4 Orbital mechanics

Even though an interstellar spacecraft would spend most of its time away from the gravity of stars and planets, knowledge of orbital mechanics is important if a spacecraft is to make a rendezvous. In this chapter we examine the essentials of orbits as well as orbital manoeuvres, in particular the Hohmann transfer orbit. We will see that the efficiency of a rocket impulse depends on where in an orbit it is performed, something known as the Oberth effect. Finally, we examine how spacecraft can accelerate or decelerate in a star system using gravity assists around planets.

4.1 Two-body motion under gravity: Keplerian orbits

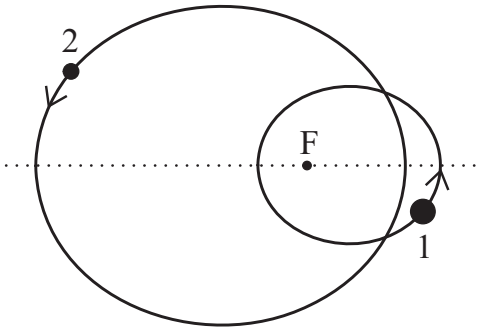
Two point masses interacting via only their mutual gravity undergo Keplerian orbits about their barycentre (centre of mass). Consider two bodies of different masses moving in closed (elliptical) orbits, as shown in figure 4.1. Viewed from their barycentre, which is also the focus F of their orbits, the more massive body (1) has a smaller orbit, but both orbits have the same shape. It can be shown that object 1 also has a Keplerian orbit *relative* to object 2, and vice versa. These relative orbits are not in inertial reference frames – the bodies accelerate one another – but the same equations of motion apply in the rest frame of each body. The two relative orbits have all the same properties as each other – size, shape, period, etc. – except for the phase, because the two bodies are on opposite sides of the barycentre. As an example of this, the orbit of the Sun viewed from the Earth is the same as the orbit of the Earth viewed from the Sun.[1]

Both the total energy and angular momentum of a two-body system interacting under gravity is conserved. The latter means that both the orbits and barycentre are all in a common plane.

Below I summarize various properties of Keplerian orbits that will be useful in our considerations of space travel. They follow from the inverse square law of gravity assuming point masses and Newtonian mechanics. The derivations can be found in all good textbooks on orbital mechanics. Extended objects and special and general relativity introduce deviations from the inverse square law and some interesting effects (e.g. orbital precession), but those will not be considered. Furthermore, as we are interested in spacecraft orbiting a much more massive planet or star, I will consider the former to have negligible mass, so the barycentre becomes coincident with the central body. The standard gravitational parameter is then

[1] The statement 'the Sun orbits the Earth' is therefore not entirely wrong. This equivalence of the relative motion partly explains why it took us so long to realize that the description of the Earth orbiting the Sun, rather than vice versa, is a better *dynamical* description of the heavens.

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The elliptical orbits of two bodies, 1 and 2, about their barycentre F. In this example body 1 has twice the mass of body 2, so its orbit is half the size. Fig. 4.1

$\mu = GM$, rather than $G(M+m)$, where G is the gravitational constant and M the mass of the central body. The quantities r and v are, respectively, the distance and velocity of the orbiting body relative to the central body.

4.1.1 All orbits

For any Keplerian orbit, the total energy per unit mass (specific energy) of the orbiting body is the sum of its kinetic and gravitational potential energies,

$$E = K + U \tag{4.1a}$$

$$= \frac{v^2}{2} - \frac{\mu}{r} \quad \text{(all orbits)}, \tag{4.1b}$$

taking the usual convention of zero potential energy at $r = \infty$. The nature of a Keplerian orbit depends on the value of E. If the energy is low enough, the objects will remain bound and the orbit will be an ellipse (the circle being a special case). With our potential energy convention, an elliptical orbit is one with negative energy E. If the energy is positive, the object is not bound and the orbit is a hyperbola. If the system has zero energy, then the orbit is a parabola. The central mass is at the focus of the ellipse, parabola, or hyperbola. For all orbits we can write the energy per unit mass in terms of the semi-major axis a (defined later),

$$E = -\frac{\mu}{2a} \quad \text{(all orbits)}. \tag{4.2}$$

From now on I will refer to the 'energy per unit mass' as simply 'energy' (you can always check the units to be sure). Combining the above two equations 4.1 and 4.2 we get the *vis-viva equation*

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad \text{(all orbits)}, \tag{4.3}$$

which is useful for computing the velocity at any point in the orbit.

All Keplerian orbits can be described geometrically by their semi-major axis a (size) and eccentricity e (shape). Solving the equations of motion, we arrive at the following equation for the motion in polar coordinates,

$$r = \frac{a(1-e^2)}{1 + e\cos\theta} \quad \text{(all orbits)}, \tag{4.4}$$

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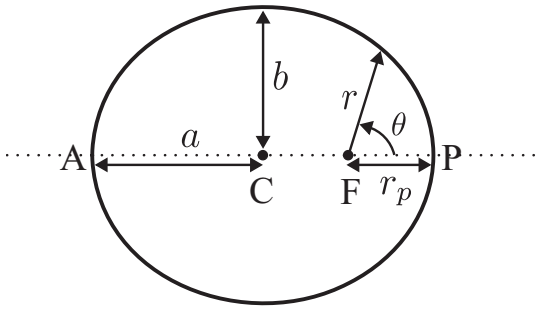


Fig. 4.2 An elliptical orbit. A = apoapsis, C = centre, F = focus, P = periapsis. AC = CP = a is the semi-major axis, b is the semi-minor axis, AF = $r_a = a(1 + e)$ is the apoapsis distance, and FP = $r_p = a(1 - e)$ is the periapsis distance. This ellipse has an eccentricity of $e = \sqrt{1 - b^2/a^2}$ (= 0.5 in this example). The true anomaly θ is measured from the periapsis.

where the angular coordinate θ is known as the true anomaly. Interestingly, although we can also solve the equations of motion for the time as a function of r, there is in general no closed-form solution for r as a function of time. The resulting equation – Kepler's equation – has to be solved numerically to get $r(t)$. The exceptions to this are circular and parabolic orbits.

Let us now look at elliptical and hyperbolic orbits separately.

4.1.2 Elliptical orbits

Elliptical orbits are those with $0 \leq e < 1$ in equation 4.4 and are closed orbits. See figure 4.2. The semi-major axis is half the length of the long axis. The semi-minor axis – half the length of the short axis – is given by $b = a\sqrt{1 - e^2}$, so $0 < b \leq a$. The eccentricity measures how elongated the ellipse is. When $e > 0$, the distance of the orbiting object from the focus (central mass) varies over the orbit. It follows from equation 4.4 that the closest approach (periapsis, distance r_p) occurs at $\theta = 0°$, and the most distant point (apoapsis, distance r_a) occurs at $\theta = 180°$, so

$$\begin{aligned} r_p &= a(1 - e) \\ r_a &= a(1 + e) \end{aligned} \quad \text{(elliptical orbits).} \tag{4.5}$$

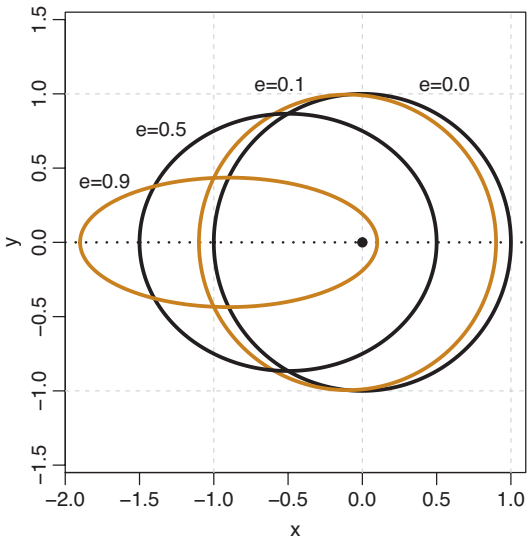
For the Earth (actually the barycentre of the Earth–Moon system), $a = 1.496 \times 10^8$ km (approximately 1 au) and $e = 0.0167$, so the periapsis is 1.471×10^8 km and the apoapsis is 1.521×10^8 km, a difference of five million kilometres. Figure 4.3 shows examples of orbits with different eccentricities. Note that an orbit with $e = 0.1$ has almost the same shape as a circular orbit ($e = 0$), but the focus is notably shifted.

