_2\Delta t\delta_y^2 q^{y,n}_{i,j_1+\frac{1}{2}}+\frac{1}{1+2r_y}D_2\Delta t\delta_y^3 f^{n+1}_{i,j_1+\frac{1}{2}}\\
&\qquad+\frac{2r_y}{1+2r_y}D_2\Delta t\delta_y^2 q^{y,n}_{i,j_1+\frac{1}{2}}-\frac{2r_yD_1D_2(\Delta t)^2}{1+2r_y}\delta_x^2\delta_y^2 q^{y,n+1}_{i,j_1}-q^{y,n+1}_{i,j_1+\frac{1}{2}}+D_1\Delta t\delta_x^2 q^{y,n+1}_{i,j_1+\frac{1}{2}}\Big]\\
&=\frac{\Delta t}{\Delta y}\Big[-\partial_t q^{y,n+1}_{i,j_1+\frac{1}{2}}+D_2\delta_y^2 q^{y,n+1}_{i,j_1+\frac{1}{2}}+D_1\delta_x^2 q^{y,n+1}_{i,j_1+\frac{1}{2}}+D_2\delta_y f^{n+1}_{i,j_1+\frac{1}{2}}\\
&\qquad-D_2\Delta t\delta_y^2\partial_t q^{y,n+1}_{i,j_1+\frac{1}{2}}-D_1D_2\Delta t\frac{2r_y}{1+2r_y}\delta_x^2\delta_y^2 q^{y,n+1}_{i,j_1}+D_2\Delta t\frac{1}{1+2r_y}\delta_y^3 f^{n+1}_{i,j_1+\frac{1}{2}}\Big]\\
&=-\frac{1}{2}\frac{(\Delta t)^2}{\Delta y}(u_{tty})^{n_1}_{i,j_1}+\frac{D_1D_2}{12}\frac{\Delta t(\Delta x)^2}{\Delta y}(u_{xxxxy})^{n+1}_{s_1,t_1}+\frac{{D_2}^2}{12}\Delta t\Delta y(u_{yyyyy})^{n}_{i,t_2}\\
&\quad-D_2^2\frac{(\Delta t)^2}{\Delta y}(u_{tyyy})^{n_2}_{i,t_3}-D_1D_2\frac{(\Delta t)^2}{\Delta y}\frac{2r_y}{1+2r_y}(u_{xxxyy})^{n+1}_{s_2,t_4}+D_2\frac{(\Delta t)^2}{\Delta y}\frac{1}{1+2r_y}(f_{yyy})^{n+1}_{i,t_5}\\
&=\mathcal{O}\Big(\Delta t+\frac{(\Delta t)^2}{\Delta y}\Big),
\end{aligned}
\tag{4.12}
$$

where $\{n_1,n_2\}\in[n,n+1],\{s_1,s_2\}\in[i-1,i+1],\{t_1,t_2,t_3,t_4,t_5\}\in[j_1,j_1+1]$.

Similarly to the first term of (4.11) as analyzing (4.9), and combining with (4.12), we obtain that on the near interface line Γ_{x,j_1}

$$
\psi^{n+1}_{i,j_1}=\mathcal{O}\Big(\Delta t+(\Delta x)^2+(\Delta y)^2+\frac{(\Delta t)^2}{\Delta y}\Big),\quad i\notin\{i_1,i_1+1\},\ \ j=j_1,
\tag{4.13}
$$

and, it is also true on the near interface line Γ_{x,j_1+1} for $j=j_1+1, i\notin\{i_1,i_1+1\}$.

Further, similarly, for the near interface nodes, $i=i_1, j\notin\{j_1,j_1+1\}$, on the near interface line $\Gamma_{i_1,y}$, we can have the estimate of

$$
\psi^{n+1}_{i_1,j}=\mathcal{O}\Big(\Delta t+(\Delta x)^2+(\Delta y)^2+\frac{(\Delta t)^2}{\Delta x}\Big),\quad i=i_1,\ \ j\notin\{j_1,j_1+1\},
\tag{4.14}
$$

and, it is also true on the near interface line $\Gamma_{i_1+1,y}$, for $i=i_1+1, j\notin\{j_1,j_1+1\}$.

For the intersection point of the near interface lines $\Gamma_{i_1,y}$ and Γ_{x,j_1} with $i=i_1, j=j_1$, from the second equation of (4.4) and (4.5), we have that

$$
\begin{aligned}
\psi^{n+1}_{i_1,j_1}=&\Big[\frac{u^{n+1}_{i_1,j_1}-u^{*}_{i_1,j_1}}{\Delta t}-\delta_y q^{y,n+1}_{i_1,j_1}+D_1\frac{\Delta t}{\Delta x}\delta_x\Big(\frac{u^{n+1}_{i_1-\frac{1}{2},j_1}-u^{*}_{i_1-\frac{1}{2},j_1}}{\Delta t}-\delta_y q^{y,n+1}_{i_1-\frac{1}{2},j_1}\Big)\Big]\\
&+\Big[-\frac{1}{\Delta y}\Big(\tilde{q}^{y,n+1}_{i_1,j_1+\frac{1}{2}}-q^{y,n+1}_{i_1,j_1+\frac{1}{2}}+D_1\frac{\Delta t}{\Delta x}\delta_x(\tilde{q}^{y,n+1}_{i_1-\frac{1}{2},j_1+\frac{1}{2}}-q^{y,n+1}_{i_1-\frac{1}{2},j_1+\frac{1}{2}})\Big)\Big].
\end{aligned}
\tag{4.15}
$$

