Notes

100.01 Heronian triangles

An integer triangle is a triangle each of whose sides is of integer length. A Heronian triangle (named after Heron of Alexandria) is an integer triangle with the additional property that its area is also an integer. As a simple example, consider any right-angled integer triangle. Since one of its perpendicular sides must be even, the area is always an integer and thus all such triangles are Heronian. A general method for determining Heronian triangles, together with some basic results, is given in [1], whilst proofs of other simple properties may be found in [2]. In this Note, we develop an alternative approach based on the angles of a Heronian triangle. In all that follows, the term 'primitive' will be applied to an integer triangle if the gcd of all its sides is 1. We start off with a lemma involving the angles of an integer triangle.

Lemma 1

(a) The cosine of each of the angles of an integer triangle is rational.

(b) If the sine of one of the angles of an integer triangle is rational then the sine of each of the other two angles is also rational.

Proof

Parts (a) and (b) are immediate consequences of the cosine rule and sine rule respectively.

Lemma 2

Let ABC be an integer triangle with sides a, b and c and let $\cos C = \frac{m}{n}$, where m, n are integers with $n > 0$, $-n < m < n$ and $\gcd(m, n) = 1$.

If $\sin C$ is rational, then there exists a positive integer r with $\gcd(r, n) = 1$ such that $\sin C = \frac{r}{n}$. Furthermore,

(a) $r^2 + m^2 = n^2$,

(b) either C or $180° - C$ is an angle of a right-angled integer triangle,

(c) n is odd and not divisible by 3,

(d) one of r, m is divisible by 4,

(e) one of r, m is divisible by 3,

(f) ab is divisible by n.

Proof

Let us first of all deal with the particular case $C = 90°$.

In this case, $n = 1$, $m = 0$ and $r = 1$ and all parts of the Theorem are clearly true.

Now assume that $C \neq 90°$ in which case both n and m are non-zero.

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Since $\sin C$ is rational, we can write $\sin C = \frac{r}{s}$, where r, s are positive integers with $\gcd(r, s) = 1$.

Thus $\frac{r^2}{s^2} + \frac{m^2}{n^2} = 1$, which can be rewritten in the form $\frac{r^2}{s^2} = \frac{n^2 - m^2}{n^2}$.

The fact that $\gcd(r^2, s^2) = \gcd(n^2 - m^2, n^2) = 1$ now implies that $r^2 = n^2 - m^2$ and $s^2 = n^2$.

Thus $s = n$ and $r^2 + m^2 = n^2$. This, together with the fact that $\gcd(m, n) = \gcd(r, n) = 1$ implies that $(r, |m|, n)$ is a primitive Pythagorean triple, thus proving part (b).

Parts (c), (d) and (e) now follow since they are all simple well-known properties of primitive Pythagorean triples, see eg [3, pp. 219-224]. In order to prove part (f), it is sufficient to note that the cosine rule may be written in the form $c^2 = a^2 + b^2 - \frac{2abm}{n}$. The final term must therefore be an integer and it follows from part (c) that ab is divisible by n.

Theorem 1

The primitive integer triangle ABC is Heronian if, and only if, the sine of each of its angles is rational.

Proof

Let Δ be the area of triangle ABC. Assume that the primitive integer triangle ABC is Heronian. Then $\sin C = \frac{2 \times \Delta}{ab}$, which is rational since Δ, a and b are all integers.

Conversely, assume now that $\sin C$ is rational. Then we may write $\cos C = \frac{m}{n}$ and $\sin C = \frac{r}{n}$, where n, m, r are as in Lemma 2.

We consider two possible cases which arise from Lemma 2(d). First of all, assume that r is divisible by 4. Then $\Delta = \frac{abr}{2n}$, and since ab is divisible by n it follows that Δ must in fact be an even integer.

Now assume that m is divisible by 4. In this case, we note that the final term in the cosine rule above must be divisible by 8 and thus $c^2 \equiv a^2 + b^2 \pmod 8$. The fact that triangle ABC is primitive implies that at least one of a, b, c must be odd. By using the fact that $x^2 \equiv 1 \pmod 8$ when x is odd, it may now be easily verified that the above congruence can only be satisfied if c is odd and one of a, b is also odd whereas the other is divisible by 4. It now follows that $\Delta = \frac{abr}{2n}$ must again be an even integer.