The specific angular momentum h of the orbiting body about the central mass is directly related to the semi-major axis and eccentricity via

$$h^2 = \mu a(1 - e^2) \quad \text{(all orbits).} \tag{4.6}$$

Thus for a given energy, a larger eccentricity has a smaller angular momentum. Also useful is Kepler's third law, which relates the semi-major axis and total mass to the orbital period T via

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Elliptical orbits with different eccentricities. All have $a = 1$ and a common focus at $(x, y) = (0, 0)$. $x = r\cos\theta, y = r\sin\theta$ are in units of the semi-major axis. Fig. 4.3

$$a^3 = \frac{\mu}{4\pi^2}T^2 \quad \text{(elliptical orbits)}. \tag{4.7}$$

4.1.3 Circular orbits

Circular orbits are special cases of elliptical orbits with $e = 0$. From equation 4.3, $r = a$ on all points of the orbit and so

$$v^2 = \frac{\mu}{a} \quad \text{(circular orbits)}. \tag{4.8}$$

This can also be found by equating the magnitudes of the gravitational and centripetal forces (both time independent). Thus objects in larger orbits have smaller velocities. The outer planets in the solar system move more slowly than the inner ones, for example. They also have smaller angular velocities, v/a. The Earth at 1 au orbits at about $29.8\,\mathrm{km\,s^{-1}}$, although this varies slightly due to the non-zero eccentricity. As the left side of equation 4.8 is twice the kinetic energy K, and the right side is the negative of the potential energy U, we can write

$$2K + U = 0 \quad \text{and} \tag{4.9a}$$

$$E = -K = \frac{U}{2} \quad \text{(circular orbits)}. \tag{4.9b}$$

This is an example of the *virial theorem*. Recall that $E < 0$. The virial theorem is instructive because it tells is that if we increase (towards zero) the total energy of an object in a circular orbit, we *decrease* the kinetic energy by this amount and *increase* the potential energy by twice this amount. Adding energy puts the object into a higher orbit but also slows it down, which might seen counter-intuitive. Viewed from the perspective of thermodynamics, such a

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system has a negative heat capacity, because adding energy reduces the temperature (a proxy for kinetic energy). This is true for all systems of gravitationally bound bodies. The virial theorem relates time-averaged energies, but always holds for circular orbits since r and v are constant.

4.1.4 Escape velocity

A useful concept is the escape velocity. For a spacecraft a distance r from a star or planet, what initial velocity does it need to just escape the gravitational field, that is, to reach infinite distance at zero velocity? We can compute this by equating the initial energy, given by equation 4.1b, with the final energy, which is zero. This gives

$$v_{\text{escape}} = \sqrt{\frac{2\mu}{r}} = \sqrt{2}\, v_{\text{circ}}(r), \tag{4.10}$$

where $v_{\text{circ}}(r)$ is the circular orbital velocity at distance r (equation 4.8). The escape velocity from the Earth's surface (ignoring the Sun and all other bodies) is $11.2\,\text{km}\,\text{s}^{-1}$. The escape velocity from the Sun starting at the distance of the Earth is $42.1\,\text{km}\,\text{s}^{-1}$, but when starting at the surface of the Sun it is $617\,\text{km}\,\text{s}^{-1}$ (ignoring the Earth and other bodies in both cases). The above equation shows us that to escape to infinity, a spacecraft initially on a circular orbit must increase its velocity in the tangential direction by $\Delta v = (\sqrt{2} - 1)v_{\text{circ}}(r)$. So a spacecraft in the Earth's orbit about the Sun ($v_{\text{circ}} = 29.8\,\text{km}\,\text{s}^{-1}$) but far from the Earth will need a prograde boost of $\Delta v = 0.414 \times 29.8\,\text{km}\,\text{s}^{-1} = 12.3\,\text{km}\,\text{s}^{-1}$ to escape the solar system. As gravity is a conservative force, once the spacecraft has acquired its escape velocity it will reach infinity no matter what direction it was initially moving in. The orbit of a spacecraft with escape velocity is a parabolic orbit, as this is the orbit with zero energy.

At $11.2\,\text{km}\,\text{s}^{-1}$, the escape velocity of the Earth is more than the maximum specific exhaust velocity available with chemical rockets, which is why large mass ratios are required for a rocket to escape from the Earth (see chapter 3). If we rewrite equation 4.10 in terms of the mean density ρ and radius r of the planet we get

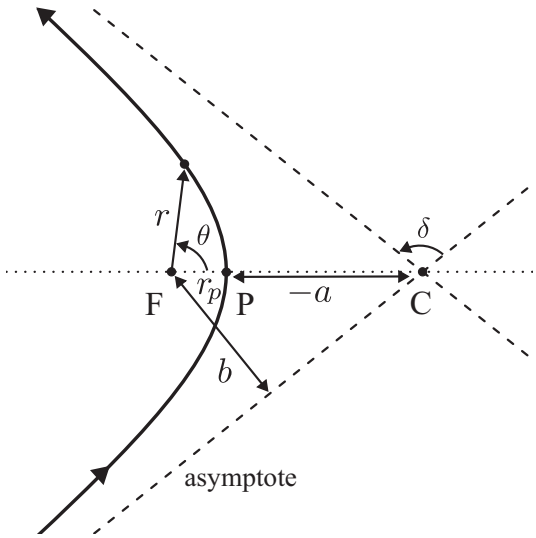
$$v_{\text{escape}} = r\sqrt{\frac{8\pi G\rho}{3}}. \tag{4.11}$$

For a given density, the larger the planet, the larger the escape velocity from its surface. Eventually this will exceed the Δv that can be provided by a chemical rocket with any plausible mass ratio. In the best case, a rocket exhausts all of its propellant near the surface of the planet, in which case Δv is given by the force-free rocket equation (equation 3.4). Setting this Δv equal to the escape velocity, we get an expression for the maximum radius a planet can have such that a rocket with effective exhaust velocity v_e and mass ratio R can still escape from it:

$$r_{\text{max}} = v_e \ln R \sqrt{\frac{3}{8\pi G\rho}}. \tag{4.12}$$

With $v_e = 5\,\text{km}\,\text{s}^{-1}$ and ρ the density of the Earth, then if we adopt a maximum plausible mass ratio of $R = 1000$, the maximum radius is $3.1\text{R}_\oplus$. Rocky exoplanets larger than the Earth have a range of densities, so the maximum radius would deviate somewhat from this. But this calculation shows that potential aliens on a very large rocky planet could not escape from its surface using chemical rockets that we know of.

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A hyperbolic orbit. F = focus, P = periapsis, C = centre. FP = r_p, the periapsis distance, PC = $-a$, the semi-major axis ($a < 0$). If the central body at F had zero mass, then the orbiting body would move on a straight line along the asymptote with impact parameter b. The mass instead turns the orbiting body through an angle δ, such that at infinity it is travelling along the outgoing asymptote. In this example $a = -2.50$ and $r_p = 0.70$ (arbitrary units), $\delta = 103°$, and $e = 1.28$. Fig. 4.4

4.1.5 Hyperbolic orbits

When an orbiting object has sufficient energy, it will not be bound to the central mass, but its path will be deflected by it. This is a hyperbolic orbit: see figure 4.4. Hyperbolic orbits have $E > 0$ and $e > 1$. It follows from equation 4.2 that they therefore have negative semi-major axes[2], $a < 0$.