For the first term of (4.15), using the second equation of (4.1), the definition of $\tilde{q}^{x,*}_{i_1+\frac{1}{2},j_1}$ of

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

(4.2), (3.10) of Lemma 3.3, the Taylor formula and equation (2.1), we have that

$$\begin{aligned}
&\frac{u_{i_1,j_1}^{n+1}-u_{i_1,j_1}^{*}}{\Delta t}-\delta_y q_{i_1,j_1}^{y,n+1}+D_1\frac{\Delta t}{\Delta x}\delta_x\left(\frac{u_{i_1-\frac{1}{2},j_1}^{n+1}-u_{i_1-\frac{1}{2},j_1}^{*}}{\Delta t}-\delta_y q_{i_1-\frac{1}{2},j_1}^{y,n+1}\right)\\
=&\partial_t u_{i_1,j_1}^{n+1}-\delta_x q_{i_1,j_1}^{x,n+1}-\delta_y q_{i_1,j_1}^{y,n+1}-f_{i_1,j_1}^{n+1}-\Delta t D_1\delta_x^2\delta_y q_{i_1,j_1}^{y,n+1}\\
&+\frac{\Delta t}{\Delta x}\Big[\partial_t q_{i_1+\frac{1}{2},j_1}^{x,n+1}-D_1\delta_x^2 q_{i_1+\frac{1}{2},j_1}^{x,n+1}-D_2\delta_y^2 q_{i_1+\frac{1}{2},j_1}^{x,n+1}-D_1\delta_x f_{i_1+\frac{1}{2},j_1}^{n+1}\\
&\qquad -D_1\Delta t\delta_x^2\partial_t q_{i_1+\frac{1}{2},j_1}^{x,n+1}-\Delta t D_1\delta_x\partial_t f_{i_1+\frac{1}{2},j_1}^{n+1}\Big]\\
=&\mathcal{O}\left(\Delta t+(\Delta x)^2+(\Delta y)^2+\frac{(\Delta t)^2}{\Delta x}\right).
\end{aligned}\tag{4.16}$$

For the second term of (4.15), using the definition $\tilde{q}_{i,j_1+\frac{1}{2}}^{y,n+1}$ of (4.2) and the second equation of (4.1), we can obtain that

$$\begin{aligned}
&-\frac{1}{\Delta y}\left(\tilde{q}_{i_1,j_1+\frac{1}{2}}^{y,n+1}-q_{i_1,j_1+\frac{1}{2}}^{y,n+1}+D_1\frac{\Delta t}{\Delta x}\delta_x(\tilde{q}_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,n+1}-q_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,n+1})\right)\\
=&-\frac{1}{\Delta y}\left[q_{i_1,j_1+\frac{1}{2}}^{y,*}+\frac{D_2}{1+2r_y}\Delta t\delta_y^2 q_{i_1,j_1+\frac{1}{2}}^{y,*}+\frac{2r_y}{1+2r_y}D_2\Delta t\delta_y^2 q_{i_1,j_1+\frac{1}{2}}^{y,n}-q_{i_1,j_1+\frac{1}{2}}^{y,n+1}\right.\\
&\left.+D_1\frac{\Delta t}{\Delta x}\delta_x\left(q_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,*}+\frac{D_2}{1+2r_y}\Delta t\delta_y^2 q_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,*}+\frac{2r_y}{1+2r_y}D_2\Delta t\delta_y^2 q_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,n}-q_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,n+1}\right)\right]\\
=&-\frac{1}{\Delta y}\left[q_{i_1,j_1+\frac{1}{2}}^{y,n}-q_{i_1,j_1+\frac{1}{2}}^{y,n+1}+D_1\Delta t\delta_x^2 q_{i_1,j_1+\frac{1}{2}}^{y,n+1}+D_2\Delta t\delta_y^2 q_{i_1,j_1+\frac{1}{2}}^{y,n+1}+D_2\Delta t\delta_y f_{i_1,j_1+\frac{1}{2}}^{n+1}\right.\\
&+\frac{\Delta t}{\Delta x}D_2\delta_y\tilde{q}_{i_1+\frac{1}{2},j_1+\frac{1}{2}}^{x,*}-D_1\frac{\Delta t}{\Delta x}\delta_x q_{i_1+\frac{1}{2},j_1+\frac{1}{2}}^{y,n+1}-D_2(\Delta t)^2\delta_y^2\partial_t q_{i_1,j_1+\frac{1}{2}}^{y,n+1}\\
&+\frac{{D_2}^2}{1+2r_y}\left(\frac{(\Delta t)^2}{\Delta x}\delta_y^3 q_{i_1+\frac{1}{2},j_1+\frac{1}{2}}^{x,n}+(\Delta t)^2\delta_y^3 f_{i_1,j_1+\frac{1}{2}}^{n+1}\right)\\
&\left.+\frac{2r_y}{1+2r_y}D_1D_2\frac{(\Delta t)^2}{\Delta x}\delta_x\delta_y^2 q_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,n}\right].
\end{aligned}\tag{4.17}$$