Note that Theorem 1 has only been proved valid for primitive integer triangles. In our next result, we show that it is possible to dispense with this condition.

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Theorem 2

The integer triangle *ABC* is Heronian if, and only if, the sine of each of its angles is rational.

Proof

Assume that the integer triangle *ABC* is Heronian. Then arguing exactly as in the proof of Theorem 1, we see that the sine of any one of its angles must be rational.

Conversely, assume that the angles of the integer triangle *ABC* have rational sines. Denoting $\gcd(a, b, c)$ by d, we note that the triangle $A'B'C'$ with sides $a' = \frac{a}{d}$, $b' = \frac{b}{d}$, $c' = \frac{c}{d}$ is a primitive integer triangle whose angles are the same as those of triangle *ABC*. Thus, by Theorem 1, $A'B'C'$ is a Heronian triangle and since the triangles are similar with integral scale factor, it follows that *ABC* is also a Heronian triangle.

Theorem 3

Let *ABC* be a Heronian triangle. Any integer triangle similar to *ABC* is also Heronian.

Proof

This follows from Theorem 2 and the fact that the angles of any triangle similar to *ABC* are the same as those of *ABC*.

Theorem 4

The area of any Heronian triangle is divisible by 6.

Proof

It is clearly sufficient to prove that the result is true for primitive Heronian triangles. We already know from the proof of Theorem 1 that the area of a primitive Heronian triangle is divisible by 2 and thus all that remains to be done is to prove that it is also divisible by 3. This can be done as in Theorem 1 by considering two separate cases, noting in particular the results of parts (c) and (e) of Lemma 2 which state that

(i) n is not divisible by 3,

(ii) one of r, m is divisible by 3.

The details are left as an exercise to the reader.

In order to be able to generate examples of Heronian triangles, we quote a result which is proved in [4] using slightly different notation.

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Theorem 5

Let m, n be integers with $n > 0$, $-n < m < n$ and $\gcd(m, n) = 1$. Let $\{\alpha, \beta\}$ be a pair of positive rationals such that $\alpha\beta = \dfrac{2(n-m)}{n}$. Writing $\alpha = \dfrac{\alpha_1}{\alpha_2}$, $\beta = \dfrac{\beta_1}{\beta_2}$, where α_1, α_2, β_1, β_2 are positive integers with $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$, we define $T_{\{\alpha,\beta\}}$ to be the integer triangle with sides

$$a = \alpha_2(\beta_1 + 2\beta_2),\ b = \beta_2(\alpha_1 + 2\alpha_2),\ c = (\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\beta_1).$$

Then:

(a) the cosine of the angle opposite the side c is $\dfrac{m}{n}$;

(b) any integer triangle containing the angle whose cosine is $\dfrac{m}{n}$ is similar to $T_{\{\alpha,\beta\}}$ for some choice of $\{\alpha, \beta\}$;

(c) $T_{\{\alpha,\beta\}}$ and $T_{\{\gamma,\delta\}}$ are similar if, and only if, $\{\alpha, \beta\} = \{\gamma, \delta\}$.

We now show how to apply our results to generate Heronian triangles.

First of all, use Theorem 5 to generate right-angled integer triangles. Simple examples of such triangles are well-known while others, whose areas may easily be evaluated are given in [4]. All such triangles are Heronian, as noted in the introduction.

For other Heronian triangles, we may proceed as follows.

(a) Let T be one of the right-angled integer triangles generated above and let C be one the acute angles contained in T.

(b) Use Theorem 5 to generate integer triangles containing the angle C.

(c) Repeat this process with the angle $180° - C$.

(d) Repeat steps (b) and (c) for the other acute angle contained in T.

(e) Repeat steps (b), (c) and (d) for other choices of T.

Lemma 1, Lemma 2 and Theorem 1 guarantee that the triangles generated by this procedure will all be Heronian and that each primitive Heronian triangle may be determined in this way. There will, however, always be repeats (see Tables 1 and 3 below). Each scalene triangle will appear three times, once for each angle, and each isosceles triangle twice.

The tables below each exhibit six examples of primitive Heronian triangles which may be generated from the 3, 4, 5 triangle. In each case, four of the triangles are relatively small whereas the final two are larger. The notation used is that of Theorems 2 and 5.