If an object has an asymptotic velocity v_∞, and so a total energy $E = v_\infty^2/2$, then from the vis-viva equation (equation 4.3) we see that

$$a = -\frac{\mu}{v_\infty^2} \quad \text{(hyperbolic orbits)}. \tag{4.13}$$

The eccentricity is defined in terms of this and the semi-minor axis b,

$$e^2 = 1 + \frac{b^2}{a^2} \quad \text{(hyperbolic orbits)}, \tag{4.14}$$

which differs from the definition for elliptical orbits. The sign of b is not defined by this equation; we take it to be positive. The quantity b is also the impact parameter, which is the closest approach the object would have to the central body in the absence of gravity. This can also be seen by substituting equations 4.14 and 4.13 into equation 4.6, from which we find that the angular momentum is $h = bv_\infty$. As with elliptical orbits, the periapsis of the hyperbolic orbit $\theta = 0$ is $r_p = a(1 - e)$, but its apoapsis is infinite.

A spacecraft making a flyby of a planet or star is actually in a hyperbolic orbit about that body (ignoring all other masses). At a large distance from the body, the energy and speed of

[2] Some sources define hyperbolic orbits to have $a > 0$ and then modify the equations.

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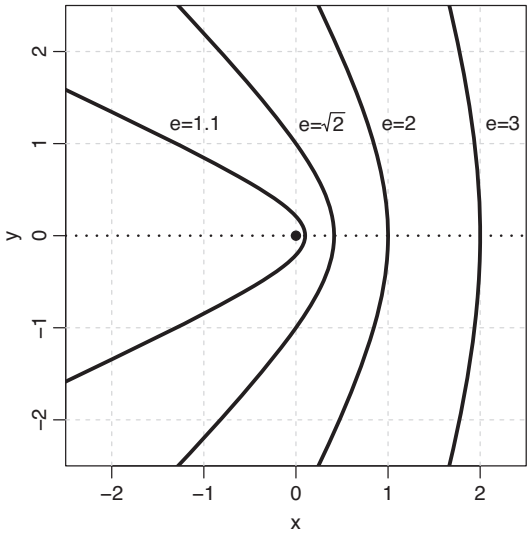


Fig. 4.5 Hyperbolic orbits with different eccentricities, but all with $a = -1$ and a common focus at $(x, y) = (0, 0)$. The Cartesian coordinates $x = r\cos\theta, y = r\sin\theta$ are in units of the semi-major axis.

the spacecraft is the same before and after the flyby, but its direction of travel has changed (figure 4.4). The deflection angle of the orbit is a property of the hyperbola and is given by

$$\delta = 2\sin^{-1}\frac{1}{e}. \tag{4.15}$$

Because hyperbolic orbits are not closed, θ in equation 4.4 varies between only $-(\delta+\pi)/2$ and $+(\delta+\pi)/2$.

Knowing the impact parameter b and the asymptotic velocity v_∞ relative to a central body, we can derive the geometric properties (a, e) of the orbit. The example in figure 4.4 is for an orbit around an Earth-mass object with $b = 2\mathrm{R}_\oplus$ and $v_\infty = 5\,\mathrm{km\,s^{-1}}$, which corresponds to $e = 1.28$ (a and r_p in the caption are then also in units of $\mathrm{R}_\oplus$). Examples of orbits with other eccentricities are show in figure 4.5. The larger the eccentricity, the closer the orbit is to a straight line.

A limiting case of a hyperbolic orbit has $e = 1$ and is a parabolic orbit. This has zero energy and is on the boundary of bound/unbound, elliptical/hyperbolic. Its velocity at periapsis is therefore equal to the escape velocity. The deflection angle is then $\delta = 180°$. After passing the central mass, the spacecraft eventually returns in the direction it came from, albeit only after infinite time. If two bodies are headed straight at each other, this can also be considered as a limiting case of a hyperbolic orbit with $b = 0$, and therefore $e = 1$. This is not a parabolic orbit but a radial orbit, for which the above equations break down due to the singularity at $r = 0$.

The velocity at infinity – also called the hyperbolic excess velocity – is used by mission planers for specifying the energy of a spacecraft relative to a body. It is sometimes specified via the characteristic energy, defined as $C_3 = v_\infty^2 = 2E$, which is twice the specific energy the spacecraft would have if it escaped to infinity. The New Horizons spacecraft to the outer solar system, for example, had $C_3 = 157.8\,\mathrm{km^2\,s^{-2}}$ with respect to the Sun at launch, the largest launch specific energy of a spacecraft so far. Voyager 1 started with a

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lower characteristic energy, but after its gravity assists it attained $v_\infty = 16.6\,\mathrm{km\,s^{-1}}$, making it the fastest spacecraft to leave the solar system so far. It is not the fastest spacecraft with respect to the Sun, however. That record goes to the Parker Solar Probe, which reached $192\,\mathrm{km\,s^{-1}}$ during a perihelion passage at a distance of $9.8\,\mathrm{R}_\odot$ ($8.8\,\mathrm{R}_\odot$ from the surface) in December 2024. As it is bound to the Sun, its characteristic energy is negative.

4.2 Changing orbits

Although Newtonian gravity appears simple (it's just $1/r^2$ after all), Keplerian orbits are non-trivial, and changing orbits can be surprisingly complicated. We won't dive deeply into the fascinating details here but will look briefly at some simple ways of moving from one orbit to another using impulsive manoeuvres. These are manoeuvres that take place instantaneously, for example via a short burst of a rocket engine. The spacecraft therefore moves instantaneously from one Keplerian orbit to another with a different energy and/or angular momentum. The change in the speed at this instant is the scalar quantity Δv. From the rocket equation (chapter 3), the total amount of propellant in a rocket determines the total Δv a rocket can deliver, and so the Δv budget of a manoeuvre is a critical aspect of mission design.

Before we get to this, let us briefly ignore gravity for a useful insight. Consider a spacecraft moving with a vectorial velocity $\mathbf{v}$ in some reference frame. The velocity now changes by $\Delta\mathbf{v}$. If the initial and final (specific) kinetic energies are K_i and K_f respectively, and the change in kinetic energy is ΔK, then we see that

$$K_i = \frac{1}{2}|\mathbf{v}|^2, \tag{4.16a}$$

$$K_f = \frac{1}{2}|\mathbf{v} + \Delta\mathbf{v}|^2, \tag{4.16b}$$

$$\Delta K = \frac{1}{2}|\Delta\mathbf{v}|^2 + \mathbf{v}\cdot\Delta\mathbf{v}. \tag{4.16c}$$

The largest change in the kinetic energy is therefore achieved when $\Delta\mathbf{v}$ is made parallel to $\mathbf{v}$, to give $\Delta E = (1/2)(\Delta v)^2 + v\Delta v$. If we instead applied the burst perpendicular to the direction of travel, we would only get $\Delta E = (1/2)(\Delta v)^2$. Thus to maximize the change in energy of the spacecraft, we should apply a given Δv in the direction of travel. As energy is a measure of the semi-major axis of an orbit, this is also the way to make the largest change in the size of an orbit. This consideration is involved in the Oberth effect, which we look at in section 4.2.3.

4.2.1 Hohmann transfer

Suppose that a spacecraft is initially in a circular orbit of radius r_1. We want to lift it onto a larger orbit of radius r_2 using an impulsive rocket. This will require at least two bursts of the rocket: one to move it from the inner orbit onto a transfer orbit, and a second to move it from the transfer orbit onto the outer orbit. Following the discussion in the previous section, the most energy efficient way to do this will be with two tangential bursts, as shown in figure 4.6.