Further, using the definition of $\tilde{q}_{i_1+\frac{1}{2},j}^{x,*}$ of (4.2), Taylor's expansion and Eq. (2.1), we have from (4.17) that

$$-\frac{1}{\Delta y}\left(\tilde{q}_{i_1,j_1+\frac{1}{2}}^{y,n+1}-q_{i_1,j_1+\frac{1}{2}}^{y,n+1}+D_1\frac{\Delta t}{\Delta x}\delta_x(\tilde{q}_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,n+1}-q_{i_1-\frac{1}{2},j_1+\frac{1}{2}}^{y,n+1})\right)$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

$$
\begin{aligned}
=&-\frac{1}{\Delta y}\Big[q^{y,n}_{i_1,j_1+\frac{1}{2}}-q^{y,n+1}_{i_1,j_1+\frac{1}{2}}+D_2\Delta t\delta_y^2 q^{y,n+1}_{i_1,j_1+\frac{1}{2}}+D_1\Delta t\delta_x^2 q^{y,n+1}_{i_1,j_1+\frac{1}{2}}+D_2\Delta t\delta_y f^{n+1}_{i_1,j_1+\frac{1}{2}}\\
&+\frac{(\Delta t)^2}{\Delta x}\Big(-D_1\delta_x\partial_t q^{y,n+1}_{i_1+\frac{1}{2},j_1+\frac{1}{2}}+D_1^2\delta_x^3 q^{y,n}_{i_1+\frac{1}{2},j_1+\frac{1}{2}}+D_1D_2\delta_x\delta_y f^{n}_{i_1+\frac{1}{2},j_1+\frac{1}{2}}\Big)\\
&+\frac{(\Delta t)^2}{\Delta x}\Big(\frac{{D_2}^2}{1+2r_y}\delta_y^3 q^{x,n}_{i_1+\frac{1}{2},j_1+\frac{1}{2}}+\frac{2r_y}{1+2r_y}D_1D_2\delta_x\delta_y^2 q^{y,n}_{i_1-\frac{1}{2},j_1+\frac{1}{2}}\Big)\\
&+(\Delta t)^2\Big(\frac{D_2^2}{1+2r_y}\delta_y^3 f^{n+1}_{i_1,j_1+\frac{1}{2}}-D_2\delta_y^2\partial_t q^{y,n+1}_{i_1,j_1+\frac{1}{2}}\Big)\Big]\\
=&\mathcal{O}\Big(\Delta t+\frac{(\Delta t)^2}{\Delta y}+\frac{(\Delta t)^2}{\Delta x\Delta y}\Big).
\end{aligned}
\tag{4.18}
$$

Using (4.16) and (4.18), from (4.15) we obtain that, at the intersection point of the near interface lines $\Gamma_{i_1,y}$ and Γ_{x,j_1}

$$
\psi^{n+1}_{i_1,j_1}=\mathcal{O}\Big(\Delta t+(\Delta x)^2+(\Delta y)^2+\frac{(\Delta t)^2}{\Delta x}+\frac{(\Delta t)^2}{\Delta y}+\frac{(\Delta t)^2}{\Delta x\Delta y}\Big),\quad i=i_1,\ j=j_1. \tag{4.19}
$$

Similarly, at other three intersection points of the near interface lines, they are also true for $i=i_1+1,j=j_1$; $i=i_1,j=j_1+1$ and $i=i_1+1,j=j_1+1$.

Using the estimate of (4.10) at the strict interior nodes, the estimates of (4.13) and (4.14) on the near interface lines, and the estimate of (4.19) at the four intersection points, from the definition of discrete L^2-norm, we have that

$$
\begin{aligned}
\|\psi^{n+1}\|^2=&\sum_{i=1}^{I}\sum_{j=1}^{J}(\psi^{n+1}_{i,j})^2\Delta x\Delta y\\
=&\sum_{i\neq i_1,i_1+1}^{I}\sum_{j\neq j_1,j_1+1}^{J}(\psi^{n+1}_{i,j})^2\Delta x\Delta y+\Delta x\sum_{j\neq j_1,j_1+1}^{J}((\psi^{n+1}_{i_1,j})^2+(\psi^{n+1}_{i_1+1,j})^2)\Delta y\\
&+\Delta y\sum_{i\neq i_1,i_1+1}^{I}((\psi^{n+1}_{i,j_1})^2+(\psi^{n+1}_{i,j_1+1})^2)\Delta x\\
&+\Delta x\Delta y\Big[(\psi^{n+1}_{i_1,j_1})^2+(\psi^{n+1}_{i_1+1,j_1})^2+(\psi^{n+1}_{i_1,j_1+1})^2+(\psi^{n+1}_{i_1+1,j_1+1})^2\Big]\\
\le&M\Big((\Delta t)^2+(\Delta x)^4+(\Delta y)^4+\frac{(\Delta t)^4}{(\Delta x)^2}+\frac{(\Delta t)^4}{(\Delta y)^2}\Big).
\end{aligned}
\tag{4.20}
$$

This ends the proof. □

We then estimate $\{\eta^{n+1}_{i,j}\}$ in (4.4). Considering (4.5), we have the following system:

$$
\mathbf{B}\eta^{n+1}=\psi^{n+1}, \tag{4.21}
$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

where **B** is a block diagonal matrix:

$$\mathbf{B}=\begin{pmatrix} \mathbf{C} & & 0 \\ & \ddots & \\ & \mathbf{C} & \\ 0 & & \ddots \\ & & \mathbf{C} \end{pmatrix}_{J\times J}, \quad \mathbf{C}=\begin{pmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{pmatrix}_{2\times 2}, \tag{4.22}$$

with

$$\mathbf{A}_1=\begin{pmatrix} 1+r_x & -r_x & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ -r_x & 1+2r_x & -r_x & 0 & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 0 & -r_x & 1+2r_x & -r_x \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 & -r_x & 1+r_x \end{pmatrix}_{i_1\times i_1}, \tag{4.23}$$

$$\mathbf{A}_2=\begin{pmatrix} 1+r_x & -r_x & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ -r_x & 1+2r_x & -r_x & 0 & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 0 & -r_x & 1+2r_x & -r_x \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 & -r_x & 1+r_x \end{pmatrix}_{(I-i_1)\times(I-i_1)}. \tag{4.24}$$

