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DUAL p-ADIC DIOPHANTINE APPROXIMATION ON MANIFOLDS

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Abstract The Generalised Baker–Schmidt Problem (1970) concerns the Hausdorff measure of the set of ψ-approximable points on a non-degenerate manifold. Beresnevich-Dickinson-Velani (in 2006, for the homogeneous setting) and Badziahin-Beresnevich-Velani (in 2013, for the inhomogeneous setting) proved the divergence part of this problem for dual approximation on arbitrary non-degenerate manifolds. The divergence part has also been resolved for the p-adic setting by Datta-Ghosh in 2022, for the inhomogeneous setting. The corresponding convergence counterpart represents a challenging open question. In this paper, we prove the homogeneous p-adic convergence result for hypersurfaces of dimension at least three with some mild regularity condition, as well as for some other classes of manifolds satisfying certain conditions. We provide similar, slightly weaker results for the inhomogeneous setting. We do not restrict to monotonic approximation functions.

keywords: Diophantine approximation; p-adic numbers; Hausdorff Dimension

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1. Introduction

The investigation of Diophantine approximation on manifolds, initiated by Mahler in 1932, is an active area of research that studies how well points on some curve or manifold can be approximated by some rational affine subspace. In this article we consider the dual approximation setting where the manifold is approximated by rational hyperplanes. Initiated by the breakthrough work of Kleinbock and Margulis [23], the approximation of real manifolds by rational hyperplanes has been extensively studied over the past two decades. See [1–5, 7, 8, 10, 11, 13, 17, 18–22, 33] and references within for just a sample of the many results. In this article, we will contribute to the theory of Diophantine approximation on manifolds in *p-adic space*. To begin we recall

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the setup for dual p-adic Diophantine approximation, before giving the state of the art of Diophantine approximation on p-adic manifolds. We then state our new results and associated corollaries.

Fix a prime p and let $|\cdot|_p$ denote the p-adic norm, $\mathbb{Q}_p$ the p-adic numbers and $\mathbb{Z}_p$ the ring of p-adic integers, that is, $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. Throughout, let $n \geq 1$ be a fixed integer and $\mathbf{a} := (a_0, , a_1, \dots, a_n) \in \mathbb{Z}^{n+1}$. Let $\Psi : \mathbb{Z}^{n+1} \to [0, \infty)$ be a *multivariable approximating function*, that is, Ψ has the property that

$$\Psi(\mathbf{a}) \to \ 0 \text{as } \|\mathbf{a}\| := \max(|a_0|, |a_1|, \dots, |a_n|) \to \ \infty.$$

Fix some $\theta \in \mathbb{Z}_p$ and define the set of p-adic dually (Ψ, θ)-approximable points as

$$D^{\theta}_{n,p}(\Psi) := \left\{ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Q}_p^n : \begin{array}{l} |a_1 x_1 + \cdots + a_n x_n + a_0 + \theta|_p < \Psi(\mathbf{a}) \\ \text{for infinitely many } \mathbf{a} = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1} \end{array} \right\}.$$

For the homogeneous setting write $D^0_{n,p}(\Psi) := D_{n,p}(\Psi)$, and for Ψ a univariable approximation function of the form $\Psi(\mathbf{r}) = \psi(\|\mathbf{r}\|)$, that is $\psi : \mathbb{N} \to [0, \infty)$, we write $D^{\theta}_{n,p}(\Psi) := D^{\theta}_{n,p}(\psi)$.

Note that, unlike classical Diophantine approximation in $\mathbb{R}$, the function Ψ depends on the $n+1$ integers $(a_0, \dots, a_n)$, including a_0. This is because $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ and so if a_0 is unbounded from above, one could obtain increasingly precise approximations of $\mathbf{x}$ for $(a_1, \dots, a_n)$ fixed, at least if $\mathbf{x} \in \mathbb{Z}_p^n$.

The concept of Hausdorff dimension and more generally Hausdorff f-measure for any dimension function $f : \mathbb{R}_+ \to \mathbb{R}_+$ can be defined over $\mathbb{Q}_p^n$ by coverings of balls with respect to the p-adic metric derived from $\|\cdot\|_p = \max|\cdot|_p$. We refer to [30] for properties of the Hausdorff measure and dimension in general metric space. Thus it makes sense to ask questions about the 'size' of $D^{\theta}_n(\Psi)$ with respect to the f-dimensional Hausdorff measure $\mathcal{H}^f$ for some dimension function f. When $f(r) = r^n$ this reduces to the size of $D^{\theta}_n(\Psi)$ in terms of the n-dimensional p-adic Haar measure $\mu_{p,n}$ up to some fixed constant.