To move out from orbit 1, we need to increase the energy of the spacecraft, so the first burst must be in the prograde direction. This will put the spacecraft on an elliptical transfer

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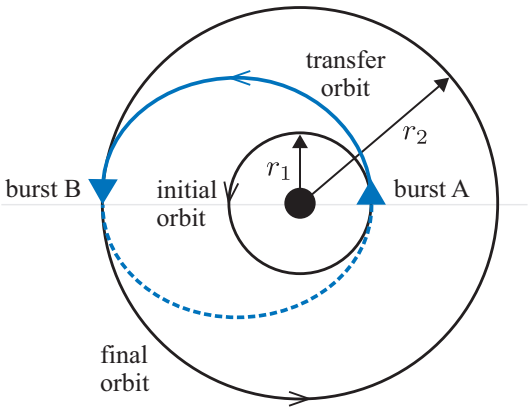


Fig. 4.6 A Hohmann transfer orbit between an inner circular orbit of radius r_1 and an outer circular orbit of r_2. The Hohmann transfer orbit is half of an elliptical orbit with periapsis r_1 and apoapsis r_2.

orbit with semi-major axis a and eccentricity e (values to be determined). The point of the burst (A) will be the periapsis of the transfer orbit, which is equal to r_1. If we arrange that the apoapsis of this transfer orbit is r_2, then apply a second Δv there (point B), we can get the spacecraft onto the outer circular orbit. What values of Δv do we need at A and B to achieve this?

This can be solved by considering the (specific) angular momenta of the three orbits involved. The angular momenta of orbits 1 and 2 are, from equation 4.6, $h_1 = \sqrt{\mu r_1}$ and $h_2 = \sqrt{\mu r_2}$, respectively. The semi-major axis of an elliptical orbit is half the sum of the periapsis and apoapsis distances, which for the transfer orbit is $a = (r_1 + r_2)/2$. Using this together with the expressions for periapsis and apoapsis from equation 4.5, equation 4.6 gives the angular momentum of the transfer orbit to be

$$h_t = \sqrt{2\mu \frac{r_1 r_2}{r_1 + r_2}}. \tag{4.17}$$

As $r_2 > r_1$ it follows that $h_2 > h_t > h_1$. As the spacecraft moves tangentially at periapsis and apoapsis, the angular momenta at these points can also be written $h = rv$, where v is the velocity. The velocity on orbit 1 before the burst is $\sqrt{\mu/r_1}$, and the velocity immediately after the burst (on the transfer orbit) is h_t/r_1. Thus the required change in velocity by the first burst is

$$\Delta v_A = \sqrt{\frac{\mu}{r_1}} \left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right). \tag{4.18}$$

This is positive and so a prograde burst, as anticipated. When the spacecraft gets to the apoapsis of its transfer orbit, its velocity is h_t/r_2. To move onto the circular orbit, it must change its velocity to $\sqrt{\mu/r_2}$, so the required change is

$$\Delta v_B = \sqrt{\frac{\mu}{r_2}} \left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}} \right). \tag{4.19}$$

This is also positive, so the second burst is also prograde. This makes sense because when it gets to apoapsis, the spacecraft needs to increase both its angular momentum and its semi-major axis (energy).

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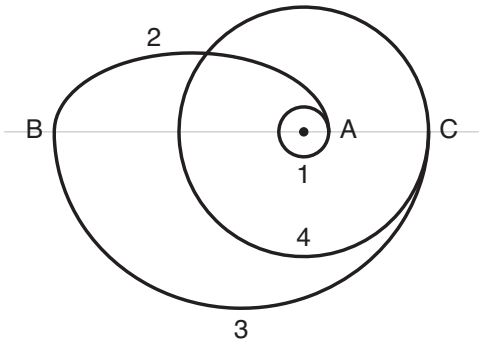


Fig. 4.7 Bielliptic transfer orbits between two circular orbits. To get from the inner orbit (1) to the outer orbit (4), a prograde boost is made at A to put the spacecraft on the elliptical orbit 2. At its apoapsis (B), another prograde boost is made to put the spacecraft on orbit 3. At its periapsis (C), a retrograde boost is made to put the spacecraft on the outer orbit. To go from the outer orbit to the inner one, the procedure is reversed.

Let's put in some numbers. Suppose we want to raise a satellite from a low Earth orbit (LEO) at $r_1 = \mathrm{R}_\oplus + 400\,\mathrm{km}$ to a geosynchronous orbit (GEO) at $r_2 = \mathrm{R}_\oplus + 35\,800\,\mathrm{km}$. The velocity on the lower circular orbit is $7.67\,\mathrm{km\,s^{-1}}$ and the first burst is $\Delta v_A = 2.40\,\mathrm{km\,s^{-1}}$. This puts the spacecraft on an elliptical orbit of semi-major axis 24 500 km and eccentricity 0.72. The second burst is $\Delta v_B = 1.46\,\mathrm{km\,s^{-1}}$, putting the satellite on a GEO with a velocity of $3.07\,\mathrm{km\,s^{-1}}$. The transfer duration – half the period of the transfer orbit – is 5.3 hr. The angular momenta of the initial, transfer, and final orbits are 5.2×10^4, 6.8×10^4, and $1.3 \times 10^5\,\mathrm{km^2 s^{-1}}$, respectively.

4.2.2 Other transfer orbits

It turns out that the Hohmann transfer of the previous section is the most energy efficient two-impulse manoeuvre between any two coplanar elliptical orbits. There is, however, a more energy efficient way of achieving the same change in orbit using three impulses. This again uses tangential boosts at an apoapsis or periapsis, but not all are prograde.

Suppose again that we want to move from an inner circular orbit of radius r_1 to an outer circular orbit of radius r_4 as shown in figure 4.7. The idea of the bielliptic transfer orbit is to again do a prograde boost (at point A), but now a larger one than in the Hohmann case in order to take the spacecraft to a larger apoapsis distance than that of the target orbit. At the apoapsis (point B), the rocket does another prograde boost to put it onto a higher energy (larger semi-major axis) and lower eccentricity (larger angular momentum) orbit 3. Point B is also the apoapsis of orbit 3. The size of this boost is chosen such that the periapsis of orbit 3 (point C) is the radius of the target circular orbit (4). As this target orbit has a smaller semi-major axis than orbit 3, a *retrograde* boost is then required at point C to put the spacecraft onto the circular orbit.

It turns out that this manoeuvre can use less energy than the Hohmann transfer. The idea is to place point B so far away from the central mass that the spacecraft is moving very slowly at this point; the boost required to put it on orbit 3 then is very small. The sum of the three Δv can then be less than the two in the Hohmann case. It turns out that if $r_4 < 11.94 r_1$, then the Hohmann transfer is more energy efficient for any r_B, whereas if $r_4 > 15.58 r_1$, then the

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bielliptic transfer is more energy efficient for any r_B. In between these values, the bielliptic transfer is more energetically favourable for larger values of r_B. Energy efficiency may not be the only consideration, of course. As it involves moving out to a larger intermediate outer orbit, the bielliptic transfer lasts much longer.