From (4.23) and (4.24), matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are symmetric positive-definite and strictly diagonal dominated and **B** then is symmetric positive-definite and strictly diagonal dominated. By Gerschgorin Theorem, it is clear from (4.23) and (4.24) that $\lambda(\mathbf{A}_1)\geq 1, \lambda(\mathbf{A}_2)\geq 1$. Thus, from (4.22), we have $\lambda(\mathbf{B})\geq 1$ and $\rho(\mathbf{B})\geq 1$. Therefore, we can get that $\|\mathbf{B}\|\geq 1$. Further, using the estimate (4.21), we have that $\|\eta^{n+1}\|_0=\|\mathbf{B}^{-1}\psi^{n+1}\|_0\leq\|\mathbf{B}^{-1}\|\|\psi^{n+1}\|_0\leq\|\psi^{n+1}\|_0$. We thus get $\|\eta^{n+1}\|\leq\|\psi^{n+1}\|$, which leads to the following estimation of truncation error η in the discrete L^2-norm.

Lemma 4.2. *Assume that the exact solution u satisfies the following regularity condition:*

$$u\in C^0([0,T];C^5(\Omega))\cap C^1([0,T];C^3(\Omega))\cap C^2([0,T];C^1(\Omega)). \tag{4.25}$$

Then we have that

$$\|\eta^{n+1}\|\leq M\Big(\Delta t+(\Delta x)^2+(\Delta y)^2+\frac{(\Delta t)^2}{\Delta x}+\frac{(\Delta t)^2}{\Delta y}\Big), \tag{4.26}$$

where $M>0$ is a positive constant independent of $\Delta t, \Delta x$, and Δy.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Finally, we consider the convergence and error estimate of the scheme (2.2)-(2.7). Let

$$\begin{aligned}&\rho_{i,j}=u_{i,j}-U_{i,j},\quad \zeta^x_{i+\frac{1}{2},j}=q^x_{i+\frac{1}{2},j}-Q^x_{i+\frac{1}{2},j}=D_1\delta_x\rho_{i+\frac{1}{2},j},\\&\zeta^y_{i,j+\frac{1}{2}}=q^y_{i,j+\frac{1}{2}}-Q^y_{i,j+\frac{1}{2}}=D_2\delta_y\rho_{i,j+\frac{1}{2}},\\&\tilde{\zeta}^x_{i_1+\frac{1}{2},j}=\tilde{q}^x_{i_1+\frac{1}{2},j}-\tilde{Q}^x_{i_1+\frac{1}{2},j},\quad \tilde{\zeta}^y_{i,j_1+\frac{1}{2}}=\tilde{q}^y_{i,j_1+\frac{1}{2}}-\tilde{Q}^y_{i,j_1+\frac{1}{2}}.\end{aligned}\tag{4.27}$$

Subtracting the first equation of Eqs. (2.3) and (2.5) from Eqs. (4.3) and (4.4), we have the error equations

$$\begin{cases}\hat{\partial}_t\rho^*_{i,j}=\delta_x\zeta^{x,*}_{i,j}+\theta^{n+1}_{i,j}, & i=1,\cdots,i_1-1,i_1+2,\cdots,I,\\ \hat{\partial}_t\rho^*_{i_1,j}=\frac{\tilde{\zeta}^{x,*}_{i_1+\frac{1}{2},j}-\zeta^{x,*}_{i_1-\frac{1}{2},j}}{\Delta x}+\theta^{n+1}_{i_1,j}, & i=i_1,\\ \hat{\partial}_t\rho^*_{i_1+1,j}=\frac{\zeta^{x,*}_{i_1+\frac{3}{2},j}-\tilde{\zeta}^{x,*}_{i_1+\frac{1}{2},j}}{\Delta x}+\theta^{n+1}_{i_1+1,j}, & i=i_1+1,\end{cases}\tag{4.28}$$

and

$$\tilde{\zeta}^{x,*}_{i_1+\frac{1}{2},j}=\zeta^{x,n}_{i_1+\frac{1}{2},j}+r_x(\zeta^{x,n}_{i_1-\frac{1}{2},j}-2\tilde{\zeta}^{x,*}_{i_1+\frac{1}{2},j}+\zeta^{x,n}_{i_1+\frac{3}{2},j})+\gamma^{n+1}_{i_1+\frac{1}{2},j},\tag{4.29}$$

where

$$\gamma^{n+1}_{i_1+\frac{1}{2},j}=2r_xD_1\Delta t\delta_x^2q^{x,n}_{i_1+\frac{1}{2},j}+(1+2r_x)D_1\Delta t\delta_xf^n_{i_1+\frac{1}{2},j}-D_1\Delta t\delta_xf^{n+1}_{i_1+\frac{1}{2},j}=\mathcal{O}(\Delta t+r_x\Delta t),\tag{4.30}$$

and with the boundary conditions $\zeta^{x,*}_{\frac{1}{2},j}=0$ and $\zeta^{x,*}_{I+\frac{1}{2},j}=0$, and