In the homogeneous setting Mahler, see for example [26], proved for $\Psi(\mathbf{a}) = \psi(\|\mathbf{a}\|) = \|\mathbf{a}\|^{-(n+1)}$ that $D_{n,p}(\Psi) = \mathbb{Q}_p^n$, providing a p-adic equivalent of Dirichlet's Theorem. By an application of the Borel Cantelli lemma, we can deduce that the set of p-adic *very well approximable points* (points in $D_{n,p}(\Psi)$ for $\Psi(\mathbf{a}) = \|\mathbf{a}\|^{-(n+1+\varepsilon)}$ for some $\varepsilon > 0$) is a nullset, where the measure here is the n-dimensional p-adic Haar measure $\mu_{p,n}$. Lutz proved the p-adic analogue of the well-known Khintchine's Theorem in the real setting [25], and the p-adic version of the classical Jarnik's Theorem in Diophantine approximation can be found in [8, Theorem 16].

2. *p*-adic Diophantine approximation on manifolds and our main result

2.1. *p*-adic metric Diophantine approximation: Introduction and brief history

For p-adic approximation on manifolds, the following results are known. We keep the statements of the known results brief and refer the reader to the relevant paper for more

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details. A key additional condition often required in the p-adic setting, in comparison the the analogous results in the real setting, is the analyticity of the curve or manifold. Let $\mathcal{M} \subset \mathbb{Q}_p^n$ be a p-adic manifold with dimension d defined by analytic map $\mathbf{g} : \mathcal{U} \subset \mathbb{Q}_p^d \to \mathbb{Q}_p^{n-d}$ via parametrization $(\mathbf{x}, \mathbf{g}(\mathbf{x}))$. Hereby analytic is defined as follows.

Definition 2.1. *A function $\boldsymbol{h} : U \subseteq \mathbb{Q}_p^m \to \mathbb{Q}_p^n$ for U open is analytic if every coordinate function can be written as a power series*

$$h_j(x_1, \dots, x_m) = \sum_{\boldsymbol{t}} a_{j,\boldsymbol{t}} x_1^{t_1} \cdots x_m^{t_m}, \qquad 1 \le j \le n,$$

converging in some p-adic ball $B(\mathbf{y}, r)$, $r = r(\mathbf{y}) > 0$ around every $\mathbf{y}$ contained in U.

Note that any analytic function $\mathbf{h} \in C^\infty(U)$. Compared to the real setting where $\mathbf{g} \in C^2$ was sufficient, for technical reasons we require the stronger condition of analytic manifolds in the p-adic setting. By the Implicit Function Theorem (the p-adic version of this following from $\mathbf{g}$ being analytic and the p-adic inverse function theorem, see Theorem 3.1) we may write

$$\mathcal{M} = \left\{(x_1, \dots, x_d, g_1(\mathbf{x}), \dots, g_{n-d}(\mathbf{x})) : \mathbf{x} = (x_1, \dots, x_d) \in \mathcal{U} \subset \mathbb{Q}_p^d\right\},$$

for analytic functions $g_i : \mathcal{U} \subset \mathbb{Q}_p^d \to \mathbb{Q}_p$, $1 \le i \le n-d$. For ease of notation write $\mathbf{g} = (g_1, \dots, g_{n-d})$.

We split the known results into three categories:

- *Extremality*: This refers to results associated with Mahler's (and subsequently Sprindzhuk's [33]) 1932 conjecture in real approximation. Specifically the size, in terms of the induced p-adic Haar measure, of the set of very well approximable points contained in some manifold is a nullset. Alongside the statement in the real setting, Sprindzhuk proved the p-adic equivalent of Mahler's conjecture in the p-adic setting [33]. Kleinbock and Tomanov [24] used similar ideas to those in [23] (in which the real case was proven) to prove extremality for all C^2 non-degenerate p-adic manifolds. See [24] for the precise definition of a C^2 function in the p-adic setting. The inhomogeneous theory for the Veronese curve preceded that of Kleinbock and Tomanov and was proven by Bernik, Dickinson, and Yuan [12].
- *Ambient measure*: These results refer to analogues of Lutz's Theorem [25] in the setting of Diophantine approximation on dependent quantities, that is the induced p-adic Haar measure of $\mathcal{D}_n^\theta(\Psi) \cap \mathcal{M}$. The complete theory for the Veronese curve was proven by Beresnevich, Bernik and Kovalevskaya [6]. Prior to this, in [9], Beresnevich and Kovalevskaya proved the complete result for p-adic normal[1.] planar curves. In the inhomogeneous setting the convergence case was proven for the Veronese curve by Ustinov [34]. The convergence case for analytic non-degenerate

[1.] see [26] for Mahler's definition of a p-adic normal function.

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manifolds was proven by Mohammadi and Salehi-Golsefidy [27] with the complementary divergence statement appearing soon after [28]. The inhomogeneous convergence statement was proven in [16].

- *Hausdorff theory*: These results refer to the Hausdorff measure and dimension of $\mathcal{D}^{\theta}_{n,p}(\Psi) \cap \mathcal{M}$. In the special case of Veronese curves, the metric theory is rather complete by results of Bernik and Morotskaya [14, 29]. For a general class of analytic manifolds, the divergence statement in both the homogeneous and inhomogeneous setting has recently been proven by Datta and Ghosh [16], see § 2.4 for their result. In this paper, we contribute to the inhomogeneous convergence case.

In full generality, a complete Hausdorff measure treatment for manifolds $\mathcal{M}$ represents a deep open problem referred to as the Generalised Baker–Schmidt Problem (GBSP) inspired by the pioneering work of Baker and Schmidt [3]. See for example [22] for the conjectured statement in $\mathbb{R}^n$. For completeness, we state the p-adic version of the GBSP here.

Problem 2.2. (Generalised Baker–Schmidt Problem for p-adics.) *Let $\mathcal{M} = \{(\mathbf{x}, g(\mathbf{x})) : \mathbf{x} \in \mathbb{Q}_p^d\}$ be a manifold of dimension d defined by analytic map $g : \mathcal{U} \subset \mathbb{Q}_p^d \to \mathbb{Q}_p^{n-d}$. Let f be a dimension function such that $r^{-d}f(r) \to \infty$ as $r \to 0$. Assume that $r \mapsto r^{-d}f(r)$ is decreasing and $r \mapsto r^{1-d}f(r)$ is increasing. Let $\theta \in \mathbb{Z}_p$ and $\Psi : \mathbb{Z}^{n+1} \to \mathbb{R}_+$ be a multivariable approximation function. Then it is conjectured that*

$$\mathcal{H}^f(D_{n,p}(\Psi) \cap \mathcal{M}) = \begin{cases} 0 & \text{if} \quad \displaystyle\sum_{\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})) < \infty\,, \\ \infty & \text{if} \quad \displaystyle\sum_{\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})) = \infty\,. \end{cases}$$

Note for dimension function $f(r) = r^s$ the assumed restrictions of the GBSP ask for $d - 1 < s < d$.

2.2. Main results

The conditions below, while restrictive, contain some interesting classes of manifolds and dimension functions. We will discuss these in more detail in § 2.3 and 6.

(Ip) Let f be a dimension function satisfying

$$f(xy) \ll x^s f(y) \text{for all } y < 1 < x, \tag{1}$$

for some $s < 2(d-1)$.

(IIp) For each $1 \le i \le n-d$, let $g_i : \mathcal{U} \to \mathbb{Q}_p$ be analytic on some open set $\mathcal{U} \subset \mathbb{Q}_p^d$ and suppose that for any $\mathbf{z} = (z_1, \ldots, z_{n-d}) \in \mathbb{Z}_p^{n-d}$ with $\|\mathbf{z}\|_p = 1$, the $d \times d$ matrix with p-adic entries

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$$M_{\mathbf{z}}(\mathbf{x}) = \left(\sum_{k=1}^{n-d} z_k \frac{\partial^2 g_k(\mathbf{x})}{\partial x_i \partial x_j} \right)_{1 \le i,j \le d},$$

has non-zero determinant for all $\mathbf{x} \in \mathcal{U} \setminus S_{\mathcal{M}}$, with some

$$S_{\mathcal{M}} := \{\mathbf{x} \in \mathcal{U} : M_{\mathbf{z}}(\mathbf{x}) \text{ is singular for some } \mathbf{z} = \mathbf{z}(\mathbf{x}) \in \mathbb{Z}_p^{n-d} \setminus \{\mathbf{0}\}\}$$

whose p-adic closure $\overline{S}_{\mathcal{M}}$ has f-measure 0.