A simpler variation of this transfer orbit is when we want to drop into the Sun from an initial circular orbit of radius r_1. To achieve this with a single boost, we need to apply a retrograde boost equal to the circular velocity, which is $\Delta v_0 = \sqrt{\mu/r_1}$. Alternatively, we could do a prograde boost into an elliptical orbit with an apoapsis distance r_2. The velocity change required for this we already calculated for the Hohmann transfer as Δv_A from equation 4.18. At the apoapsis of this elliptical orbit (point B in figure 4.6), the Δv required to remove the angular momentum and so drop into the Sun is equal to the velocity at that point. This is the term we subtracted in equation 4.19 (or we can use the vis-viva equation), and is $\Delta v'_B = \sqrt{\mu}\sqrt{2(r_1/r_2)/(r_1 + r_2)}$. Thus the ratio of the total Δv from this two-burst manoeuvre to the Δv needed in the one-burst 'direct drop' manoeuvre is

$$Q = \frac{\Delta v_A + \Delta v'_B}{\Delta v_0} = \sqrt{2\left(\frac{r_1}{r_2} + 1\right)} - 1. \tag{4.20}$$

By inspection we see that for $r_2 > r_1$ we get $Q < 1$, that is, the two-burst manoeuvre requires a smaller Δv. Thus by boosting to a larger orbit first, we require less energy to drop into the Sun. The smallest value of Q is achieved as r_2/r_1 tends to infinity, in which case $Q = \sqrt{2} - 1$: the two-burst manoeuvre only needs 41% the Δv as a direct drop into the Sun. It is somewhat counter-intuitive that the more energy efficient way to get to the Sun is to first boost to an arbitrarily large orbit. This will, however, require an arbitrarily long time.

The manoeuvres looked at so far have only involved a Δv perpendicular to the radial vector (and so only at apoapsis and periapsis) and in the orbital plane. Boosts in other directions are of course possible and have a range of applications, for example to intercept an object on another orbit, or to change the inclination of the orbit. Even more complicated manoeuvres occur when we consider not instantaneous bursts but long-duration burns of an engine (e.g. an ion engine; chapter 6), or the application of other external forces (e.g. a solar sail; chapter 10). The problems can become quite complicated and will often require numerical methods to solve.

4.2.3 The Oberth effect

We saw from equation 4.16c that the largest change in the kinetic energy of a spacecraft is achieved when applying a boost (of magnitude Δv) in the direction of travel. This equation also shows that a given Δv corresponds to a larger change in kinetic energy the larger the initial velocity. The Δv is the same for all inertial reference frames (neglecting special relativity), but the corresponding change in kinetic energy is not. This follows from the quadratic dependence of kinetic energy on velocity. Consequently, the faster a spacecraft is moving relative to a star when the Δv is applied, the larger its change in kinetic energy. This is the Oberth effect. It is particularly relevant when changing orbits. For example, when a spacecraft is in an elliptical orbit and wants to transfer to the highest possible orbit using a given Δv, it should apply the boost when it is moving fastest, which is at periapsis.

An illustration of the Oberth effect relevant to interstellar travel is when a spacecraft is in a parabolic orbit about a star and wants to use a single boost to escape and achieve the

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maximum velocity at infinity. Suppose the periapsis distance is r_i, where the velocity is v_i. The total energy of a parabolic orbit is zero, so it follows from the vis-viva equation that $v_i^2 = 2\mu/r_i$. This is the escape velocity (section 4.1.4), meaning the spacecraft would reach infinity with zero velocity. If this spacecraft flew to infinity and applied its boost, its final velocity would be Δv. If it instead applies the same boost at periapsis, then the total energy immediately after the boost is the sum of the potential energy ($-\mu/r_i = -v_i^2/2$) and the now-increased kinetic energy, which is

$$E_i = -\frac{v_i^2}{2} + \frac{1}{2}(v_i + \Delta v)^2 = v_i \Delta v + \frac{(\Delta v)^2}{2}. \tag{4.21}$$

Upon reaching infinity in this case, the final energy is just the kinetic energy $(1/2)v_f^2$, where v_f is the final velocity. By conservation of energy, this final energy equals E_i, so

$$v_f = \Delta v \sqrt{1 + \frac{2v_i}{\Delta v}}. \tag{4.22}$$

This tells us that doing the boost at periapsis rather than at infinity increases the final velocity by a factor of $\sqrt{1 + 2v_i/\Delta v}$.

Suppose we arrange for a rocket to make a parabolic flyby of the Sun at 10 solar radii (9 solar radii from the surface), which corresponds to $v_i = 195.4\,\mathrm{km\,s^{-1}}$ (the escape velocity). If the rocket can provide $\Delta v = 5\,\mathrm{km\,s^{-1}}$, then applying this boost at perihelion results in a velocity at infinity of $44.5\,\mathrm{km\,s^{-1}}$, as opposed to $5\,\mathrm{km\,s^{-1}}$ if we do the boost at infinity. With $\Delta v = 10\,\mathrm{km\,s^{-1}}$ applied at perihelion, the final velocity is $63.3\,\mathrm{km\,s^{-1}}$. Thus the Oberth effect can give large boosts and so allow fuel to be used more efficiently.

We can apply the same idea to escaping from an elliptical orbit. Both the orbital velocity and the escape velocity are largest at periapsis and smallest at apoapsis. If we compute the difference between the escape velocity and orbital velocity at each apsis, we find that the Δv required to escape is always smaller at periapsis than at apoapsis (see exercises). Furthermore, the difference increases the larger the eccentricity. Thus to escape from an elliptical orbit using minimum Δv, we should apply the boost at periapsis.

The above effect is of course reversible. If a spacecraft is initially unbound to a star and wants to go into orbit about it, it should apply its braking Δv at periapsis in order to minimize the Δv required.

As a parabolic orbit can be thought of as the limiting case of an elliptical orbit with an infinite semi-major axis and high eccentricity, the parabolic example above is the best we could achieve in trying to escape from the Sun with the Oberth effect. In practice we would start in an elliptical orbit, in which case both the perihelion velocity and final velocity would be lower. But not by much (see exercises). For example, if we start at the Earth's orbit and do a large retrograde boost to drop to 10 solar radii, the perihelion velocity is $193.2\,\mathrm{km\,s^{-1}}$. Following a $5\,\mathrm{km\,s^{-1}}$ boost, the velocity at infinity is $44.2\,\mathrm{km\,s^{-1}}$, hardly any smaller than in the parabolic case above.

4.3 Gravity assists

An unbound spacecraft flying close to a planet will have its direction changed by their gravitational interaction. Ignoring all other masses, this two-body interaction is a hyperbolic orbit of the spacecraft about the planet (section 4.1.5). Energy is conserved, so the magnitude

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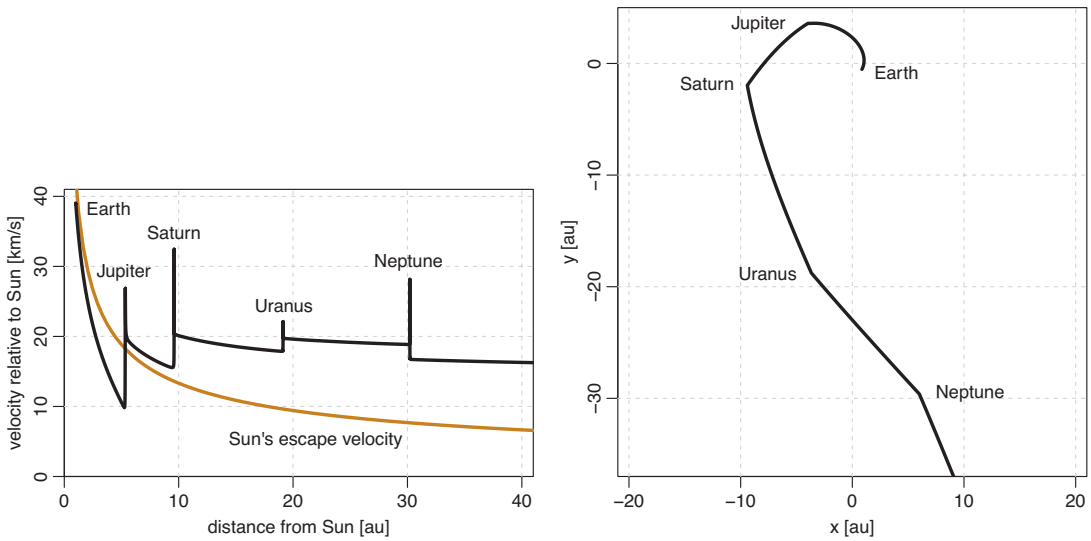