$$\begin{cases}\check{\partial}_t\rho^{n+1}_{i,j}=\delta_y\zeta^{y,n+1}_{i,j}+\eta^{n+1}_{i,j}, & j=1,2,\cdots,j_1-1,j_1+2,\cdots,J,\\ \check{\partial}_t\rho^{n+1}_{i,j_1}=\frac{\tilde{\zeta}^{y,n+1}_{i,j_1+\frac{1}{2}}-\zeta^{y,n+1}_{i,j_1-\frac{1}{2}}}{\Delta y}+\eta^{n+1}_{i,j_1}, & j=j_1,\\ \check{\partial}_t\rho^{n+1}_{i,j_1+1}=\frac{\zeta^{y,n+1}_{i,j_1+\frac{3}{2}}-\tilde{\zeta}^{y,n+1}_{i,j_1+\frac{1}{2}}}{\Delta y}+\eta^{n+1}_{i,j_1+1}, & j=j_1+1,\end{cases}\tag{4.31}$$

and

$$\tilde{\zeta}^{y,n+1}_{i,j_1+\frac{1}{2}}=\zeta^{y,*}_{i,j_1+\frac{1}{2}}+r_y(\zeta^{y,*}_{i,j_1-\frac{1}{2}}-2\tilde{\zeta}^{y,n+1}_{i,j_1+\frac{1}{2}}+\zeta^{y,*}_{i,j_1+\frac{3}{2}})+\varepsilon^{n+1}_{i,j_1+\frac{1}{2}},\tag{4.32}$$

where

$$\varepsilon^{n+1}_{i,j_1+\frac{1}{2}}=2r_yD_2\Delta t\delta_y^2q^{y,n}_{i,j_1+\frac{1}{2}}=\mathcal{O}(r_y\Delta t),\tag{4.33}$$

and with the boundary conditions $\zeta^{y,n+1}_{i,\frac{1}{2}}=0$ and $\zeta^{y,n+1}_{i,J+\frac{1}{2}}=0$.

Similarly to Lemmas 3.3-3.7, and using the estimates of $\gamma_{i_1+\frac{1}{2},j}$ $\varepsilon_{i,j_1+\frac{1}{2}}$, and the truncation estimates in Lemma 4.1 and Lemma 4.2, we can finally obtain the convergence theorem:

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Theorem 4.1. (Convergence) *Assume the conditions of Lemma* 4.2 *hold. The scheme* (2.2)-(2.7) *over multi-block sub-domains has the error estimate in the* L^2*-norm as*

$$\| u^n - U^n \| \leq M\left(\Delta t + (\Delta x)^2 + (\Delta y)^2 + \frac{(\Delta t)^2}{(\Delta x)^{\frac{3}{2}}} + \frac{(\Delta t)^2}{(\Delta y)^{\frac{3}{2}}} \right), \tag{4.34}$$

for $n \geq 1$, *where* $M > 0$ *is a constant independent of* Δt, Δx *and* Δy.

Remark 4.1. We prove the convergence of our scheme with the estimate of (4.34) over multi-block sub-domains due to the interfaces, which reaches the optimal order of $\mathcal{O}(\Delta t + (\Delta x)^2 + (\Delta y)^2)$ when using $\Delta t = \mathcal{O}((\Delta x)^{3/2}), \Delta t = \mathcal{O}((\Delta y)^{3/2})$. The regularity conditions in Lemmas 4.1-4.2 and Theorem 4.1 are a little bit higher than the classical ones due to the use of the flux schemes on the interfaces. This proved error estimate (4.34) in L_2-norm of our mass-preserving S-DDM scheme in Theorem 4.1 is much better than that of $\mathcal{O}(\Delta t + h^2 + \sqrt{h^{-1}}(\Delta t + h^2))$ in L_2-norm obtained in Theorem 6.4 for their schemes on Zigzag-shaped sub-domains in [14], where the schemes in [14] do not preserve mass.

Remark 4.2. On the other hand, our mass-preserving S-DDM scheme (2.2)-(2.7) and its analysis can be extended to the Parabolic equations with reaction terms and with Dirichlet boundary conditions and to the three dimensional Parabolic equations, which will be further studied in our next work.

5 Numerical experiments

In the section, we present numerical experiments to illustrate the performance of the scheme.

Example 5.1. Firstly, we consider the parabolic equation in two dimensions with Neumann boundary condition and without right side function. The exact solution is

$$u = \frac{1}{2}\left(e^{-2D\pi^2 t}\cos(\pi x)\cos(\pi y) + 1\right). \tag{5.1}$$

The domains $\Omega = [0,1] \times [0,1]$ and $D = D_1 = D_2$. Take uniform mesh steps $\Delta x = \Delta y = h$ and divide the domain into 2×2 sub-domains. We focus on the accuracy, orders of convergence in time and space, and mass conservation.

Let error $e_h = \max_{0 \leq l \leq n} \| u^l - U^l \|$. Table 1 presents the errors computed by our scheme and the order of convergence in space step at $t = 0.1$. The space step size h is selected from $1/10$ to $1/40$, while the time step size is taken as $\Delta t = 1/100000$ for obtaining the space step error order. From Table 1, we can see clearly that our scheme has second-order convergence in L^2-norm in spatial step for the cases of different diffusions.

The time order of convergence of the scheme at time $t = 0.1$ is presented in Table 2. Taking $h = \Delta t$, Table 2 shows clearly that the scheme is of first-order convergence in time step for $D = 2E-2, 2E-1$ and has much better convergence for $D = 2$.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Table 1: Errors and ratios of convergence in space for different diffusion D.