Note while for fixed $\mathbf{z}$ the accordingly defined set $S_{\mathcal{M}}(\mathbf{z})$ is closed, this seems not necessarily true for $S_{\mathcal{M}}$. Hence for technical reasons we take the closure in (IIp).

Theorem 2.3 *Let f be a dimension function satisfying (Ip) and $\mathcal{M} = \{(\mathbf{x}, \boldsymbol{g}(\mathbf{x})) : \mathbf{x} \in \mathcal{U}\}$ be a manifold of dimension d defined by analytic $\boldsymbol{g} : \mathcal{U} \subset \mathbb{Q}_p^d \to \mathbb{Q}_p^{n-d}$ satisfying (IIp). Let $\Psi : \mathbb{Z}^{n+1} \to \mathbb{R}_+$ be a multi-variable approximation function satisfying*

$$\Psi(\mathbf{a}) < \|\mathbf{a}\|^{-1}, \quad \text{and} \quad p\Psi(p\mathbf{a}) \le \Psi(\mathbf{a}) \quad \forall\, \mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}. \tag{2}$$

Then $\mathcal{H}^f(D_{n,p}(\Psi) \cap \mathcal{M}) = 0$ if

$$\sum_{\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})) < \infty\,.$$

Remark 2.4. Note that (2) is clearly satisfied for the standard power functions $\Psi(\mathbf{a}) = \|\mathbf{a}\|^{-\tau-1}, \tau \ge n$. The limitation $\Psi(\mathbf{a}) < \|\mathbf{a}\|^{-1}$ is mild. Indeed, if $\Psi(\mathbf{a}) > \|\mathbf{a}\|^{-1} > \|\mathbf{a}\|^{-n-1}$ for all sufficiently large $\|\mathbf{a}\|$, then by Dirichlet's Theorem $D_{n,p}(\Psi) = \mathbb{Q}_p^n$. Assuming this, the right condition of (2) additionally holds more generally for example if $-\log \Psi(\mathbf{a})/\|\mathbf{a}\|$ is non-decreasing.

Note that Theorem 2.3 only treats the homogeneous case. We want to further establish related results on inhomogeneous approximation. To this end, we introduce the following set

$$D_{n,p}^{\theta}(\Psi, Z) := \left\{ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Q}_p^n : \begin{array}{l} |a_1 x_1 + \cdots + a_n x_n + a_0 + \theta|_p < \Psi(\mathbf{a}) \\ \text{for infinitely many } (a_0, a_1, \dots, a_n) \in Z \end{array} \right\}$$

for a subset $Z \subseteq \mathbb{Z}^{n+1}$. In particular we will consider the subsets of $\mathbb{Z}^{n+1}$ defined by

$$Z(1) = \{\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{Z} \times (\mathbb{Z}^n \setminus \{\mathbf{0}\}) : \gcd(\mathrm{a_i}, \mathrm{a_j}, \mathrm{p}) = 1, \quad 0 \le \mathrm{i} < \mathrm{j} \le \mathrm{n}\}\,,$$
$$Z(2) = \{\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{Z} \times (\mathbb{Z}^n \setminus \{\mathbf{0}\}) : \gcd(\mathrm{a_0}, \dots, \mathrm{a_n}, \mathrm{p}) = 1\}\,.$$

That is, the sets where p divides at most one a_i and not all a_i, respectively. Notice that

$$Z(1) \subseteq Z(2) \subseteq \mathbb{Z} \times (\mathbb{Z}^n \setminus \{\mathbf{0}\})\,.$$

For $Z = \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}$ we obviously have $D_{n,p}^{\theta}(\Psi, Z) = D_{n,p}^{\theta}(\Psi)$.