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α	β	a	b	c	a', b', c'	$\Delta A'B'C' = \frac{1}{2}a'b' \sin C$
$\frac{2}{5}$	$\frac{2}{1}$	20	12	16	5, 3, 4	6
$\frac{2}{3}$	$\frac{6}{5}$	48	40	40	6, 5, 5	12
$\frac{2}{7}$	$\frac{14}{5}$	168	80	136	21, 10, 17	84
$\frac{4}{5}$	$\frac{1}{1}$	15	14	13	15, 14, 13	84
$\frac{1}{2}$	$\frac{8}{5}$	36	25	29	36, 25, 29	360
$\frac{5}{8}$	$\frac{32}{25}$	656	525	541	656, 525, 541	137 760

TABLE 1: $\cos C = \frac{3}{5}$, $\sin C = \frac{4}{5}$, $m = 3, n = 5$

α	β	a	b	c	a', b', c'	$\Delta A'B'C' = \frac{1}{2}a'b' \sin C$
$\frac{8}{5}$	$\frac{2}{1}$	20	18	34	10, 9, 17	36
$\frac{4}{5}$	$\frac{4}{1}$	30	14	40	15, 7, 20	42
$\frac{2}{5}$	$\frac{8}{1}$	50	12	58	25, 6, 29	60
$\frac{1}{1}$	$\frac{16}{5}$	26	15	37	26, 15, 37	156
$\frac{3}{2}$	$\frac{32}{15}$	124	105	205	124, 105, 205	5208
$\frac{3}{13}$	$\frac{208}{15}$	3094	435	3373	3094, 435, 3373	538 356

TABLE 2: $\cos C = -\frac{3}{5}$, $\sin C = \frac{4}{5}$, $m = -3, n = 5$

α	β	a	b	c	a', b', c'	$\Delta A'B'C' = \frac{1}{2}a'b' \sin C$
$\frac{2}{5}$	$\frac{1}{1}$	15	12	9	5, 4, 3	6
$\frac{1}{4}$	$\frac{8}{5}$	72	45	45	8, 5, 5	12
$\frac{1}{10}$	$\frac{4}{1}$	60	21	45	20, 7, 15	42
$\frac{1}{5}$	$\frac{2}{1}$	20	11	13	20, 11, 13	66
$\frac{1}{3}$	$\frac{6}{5}$	48	35	29	48, 35, 29	504
$\frac{5}{13}$	$\frac{26}{25}$	988	775	593	988, 775, 593	229 710

TABLE 3: $\cos C = \frac{4}{5}$, $\sin C = \frac{3}{5}$, $m = 4, n = 5$

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α	β	a	b	c	a', b', c'	$\Delta A'B'C' = \frac{1}{2}a'b' \sin C$
$\frac{9}{5}$	$\frac{2}{1}$	20	19	37	20, 19, 37	114
$\frac{6}{5}$	$\frac{3}{1}$	25	16	39	25, 16, 39	120
$\frac{1}{1}$	$\frac{18}{5}$	28	15	41	28, 15, 41	126
$\frac{3}{5}$	$\frac{6}{1}$	40	13	51	40, 13, 51	156
$\frac{1}{6}$	$\frac{108}{5}$	708	65	761	708, 65, 761	13 806
$\frac{7}{8}$	$\frac{144}{35}$	1712	805	2405	1712, 805, 2405	413 448

TABLE 4: $\cos C = -\frac{4}{5}$, $\sin C = \frac{3}{5}$, $m = -4$, $n = 5$

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100.02 Square pyramidal numbers and the multiplication table

A square pyramidal number represents the number of stacked spheres in a pyramid with a square base. The number of spheres in each layer of the pyramid is just the square of the number of spheres of the layer edge, and the square pyramidal number is the sum of the numbers of spheres in each layer. The first few square pyramidal numbers are: 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240.

In this Note we derive the summation formula for the first N squared natural numbers using the properties of a matrix with elements i times j in the ith row and jth column. This matrix is also known as the multiplication table. Our derivation of the square pyramidal number formula takes advantage of the multiplication table properties.

Let us imagine an $N \times N$ matrix $c_{i,j}$ with $i, j = 1, \ldots, N$ in which the elements $c_{i,j}$ are equal to the products of i and j: $c_{i,j} = ij$. We will use this

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