$D\backslash h$		1/10	1/20	1/30	1/40
2E-2	e_h	3.9669E-5	9.9429E-6	4.4223E-6	2.4890E-6
	ratio	-	1.9963	1.9982	1.9981
2E-1	e_h	3.3254E-4	8.3439E-5	3.7208E-5	2.1019E-5
	ratio	-	1.9947	1.9918	1.9890
2	e_h	5.699E-4	1.4450E-4	6.6053E-5	3.8620E-5
	ratio	-	1.9796	1.9307	1.9036

Table 2: Errors and ratios of convergence in time for different diffusion D.

$D\backslash \Delta t$		1/50	1/100	1/150	1/200
2E-2	e_h	3.3475E-5	1.6835E-5	1.1157E-5	8.3407E-6
	ratio	-	0.9916	1.0146	1.0113
2E-1	e_h	3.3658E-3	1.9595E-3	1.29999E-3	9.4458E-4
	ratio	-	0.7804	1.0122	1.10959
2	e_h	9.3206E-2	5.5072E-2	2.7380E-2	1.4350E-2
	ratio	-	0.87777	1.7235	2.2458

Table 3: Errors of mass for different diffusion D and different Δt.

$\Delta t\backslash D$	0.01	0.1	1	10
1/100	7.7716E-16	1.9984E-15	1.1713E-14	2.0939E-13
1/500	2.5535E-15	4.1078E-15	1.1990E-14	3.1009E-13
1/1000	2.2760E-15	7.8826E-15	2.2204E-14	1.1369E-13
1/2000	6.7168E-15	5.7065E-14	1.7486E-14	4.4398E-13

Then, considering conservation of mass of our scheme, we let $\text{Mass}_\text{n}=\sum_{i,j}C^n_{i,j}\Delta x\Delta y$, $\text{Mass}_0=\sum_{i,j}C^0_{i,j}\Delta x\Delta y$, and $\text{MassError}=|\text{Mass}_\text{n}-\text{Mass}_0|$. Take the space step size $h=1/40$ and the time step Δt from 1/100 to 1/2000. Table 3 presents the errors of mass at time $t=0.2$ for different diffusions and with different time steps. From Table 3, it is clear that our scheme satisfies conservation of mass for the cases of different diffusion D and different time step, since the errors of mass have reached the machine accuracy 10^{-13}.

Example 5.2. Consider the parabolic equation with a right side function

$$\begin{cases} \frac{\partial u}{\partial t}=\nabla\cdot(D\nabla u)+f(x,y,t), & (x,y,t)\in\Omega\times(0,T], \\ \nabla u\cdot\vec{n}=0, & (x,y)\in\partial\Omega, \\ u(x,y,0)=x^2(1-x)^2y^2(1-y)^2, & (x,y)\in\Omega, \end{cases} \tag{5.2}$$

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Table 4: Errors and ratios of space convergence for different diffusion D and with a right side function.

$D\backslash h$		1/10	1/20	1/30	1/40
2E-2	e_h	1.1626E-5	3.0109E-6	1.3511E-6	7.6523E-7
	ratio	-	1.9606	1.9762	1.9762
2E-1	e_h	6.4453E-5	1.6016E-5	7.0312E-6	3.9063E-6
	ratio	-	2.0088	2.0303	2.0432
2	e_h	2.8828E-4	7.2266E-5	3.2031E-5	1.7969E-5
	ratio	-	1.9961	2.0067	2.0094

Table 5: Errors and ratios of time convergence for different diffusion D and with a right side function.

$D\backslash \Delta t$		1/50	1/100	1/150	1/200
2E-2	e_h	7.7449E-6	3.7660E-6	2.4872E-6	1.8567E-6
	ratio	-	1.0402	1.0231	1.0163
2E-1	e_h	8.4480E-5	4.2234E-5	2.8160E-5	2.1122E-5
	ratio	-	1.0002	0.9997	0.9996
2	e_h	5.9914E-5	2.7320E-5	1.7611E-5	1.2988E-5
	ratio	-	1.1339	1.0813	1.0584

Table 6: Stability: L^2 error of the solution at different t with spatial step $h=1/200$.

t	$D\backslash \Delta t$	1/400	1/200	1/100	1/50	1/20	1/10
0.5	2E-2	6.5426E-7	1.2872E-6	2.5487E-6	5.0531E-6	1.2500E-5	2.4219E-5
	2E-1	1.2675E-5	2.5257E-5	5.0281E-5	9.9777E-5	2.4414E-4	4.7344E-4
	2	2.1842E-5	4.3199E-5	8.5422E-5	1.6902E-4	4.1914E-4	8.3867E-4
1	2E-2	9.4418E-7	1.8567E-6	3.6779E-6	7.3027E-6	1.7969E-5	3.5547E-5
	2E-1	1.0595E-5	2.1122E-5	4.2109E-5	8.3848E-5	2.0742E-4	4.1016E-4
	2	6.8813E-6	1.2988E-5	2.5129E-5	4.9168E-5	1.1953E-4	2.3242E-4

where

$$f(x,y,t)=-2De^{-2Dt}\Big(x^2(1-x)^2y^2(1-y)^2+x^2(1-x)^2 (6y^2-6y+1)+y^2(1-y)^2(6x^2-6x+1)\Big). \tag{5.3}$$

The ratios of space convergence at time $t=1$ are listed in Table 4. The space step size h is selected from 1/10 to 1/40, and the time step size is used as $\Delta t=1/10000$. It is clearly shown that our scheme is of second-order convergence in L^2-norm in spatial step for the case with a right side function. The ratios of time convergence of the scheme at time $t=1$ are listed in Table 5, by taking $h=\Delta t$. It shows that the scheme is of first-order convergence in time step for the case with a right side function.