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Theorem 2.5 *Let f be a dimension function satisfying (Ip) and $\mathcal{M} = \{(\mathbf{x}, \boldsymbol{g}(\mathbf{x})) : \mathbf{x} \in \mathbb{Q}_p^d\}$ be a manifold of dimension d defined by analytic $\boldsymbol{g} : \mathcal{U} \subset \mathbb{Q}_p^d \to \mathbb{Q}_p^{n-d}$ satisfying (IIp). Let $\theta \in \mathbb{Z}_p$ and $\Psi : \mathbb{Z}^{n+1} \to \mathbb{R}_+$ be a multivariable approximation function with $\Psi(\mathbf{a}) < \|\mathbf{a}\|^{-1}$. Then*

(i) $\mathcal{H}^f(D^{\theta}_{n,p}(\Psi, Z(1)) \cap \mathcal{M}) = 0$ *if*

$$\sum_{\mathbf{a} \in Z(1)} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})) < \infty.$$

(ii) $\mathcal{H}^f(D^{\theta}_{n,p}(\Psi, Z(2)) \cap \mathcal{M}) = 0$ *if* $|\theta|_p \neq 1$ *and*

$$\sum_{\mathbf{a} \in Z(2)} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})) < \infty\,.$$

According to (i), restricting the integer vectors to the smallest set $Z(1)$, we have the convergence claim as predicted by Problem 2.2. In the classical homogeneous setting, the same is true when summing over $Z(2)$, by (ii). The additional condition on θ in (ii) is an artifact of our method, we do not have a satisfactory explanation for its occurrence.

2.3. Remarks on Conditions (Ip)-(IIp)

Condition (Ip): By writing $r_0 = y, r_1/r_0 = x$, condition (Ip) is checked to be equivalent to $f(r_0)/r_0^s \gg f(r_1)/r_1^s$ for any $0 < r_0 < r_1$ with the same uniform implied constant, in particular if $\ll$ is strengthened by $\leq$ in (Ip), then the map $r \to f(r)r^{-s}$ is non-increasing. Examples of dimension functions f satisfying (Ip) are $f(r) = r^s$ for some $0 < s < 2(d-1)$. Note that $s > d$ is not of interest as the entire manifold $\mathcal{M}$ has $\mathcal{H}^s(\mathcal{M}) = 0$. Condition (Ip) is always true when $d \geq 3$ and $r^{-d}f(r)$ is decreasing (a general assumption in Problem 2.2), see [22, Section 1.2] for hypersurfaces, which can readily be generalised as remarked in [20].

Condition (IIp): This condition is a little more technical. It has the most natural interpretation when $\mathcal{M}$ is a hypersurface. Then (IIp) is equivalent to asking for the Hessian of $\mathbf{g} = g : \mathcal{U} \subset \mathbb{Q}_p^{n-1} \to \mathbb{Q}_p$, denoted by $\nabla^2 g$, to be singular only on a set of $\mathcal{H}^f$ measure zero. That is,

$$\mathcal{H}^f\left(\left\{\mathbf{x} \in \mathcal{U} : \nabla^2 g(\mathbf{x}) \text{is singular}\right\}\right) = 0\,. \tag{3}$$

Combining with what was noticed above for the (Ip) statement, we have that (Ip) and (IIp) are satisfied for any hypersurface of dimension at least three (thus ambient space has a dimension at least four) satisfying (3). For $\mathcal{M}$ not a hypersurface, a detailed discussion on condition (IIp), in the real case, is provided in [20]. In short, in [20] some more classes of manifolds of codimension exceeding one are provided (the concrete examples found have codimension two or three), on the other hand, this condition induces some rigid restrictions on the dimension pairs (n, d). For the p-adic case certain aspects remain valid, however there are some notable differences to the real case, we provide more details in § 6.

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2.4. Corollaries to Theorem 2.3 combined with results of Datta and Ghosh

The following result is a special case of a more general results from [16], for s-dimensional Hausdorff measure.