Fig. 4.8 Gravity assists of Voyager 2. The left panel shows how the magnitude of the velocity of the spacecraft relative to the Sun is modified by close encounters of the four outer planets. The lower smooth line shows the Sun's escape velocity (equation 4.10). Already after its gravity assist at Jupiter, Voyager 2 had enough velocity to escape from the solar system. The right panel shows the trajectory in the plane of the solar system, with the Sun at (0, 0) and the positive x-axis the line of zero ecliptic longitude. Data obtained from JPL Horizons.

of the asymptotic velocity of the spacecraft relative to the planet will be unchanged by the encounter, but the direction of travel will change. When both planet and spacecraft are moving relative to a star, then the magnitude of the velocity of the spacecraft *relative to the star* can also change. This is the principle of the gravity assist, and it has been used many times in interplanetary flight. Famously, the gravity assist was used to give the Pioneer and Voyager spacecraft enough velocity to escape from the solar system through flybys of the outer planets, as shown in figure 4.8 for Voyager 2. This figure shows how the spacecraft's heliocentric velocity varies with its distance from the Sun. At each planet, the spacecraft received a significant velocity boost (left) and change in direction (right). Between these close encounters only the Sun's gravity is acting, so the velocity drops according to the vis-viva equation (equation 4.3). Note that the velocity after the Neptune encounter is actually lower than before the encounter; gravity assists can decelerate spacecraft too. This was done to allow Voyager to make a close pass of Neptune's moon Triton.

4.3.1 Concept

Before examining in detail how the gravity assist works, it is useful to look at a simpler system that demonstrates the same principle. Consider standing at rest on a station platform as a train approaches at velocity u_t. A ball is thrown at velocity u_b along the track towards the approaching train. The ball hits the front of the train and bounces off elastically. What is the final velocity of the ball?

The situation is illustrated in figure 4.9. The initial situation in the rest frame of the station is shown in the top left. In the rest frame of the train (top right), the ball approaches the train

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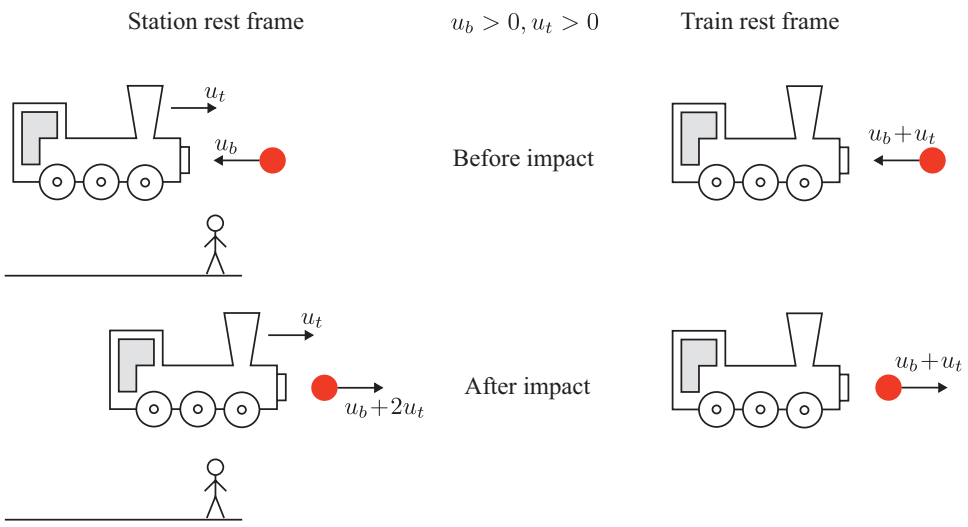


Fig. 4.9 The concept of gravity assist in one dimension is the same as how a ball gains speed when bounced elastically off a (massive) moving train. In the rest frame of the station (left), the train is initially moving to the right at speed u_t and the ball to the left at speed u_b. After the bounce, the speed of the train is essentially unchanged, but the ball picks up twice the speed of the train. In the rest frame of the train (right) the bounce is symmetric.

at a velocity $u_b + u_t$. After bouncing elastically, the ball recedes at the same velocity but in the opposite direction (bottom right). Back in the station's rest frame (bottom left), the train is still moving at velocity u_t, so the ball is now moving at velocity $u_b + 2u_t$. Thus we see that in the station's rest frame, the ball changes direction and picks up an additional velocity of twice that of the train.[3]

The principle of the gravity assist is very similar to this, with the spacecraft playing the role of the ball and the planet the role of the train. The interaction is viewed from the rest frame of the star (station). Instead of the deflection occurring by direct impact, that is, electrostatic forces in the material of the ball and train, it occurs by gravity.

In what follows I neglect the gravity of the star, so the planet is taken to be moving on a straight, unaccelerated path. This is a reasonable approximation when the timescale of the flyby is short. The spacecraft is not bound to the planet, so its flyby is a hyperbolic orbit about the planet as seen from the planet's rest frame. This encounter changes the direction of travel of the spacecraft, but not its asymptotic velocity, just as we saw with the ball in the train's rest frame. But in the star's rest frame, because the planet is moving, the velocity of the spacecraft is changed.

In the star's rest frame, let the planet's two-dimensional velocity be $\mathbf{u}_p = (u_{p,x}, u_{p,y})$, and the initial velocity of the spacecraft well before the encounter be $\mathbf{u}_s = (u_{s,x}, u_{s,y})$. In the planet's rest frame, the initial velocity of the spacecraft is $\mathbf{u}_{sp} = \mathbf{u}_s - \mathbf{u}_p$, and the encounter serves just to turn the spacecraft's velocity by an angle δ (see figure 4.4). This rotation can be represented by the rotation matrix

[3] I have assumed that the train is much more massive than the ball, and so the change in the train's momentum is negligible.

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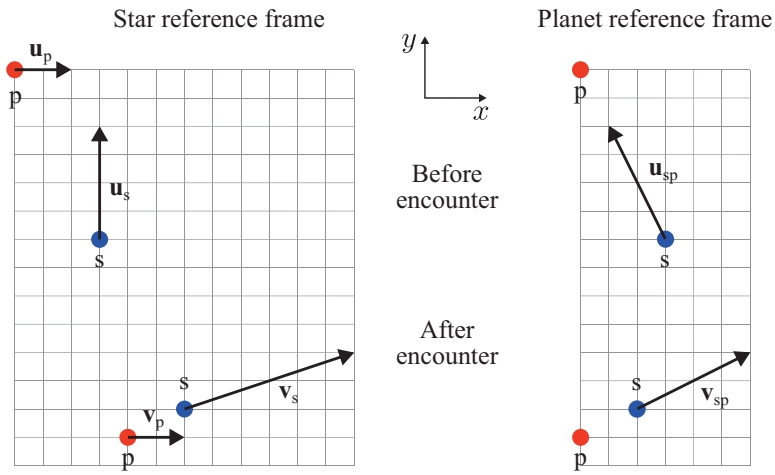


Fig. 4.10 The gravity assist of a spacecraft (s) around a planet (p) as viewed in the star's reference frame (left) and the planet's reference frame (right). The arrows show the magnitude and direction of the velocity. The upper diagrams show the situation 1.5 time units before the encounter, and the lower diagrams 0.5 time units after the encounter. In the planet's reference frame, the encounter changes the direction of the spacecraft by 90°, but it does not change the magnitude of its velocity. In the star's reference frame both change.

$$\mathrm{R} = \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix}. \tag{4.23}$$

Thus the asymptotic velocity of the spacecraft in the planet's rest frame after the encounter is given by the matrix product $\mathbf{v}_{sp} = \mathrm{R}\mathbf{u}_{sp}$. As the velocity of the planet is unchanged (we assume it to be massive), the post-encounter asymptotic of the spacecraft in the star's rest frame is $\mathbf{v}_s = \mathrm{R}\mathbf{u}_{sp} + \mathbf{u}_p$.