Then, we test the stability of the scheme. Table 6 shows the numerical results on the

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

Table 7: Errors of mass for different diffusion D and a right side function.

$\Delta t \backslash D$	0.01	0.1	1	10
1/100	9.8629E-18	2.8786E-17	5.9848E-17	3.6212E-16
1/500	2.5513E-17	4.6051E-17	6.6136E-17	2.8796E-16
1/1000	3.5782E-17	6.4131E-17	1.1633E-16	9.7145E-17
1/2000	7.9946E-17	5.9468E-16	8.8579E-17	3.4022E-16

values of $\frac{\Delta t}{h}$, where it reaches $\frac{\Delta t}{h}=20$ in the last column (i.e. $\frac{\Delta t}{h^2}=4000$). We can see clearly that the scheme is stable. The table supports the stability result of our scheme (Theorem 3.2).

Finally, for the problem with a right side function, let the numerical mass at t^n be $\text{Mass}_\text{n}=\sum_{i,j}C_{i,j}^n\Delta x\Delta y+\Delta t\sum_{l=1}^n\sum_{i,j}f_{i,j}^l\Delta x\Delta y$. Then, the error of mass is $\text{MassError}=|\text{Mass}_\text{n}-\text{Mass}_0|$. Errors of mass at $t=1$ are given in Table 7. We can see clearly that our scheme preserves mass for the problem with a right side function.

Remark 5.1. From numerical results in Examples 5.1 and 5.2, our scheme reaches second order in space, and it reaches first order in time even using $\Delta t=h$, which implies that numerical tests have shown a better rate of convergence than the theoretical results obtained.

Example 5.3. In the third example, we compute the shape of heat propagation. The initial heat distribution solution is given as 1 at space points A, B, C and D and 0 for all other points (see, Fig. 3).

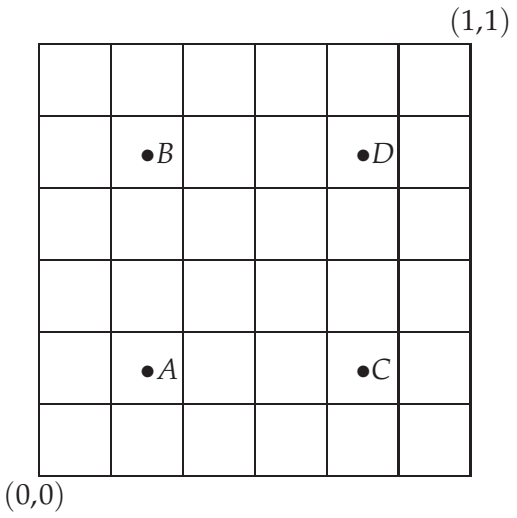
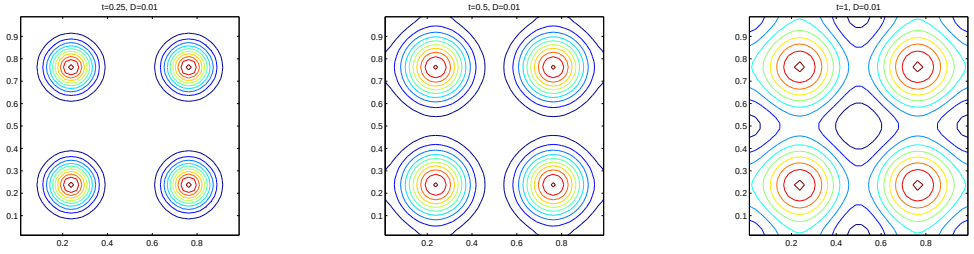


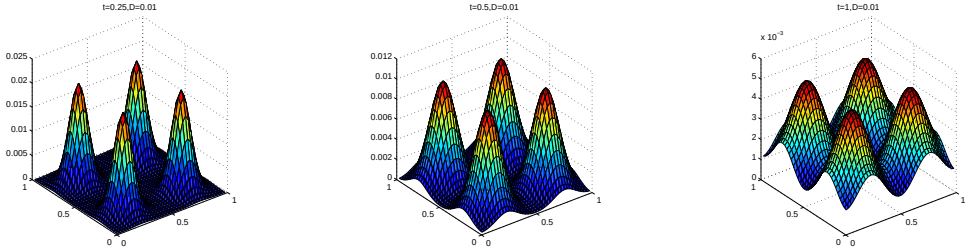
Figure 3: The initial heat distribution in the domain.

For the problems with different diffusions $D=0.01$, and 1, the numerical heat distributions are shown in Figs. 4 and 5, where we use $h=1/40$ and $\Delta t=1/1000$. In Fig. 4, we