Theorem 2.6 (Datta and Ghosh 2022) *Suppose that $\boldsymbol{g} : \mathcal{U} \subset \mathbb{Q}_p^d \to \mathbb{Q}_p^n$ and satisfies*

(I) $\boldsymbol{g}$ is an analytic map and can be extended to the boundary of $\mathcal{U} \subset \mathbb{Q}_p^d$ an open ball,
(II) Assume that $1, x_1, \dots, x_d, g_1(\mathbf{x}), \dots, g_{n-d}(\mathbf{x})$ are linearly independent functions over $\mathbb{Q}_p$ on any open subset of $\mathcal{U} \ni \mathbf{x}$,
(III) Assume that $\|\boldsymbol{g}(\mathbf{x})\|_p \leq 1$ and $\|\nabla \boldsymbol{g}(\mathbf{x})\|_p \leq 1$ for any $\mathbf{x} \in \mathcal{U}$, and that the second order difference quotient[2.] is bounded from above by $\frac{1}{2}$ for any multi-index and any triplets $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathcal{U}$.

Let

$$\Phi(\mathbf{a}) = \phi(\|\mathbf{a}\|), \quad \forall \mathbf{a} \in \mathbb{Z}^{n+1}$$

for $\phi : \mathbb{N} \to \mathbb{R}_+$ a positive non-increasing function, and assume that $s > d - 1$. Then

$$\mathcal{H}^s(D_{n,p}(\Phi) \cap \mathcal{M}) = \mathcal{H}^s(\mathcal{M}) \quad \text{if} \quad \sum_{\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{0\}} \Phi(\mathbf{a})^{s+1-d} = \infty.$$

Remark 2.7. Note that in [16] the set

$$W^{\mathbf{g}}_{\Phi,\theta} = \{\mathbf{x} \in \mathcal{U} : (\mathbf{x}, \mathbf{g}(\mathbf{x})) \in D^{\theta}_{n,p}(\Phi)\}$$

is considered. Since $\nabla \mathbf{g}$ is bounded on $\mathcal{U}$ there exists a bi-Lipschitz map between the two sets $W^{\mathbf{g}}_{\Phi,\theta}$ and $D^{\theta}_{n,p}(\Phi) \cap \mathcal{M}$ (or rather their complements) and so full measure results are equivalent. See the start of § 5 for further details of such equivalence.

Remark 2.8. In [16], the divergence case in the inhomogeneous setting is rather general. In particular, they prove the result for some general analytic function $\boldsymbol{\Theta} : \mathcal{U} \to \mathbb{Z}_p$ satisfying certain conditions, see [16, Condition (I5)] for more details. In Theorem 2.6 and our application below, since we relate to Theorem 2.3, we consider the homogeneous setting $\boldsymbol{\Theta}(\mathbf{x}) = 0$.

Combining our convergence result Theorem 2.3 with Theorem 2.6, we have the following conclusion on the homogeneous case and s-dimensional Hausdorff measure.

[2.] See [16, Section 5] or [24] for the definition of second order difference quotients.

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Theorem 2.9 *Let $\mathcal{M}$ be as in Theorem 2.5 and $f(r) = r^s$ be a dimension function with*

$$d - 1 < s < 2(d - 1).$$

Let $\boldsymbol{g} : \mathcal{U} \subset \mathbb{Q}_p^d \to \mathbb{Q}_p^n$ be an analytic map satisfying (IIp), (I), (II), and (III). Let

$$\Psi(\mathbf{a}) = \frac{\psi(\|\mathbf{a}\|)}{\|\mathbf{a}\|}, \tag{4}$$

for a monotonic decreasing function $\psi : \mathbb{N} \to \mathbb{R}_+$ tending to zero. Then

$$\mathcal{H}^s(D_{n,p}(\Psi) \cap \mathcal{M}) = \begin{cases} 0 & \text{if} \quad \sum\limits_{\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}} \Psi(\|\mathbf{a}\|)^{s+1-d} < \infty, \\ \mathcal{H}^s(\mathcal{M}) & \text{if} \quad \sum\limits_{\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \{\mathbf{0}\}} \Psi(\|\mathbf{a}\|)^{s+1-d} = \infty. \end{cases}$$