Consider the case $\mathbf{u}_s = (0, u_{s,y})$, $\mathbf{u}_p = (u_{p,x}, 0)$ shown in figure 4.10. The flyby is arranged to give a deflection angle of $\delta = -90°$ (a clockwise rotation), so

$$\mathrm{R} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}. \tag{4.24}$$

In the planet's rest frame, the spacecraft's velocity is $\mathbf{u}_{sp} = (-u_{p,x}, u_{s,y})$ before the encounter and $\mathbf{v}_{sp} = \mathrm{R}\mathbf{u}_{sp} = (u_{s,y}, u_{p,x})$ after the encounter. In the star's rest frame, the velocity of the planet after the encounter is $\mathbf{v}_s = (u_{s,y} + u_{p,x}, u_{p,x})$. With $u_{s,y} = 2$ and $u_{p,x} = 1$, we get $\mathbf{u}_{sp} = (-1, 2)$ and $\mathbf{v}_{sp} = (2, 1)$, drawn to scale in the figure. The final velocity of the spacecraft in the star's rest frame is $\mathbf{v}_s = (3, 1)$ giving a speed of $\sqrt{10} = 3.2$, compared to an initial speed of 2. The gravity assist has increased the speed of the star.

The direction of travel of the spacecraft in the star's rest frame after the encounter is $\theta' = \mathrm{atan}(v_{s,y}/v_{s,x})$, where $v_{s,x}$ and $v_{s,y}$ are the x and y components of the spacecraft's velocity, respectively. Angles are measured anticlockwise from the x-axis. Thus the deflection angle in the star's rest frame is $\theta' - \theta$, where $\theta = \mathrm{atan}(u_{s,y}/u_{s,x})$ is the initial direction of travel. For the numerical example just given, the initial direction of travel is $\theta = +90°$ and the final direction of travel is $\theta' = \mathrm{atan}\, 1/3 = +18°$. Hence the deflection angle in the star's rest frame is $-72°$.

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Note that the gravity of the star has played no part in this calculation. The star is just the frame of reference from which we observe the encounter. Planning a real gravity assist is of course more complicated because it takes place over a finite time during which the planet will move on its orbit. But this is not the cause of the 'gravity' assist. Indeed, we have not even explicitly considered the gravity in the spacecraft–planet interaction. The curved hyperbolic orbit will appear in the closest parts of the encounter, but it does not affect the asymptotic velocities considered here.

The interaction has changed the spacecraft's energy and momentum in the star's rest frame. These are taken from the energy and momentum of the planet in this same frame. In reality that corresponds to the orbital energy and angular momentum of the planet about the star. A gravity assist around a planet that accelerates the spacecraft therefore leads to the planet losing energy (relative to the star) and going into a smaller orbit. In practice this change is tiny, due to the enormous difference in mass of the planet and the spacecraft.

In the above example, the clockwise turn of the spacecraft would be achieved by the spacecraft encountering the planet from behind its direction of travel, just as it flies past (top left of figure 4.10). Gravity pulls the spacecraft to give it a velocity component in the direction the planet is moving and so the spacecraft picks up speed relative to the star. If the spacecraft instead passed in front of the planet, it would turn in the opposite direction (to the left in figure 4.10) and so would lose speed. To compute this, with $\delta = +90°$ (an anticlockwise rotation), the final velocity of the spacecraft in the planet's rest frame is $\mathbf{v}_{sp} = (-2, -1)$. In the star's rest frame it is $\mathbf{v}_s = (-1, -1)$, giving a speed of $\sqrt{2} = 1.4$, a direction of $\theta' = \operatorname{atan} 1/1 = -45°$ and thus a deflection angle of $-135°$. This is a general feature of gravity assists: to increase its speed (in the star's rest frame), the spacecraft needs to pass behind the planet in its orbit about the star; to decrease its speed the spacecraft needs to pass in front of the planet.

4.3.2 Gravity assists on real bodies

The size of the spacecraft's deflection in the planet's rest frame is given by equation 4.15, which depends only on the eccentricity e. The smaller the eccentricity – closer to 1 – the larger the deflection angle (apparent in figure 4.5). From equations 4.13 and 4.14 we see that

$$e^2 - 1 = \frac{b^2 v_\infty^4}{\mu^2}. \tag{4.25}$$

To achieve a given eccentricity (and thus deflection angle), then the larger the velocity, the smaller the impact parameter must be. Similarly, increasing the mass of the planet, decreasing the impact parameter, or decreasing the velocity of the spacecraft all produce a smaller eccentricity and thus a larger deflection angle.

Suppose we want to use a gravity assist to turn a spacecraft by $\delta = 90°$ (in the planet's rest frame). This corresponds to an orbit with $e = \sqrt{2}$ and so, from equation 4.25, $b = \mu/v_\infty^2 = -a$. With $v_\infty = 5\,\mathrm{km\,s^{-1}}$, and using Jupiter for the gravity assist (mass of $1.9 \times 10^{27}\,\mathrm{kg}$), we require an impact parameter of $b = 5.1 \times 10^6\,\mathrm{km}$, which is 71 times Jupiter's equatorial radius. Because of Jupiter's gravity, the spacecraft will approach closer to Jupiter than the

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impact parameter ('gravitational focusing'). The closest approach is the periapsis distance,[4] $r_p = a(1-e) = \mu(e-1)/v_\infty^2$, which in this case is $(\sqrt{2}-1)b = 2.1 \times 10^6$ km, or 29 times Jupiter's equatorial radius.

If we recompute what we need to get this same deflection at an asteroid of radius 100 km and density 2500 kg m^{-3} (mass of just over 10^{19} kg) we get $b = 28$ m and $r_p = 12$ m. The spacecraft would hit the asteroid, so such a turn is not possible with such a low mass object. In fact, at this density, and taking into account the gravitational focusing, such an asteroid would have to have a radius of at least 9300 km (see exercises). Gravity assists are possible at large asteroids (the largest is Ceres with a radius of around 470 km and with a mean density of 2200 kg m^{-3}), but only for smaller deflection angles. The Earth is smaller than the above limit, but it has a higher density. The impact parameter to achieve a 90° turn around the Earth at $v_\infty = 5$ km s^{-1} is 15 900 km, and results in a closest approach to the Earth of just 6606 km. This is not much more than the Earth's mean radius of 6371 km. The Cassini–Huygens spacecraft did a flyby of the Earth in August 1999 at an altitude of just 1170 km, giving it a Δv of 5.5 km s^{-1} to help it get it to Saturn.

The final velocity of the spacecraft relative to the star depends on the velocity of the planet through the vectorial addition of the velocities. The maximum change in velocity occurs when the spacecraft and planet are moving (anti-)parallel and the deflection angle is 180°. This corresponds to a parabolic orbit with $e = 1$. When planet and spacecraft are moving anti-parallel (near head-on collision) with (positive) speeds u_p and u_s, respectively, the spacecraft's speed increases by twice the planet's speed, to $2u_p + u_s$. This is exactly analogous to the ball bouncing off the train in figure 4.9. When they are instead moving parallel – so that one catches up with the other – the final velocity of the spacecraft relative to the star is $2u_p - u_s$, which could be negative.