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

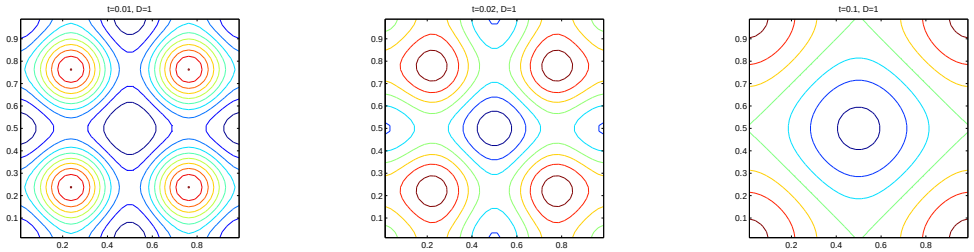


(a) The contour plots of concentration at $t=0.25,0.5,1$

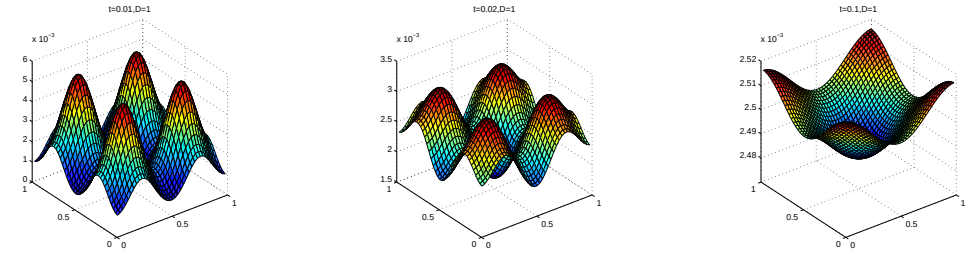


(b) The surfaces plots of concentration at $t=0.25,0.5,1$

Figure 4: The problem with the diffusion $D=0.01$ at different time.



(a) The contour plots of concentration at $t=0.01,0.02,0.1$



(b) The surfaces plots of concentration at $t=0.01,0.02,0.1$

Figure 5: The problem with the diffusion $D=1$ at different time.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

can see that the highest heats at initial gradually decrease and propagate smoothly in all directions.

Fig. 5 shows that heat distributions at different times when the diffusion, $D = 1$, is large. When the heat solutions cross the interfaces of sub-domains, the shapes of solutions move smoothly without any numerical oscillation. Our mass-preserving S-DDM scheme works excellently for the problem as well.

Acknowledgments

The authors thank the reviewers for the valuable suggestions which have helped to improve the paper greatly. This work was supported partially by Natural Sciences and Engineering Research Council of Canada and by Natural Science Foundation of China under grant 11271232.

References

[1] K. Aziz and A. Settari, Petroleum Reservoir Simulation, Applied Science Publisher, Ltd., London, 1979.
[2] J. Bear, Hydraulics of Groundwater, McGraw-Hill, New York, 1978.
[3] Z. X. Chen, G. R. Huan and Y. L. Ma, Computational Methods for Multiphase Flows in Porous Media, Computational Science and Engineering Series, SIAM, Philadelphia, 2 (2006).
[4] C. N. Dawson, Q. Du and T. F. Dupont, A finite difference domain decomposition algorithm for numerical solution of heat equations, Math. Comp., 57 (1991), 63-71.
[5] C. Du and D. Liang, An Efficient S-DDM Iterative Approach for Compressible Contamination Fluid Flows in Porous Media, J. Comput. Phys., 229 (2010), 4501-4521.
[6] Q. Du, M. Mu and Z. N. Wu, Efficient parallel algorithms for parabolic problems, SIAM J. Numer. Anal., 39 (2001), 1469-1487.
[7] M. Dryja and X. Tu, A domain decomposition discretization of parabolic problems, Numer. Math., 107 (2007), 625-640.
[8] S. Gaiffe, R. Glowinski and P. Lemonnier, Domain decomposition and splitting methods for parabolic equations via a mixed formula, in the 12th international conference on Domain Decomposition, Chiba, Japan, 1999.
[9] Y. A. Kuznetsov, New algorithms for approximate realization of implicit difference scheme, Soviet J. Numer. Anal. Math. Modelling, 3 (1988), 99-114.
[10] D. Liang and C. Du, The efficient S-DDM scheme and its analysis for solving parabolic equations, J. Comput. Phys., 272 (2014), 46-69.
[11] P. Lions, On the Schwarz alternating method I., In Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant and J. Periaux, eds., SIAM, Philadelphia, 1988, 1-42.
[12] P. Lions, On the Schwarz alternating method II., In Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, G. A. Meurant, J. Periaux and O. Widlund, eds., SIAM, Philadelphia, 1989, 47-70.
[13] W. Rivera, J. Zhu and D. Huddleston, An efficient parallel algorithm with application to computational fluid dynamics, Comput. Math. Appl., 45 (2003), 165-188.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press

[14] H. S. Shi and H. L. Liao, Unconditional stability of corrected explicit/implicit domain decomposition algorithms for parallel approximation of heat equations, SIAM J. Numer. Anal., 44 (2006), 1584-1611.
[15] Z. Q. Sheng, G. W. Yuan and X. D. Hang, Unconditional stability of parallel difference schemes with second order accuracy for parabolic equation, Appl. Math. Comput., 184 (2007), 1015-1031.
[16] B. F. Smith, P. Bjørstad and W. Gropp, Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations, Cambridge University Press, 1996.
[17] P. N. Vabishchevich, Domain decomposition methods with overlapping subdomains for the time-dependent problems of mathematical physics, Comput. Methods Appl. Math., 8 (2008), 393-405.
[18] S. H. Zhu, Conservative domain decomposition procedure with unconditional stability and second-order accuracy, Appl. Math. Comput., 216 (2010), 3275-3282.
[19] Y. Zhuang and X. H. Sun, Stabilitized explicit-implicit domain decomposition methods for the numerical solution of parabolic equations, SIAM J. Sci. Comput., 24 (2002), 335-358.

https://doi.org/10.4208/cicp.070814.190615a Published online by Cambridge University Press