Proof. The lower bound on s is due to Theorem 2.6, and the upper bound is due to Theorem 2.3. The conditions on $\mathbf{g}$ are a combination of requirements from both theorems. We identify $\phi(\|\mathbf{a}\|) = \psi(\|\mathbf{a}\|)/\|\mathbf{a}\|$ so that $\Phi(\mathbf{a}) = \Psi(\mathbf{a})$, noting that ϕ is clearly decreasing as well. Thus on the one hand we may apply Theorem 2.6, on the other hand, we may deduce that

$$\Psi(\mathbf{a}) = \frac{\psi(\|\mathbf{a}\|)}{\|\mathbf{a}\|} = \frac{\psi(\|\mathbf{a}\|)}{p\|p^{-1}\mathbf{a}\|} \leq \frac{\psi(\|p^{-1}\mathbf{a}\|)}{p\|p^{-1}\mathbf{a}\|} = p^{-1}\Psi(p^{-1}\mathbf{a}),$$

and so Theorem 2.3 is applicable. □

Since $\mathcal{H}^s(\mathcal{M}) = 0$ for $s > d$, if $d \geq 2$ the theorem covers the whole interesting range $(d - 1, d]$ for s, apart from the endpoint $s = d = 2$ when $d = 2$.

Remark 2.10. By the above remarks on (Ip), (IIp) in § 2, the above result gives us a zero-full dichotomy for homogeneous dual approximation on sufficiently curved hypersurfaces of n-dimensional p-adic space with $n \geq 3$. Note that individually each result (the convergence and divergence case) extends beyond the above theorem. For example, [16, Theorem 1.1] allows for the inhomogeneous setting, but does not allow for multivariable approximation. Additionally, we have restricted the range of admissible approximation functions to be those of the form (4). Note that such a form of approximation function is permissible in applying Theorems 2.3.

Denoting by $\dim_{\mathcal{H}}$ the Hausdorff dimension, we deduce the following immediate corollary.

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Corollary 2.11. *Let* $\boldsymbol{g}$ *be as in Theorem 2.9 and* $d \geq 2$. *Suppose that*

$$\Psi(\mathbf{a}) = \|\mathbf{a}\|^{-1-\tau},$$

for some $\tau > n$. *Then*

$$\dim_{\mathcal{H}}(D_{n,p}(\Psi) \cap \mathcal{M}) = d - 1 + \frac{n+1}{1+\tau}.$$

Proof. We first check that all the requirements of Theorem 2.9 are satisfied. Note that $s = d - 1 + \frac{n+1}{\tau+1} < 2(d-1)$ whenever $\tau > \frac{n+1}{d-1} - 1$, and that $n \geq \frac{n+1}{d-1} - 1$ for all $d \geq 2$, hence our assumption $\tau > n$ implies $s < 2(d-1)$. The remaining conditions are obvious. So it remains to find when the critical sum converges/diverges. Note that

$$\sum_{\mathbf{a}\in\mathbb{Z}^{n+1}} \|\mathbf{a}\|^{-(1+\tau)(s+1-d)} = \sum_{r=1}^{\infty} \sum_{\mathbf{a}\in\mathbb{Z}^{n+1}:\|\mathbf{a}\|=r} r^{-(1+\tau)(s+1-d)} \asymp \sum_{r=1}^{\infty} r^{n-(1+\tau)(s+1-d)}$$

and the summation on the right hand side converges for $s > d - 1 + \frac{n+1}{1+\tau}$ and diverges when $s \leq d - 1 + \frac{n+1}{1+\tau}$. □

Remark 2.12. Note trivially for $\tau \leq n$ we have that $D_{n,p}(\Psi) = \mathbb{Q}_p^n$ by the p-adic version of Dirichlet's Theorem, and so $\dim_{\mathcal{H}}(D_{n,p}(\Psi) \cap \mathcal{M}) = \dim_{\mathcal{H}} \mathcal{M} = d$.

Remark 2.13. Note that in contrast to [22] where $d \geq 3$ is required, we get a claim for $d = 2$ here. The reason is that we assume strict inequality $\tau > n$, so that the parameter range for s satisfying (Ip) can be improved. Any such improvement leads to the implementation of the case $d = 2$.