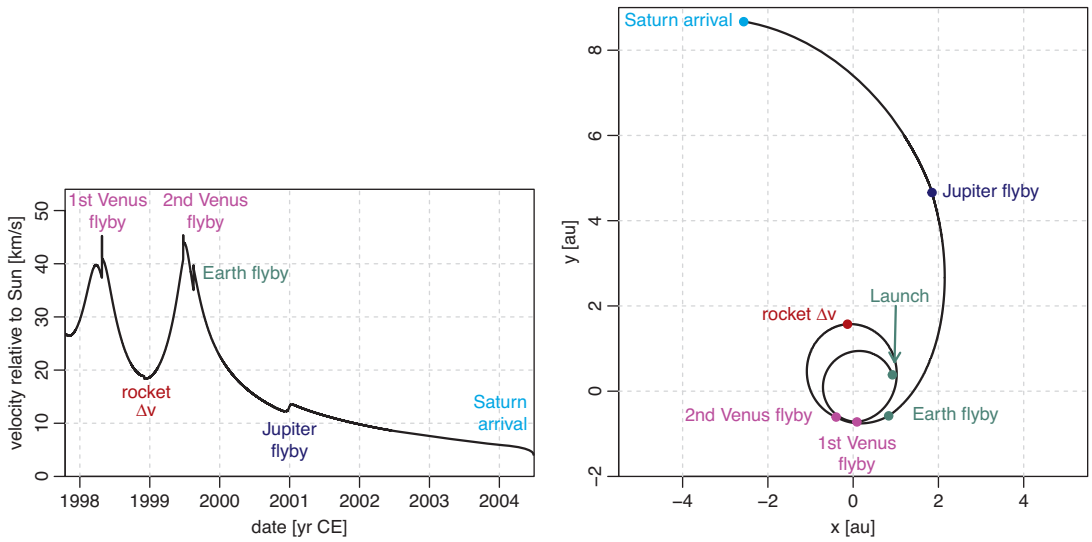
Gravity assists can provide very large changes in velocity, of several km s^{-1}. A spacecraft orbiting the Sun at the Earth's distance needs $\Delta v = 12.3$ km s^{-1} to escape from the Sun (section 4.1.4). To achieve this with a chemical rocket with $v_e = 4.5$ km s^{-1} would require a mass ratio of 15.5, from the rocket equation. If we can instead achieve some of this using one or more gravity assists about planets, the rocket can be smaller and cheaper, or the scientific payload larger. For this reason, many interplanetary missions have used gravity assists. As shown earlier in figure 4.8, Voyager 2 used them at each of the planets – Jupiter, Saturn, Uranus, and Neptune – that it visited in its grand tour.[5] At 720 kg launch mass, Voyager 2 was a relatively light spacecraft and went directly to Jupiter without a gravity assist. Cassini–Huygens, in contrast, had a launch mass of 5700 kg, so to get to Jupiter (on its way to Saturn) it needed to do two gravity assists at Venus (both Δv around 7 km s^{-1}) and one at the Earth (5.5 km s^{-1}). Its trajectory and change in velocity are shown in figure 4.11. As the gravity assist may not put the spacecraft on the exact trajectory required, rocket boosts are often used in between to make fine corrections to the path. This was the case with Cassini, which used

[4] For hyperbolic orbits we can also write this as

$$\frac{r_p}{b} = \sqrt{\frac{e-1}{e+1}}.$$

[5] Voyager 2 relied on a rare alignment of the outer planets to allow it to visit these planets in succession without needing a large rocket. Such an alignment – by which we do not mean all planets lying in a straight line – only occurs every 175 years.

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Gravity assists of Cassini–Huygens at Venus (twice), the Earth, and Jupiter. The spacecraft rockets also gave it a Δv between the two Venus flybys. The right panel shows the trajectory in the plane of the solar system, with the Sun at (0, 0) and the positive x-axis the line of zero ecliptic longitude. The motion of the spacecraft is anticlockwise. Data obtained from JPL Horizons.

Fig. 4.11

a relatively large boost after its first Venus encounter to put it on course for a second Venus encounter.

As a large Δv is also required to get to the inner solar system, gravity assists are also used for missions to Mercury and the Sun. The BepiColombo mission to go into orbit about Mercury, for example, has done nine gravity assists in total at the Earth, Venus, and Mercury between its launch in October 2018 and its orbital insertion planned for November 2026.

Are gravity assists useful for interstellar travel? Gravity assists about planets in the solar system could theoretically give a spacecraft a Δv of up to twice the orbital velocity of the planet in the case of a head-on encounter and 180° turn. The largest velocity change would be with Mercury, which could be up to twice its orbital speed, $2 \times 48\,\mathrm{km\,s^{-1}}$. To avoid hitting Mercury this could only be achieved with a large impact parameter with a small initial velocity, and the latter is the opposite of what we want for interstellar travel. Furthermore, the spacecraft would then lose a lot of velocity escaping from the deep gravitational potential close to the Sun. In practice, gravity assists off solar system planets will only increase the velocity at infinity of a spacecraft by a few tens of kilometres per second, so are of no benefit for getting to the nearest stars quickly. But they could be important for manoeuvring a spacecraft in a target star system after rendezvous.

4.4 Summary

We have looked at some of the properties of orbits and how to move between them using impulsive tangential boosts. The main orbits of interest are bound elliptical orbits and unbound hyperbolic orbits.

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The Oberth effect shows that the maximum change in the kinetic energy of a rocket in some reference frame is achieved by firing its rockets when it is moving fastest (equation 4.22). For a spacecraft moving on an elliptical orbit, this means applying the Δv at periapsis. The Oberth effect is simply a consequence of the quadratic dependence of kinetic energy on velocity, and is useful for maximizing the efficient use of propellant.

The flyby of a spacecraft past a planet is a hyperbolic orbit. When this is viewed from the perspective of a third body, specifically the central star, we get the concept of the gravity assist: what is a symmetric inbound–outbound motion in the planet's rest frame is asymmetric in the star's frame because the planet is moving (figure 4.10). The net effect is a change in the velocity of the spacecraft relative to the star. In order for this flyby to accelerate the spacecraft, it must pass behind the planet; to decelerate it, the spacecraft must pass in front of the planet. The spacecraft exchanges energy and momentum with the planet, but the latter is so massive that the change in its orbit is unmeasurable. Gravity assists are frequently used in interplanetary missions to manoeuvre spacecraft and to boost spacecraft above the solar system's escape velocity. In practice, gravity assists are limited to velocity changes of a few tens of kilometres per second in the best cases, so are of little use for reaching high interstellar velocities.

4.5 Exercises

1. In the biellptical transfer orbit of figure 4.7, the inner orbit has a radius of 1 au and the outer orbit a radius of 5 au. If we place point B at 10 au from the central mass, what are the semi-major axes and eccentricities of orbits 2 and 3?
2. In section 4.2.2, we consider the orbit transfer required to drop into the Sun efficiently. Derive equation 4.20.
3. In section 4.2.3 on the Oberth effect, it is stated that it is best to escape from an elliptical orbit by applying Δv at periapsis rather than apoapsis. Derive an expression for the velocities at infinity when applying Δv at (i) periapsis and (ii) at apoapsis. Plot how these vary with eccentricity and prove that the velocity in case (i) is never smaller than the velocity in case (ii).
4. We discussed in section 4.2.3 how to use the Oberth effect to maximize the velocity at infinity for a given boost by first diving close to the Sun to increase the pre-boost velocity (v_i in equation 4.22). For a given perihelion distance r_p, this can be achieved with a highly elliptical orbit about the Sun, that is, one with a large aphelion distance r_a. Show, however, that the perihelion velocity is more or less independent of r_a when $r_a >> r_p$.
5. Show that for a gravity assist to achieve a turn angle δ around a planet of mean density ρ when the asymptotic spacecraft velocity is v_∞, the planet must have a minimum radius of

$$r_{\min} = v_\infty \sqrt{\frac{3}{4\pi G\rho(e-1)}}$$

to avoid colliding with the planet, where $e = 1/\sin(\delta/2)$. What is the maximum value of v_∞ for achieving $\delta = 90°$ at Jupiter?

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