3. Preliminaries and the main lemma

3.1. Preliminaries

We first note that for self-mappings (that is $m = n$) of analytic functions as in the Definition 2.1, the inverse function theorem holds [32, Section 9 p. 113]. The claim is formulated more generally for any ultrametric space. We notice that as explained in [32] the notion of analyticity implies infinite differentiability in a strong sense. Hereby we call a function $\phi : U \subseteq \mathbb{Q}_p^m \to \mathbb{Q}_p^n$ strong differentiable at $\mathbf{a}$ if there is a linear function $L : \mathbb{Q}_p^m \to \mathbb{Q}_p^n$ such that

$$\lim_{\|\mathbf{h}\|_p \to 0, \mathbf{h}\neq 0} \frac{\|\phi(\mathbf{a}+\mathbf{h}) - \phi(\mathbf{a}) - L\mathbf{h}\|_p}{\|\mathbf{h}\|_p} = 0. \tag{5}$$

In the case $m = n = 1$, this is equivalent to the existence of the limit

$$\lim_{(x,y)\to(a,a), x\neq y} \frac{|\phi(x) - \phi(y)|_p}{|x-y|_p}.$$

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Such a derivative is uniquely determined (if it exists) and we denote it $L\mathbf{a} = D\phi(\mathbf{a})$. Notice that in the p-adic setting this is indeed stronger than the typical notion of differentiability where $L\mathbf{a}$ is involved instead of $L\mathbf{h}$ in the numerator of (5), and one has to be careful about transferring real/complex analysis claims to the p-adic world. See [31, Example 26.6] for a famous example of a function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ with "ordinary" derivative $Df \equiv 1$ which is not injective in any neighborhood of 0. However, with the above notion of strong differentiability most common real analysis facts can be conserved. If we assume our function to be analytic then the inverse function theorem extends to the p-adic settings as claimed in Serre's book [32].

Theorem 3.1 (p-adic Inverse Function Theorem) *Assume the function $\phi : U \subseteq \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ for $\mathbf{0} \in U$ open is analytic. Then if $D\phi(\mathbf{0})$ induces a linear isomorphism, then ϕ is a local isomorphism.*

We remark that when $n = 1$, in fact, the function $\phi/\phi'(0)$ is a local isometry, i.e. $|\phi(x) - \phi(y)|_p/|x - y|_p = |D\phi(a)|_p$ is constant for x, y, $x \neq y$, close enough to a, see [31, Proposition 27.3]. We further point out that Theorem 3.1 is the only reason why we require our parametrising function $\mathbf{g}$ to be analytic, for all other arguments C^2 would suffice.

From the inverse function theorem it can be shown that some mean value estimate similar to the real case holds.

Theorem 3.2 (p-adic Mean Value Theorem) *For ϕ as in Theorem 3.1 and small enough r we have*

$$\phi(B(\mathbf{x}, r)) \subseteq B(\phi(\mathbf{x}), \|D\phi(\mathbf{x})\|_p \cdot r).$$

In other words a ball of radius r (with respect to $\|.\|_p$-norm) around $\mathbf{x} \in \mathbb{Q}_p^n$ is mapped into a ball of radius $\ll_{\mathbf{x}} r$. In fact, the mean value estimate only requires strong differentiability.

3.2. The main lemma

First assume $S_{\mathcal{M}}$ is empty. We prove that we can restrict to the case where $|\det M_{\mathbf{z}}(\mathbf{x})|_p$ is globally bounded away from 0, meaning a uniform bound for all pairs $(\mathbf{x}, \mathbf{z}) \in \mathcal{U} \times \{\mathbf{z} \in \mathbb{Z}_p^{n-d} : \|\mathbf{z}\|_p = 1\}$ exists. For any compact subset $K \subseteq \mathcal{U}$ we have that that $K \times \mathbb{Z}_p^{n-d}$ is a compact subset of $\mathbb{Q}_p^n$. Hence by continuity of $|\det M_{\mathbf{z}}(\mathbf{x})|_p$ as a map $K \times \mathbb{Z}_p^{n-d} \subseteq \mathbb{Q}_p^n \to [0, \infty)$, its image is a compact real interval which by (IIp) is bounded away from 0. If $\mathcal{U}$ is compact, such as any ball in $\mathbb{Z}_p^d$ as they are the clopen (closed and open), then we are done. Otherwise we may write $\mathcal{U}$ as a countable union of open balls $\mathcal{U}_k \subseteq \mathcal{U}$. To see this, since $\mathcal{U}$ is open we can first write $\mathcal{U}$ as a union of open balls $\mathcal{U}_{\mathbf{x}}$ over $\mathbf{x} \in \mathcal{U}$. Then for any $\mathcal{U}_{\mathbf{x}}$ we can find a subball $V_{\mathbf{x}} = V_{\mathbf{c},r}$ with rational center $\mathbf{c} \in \mathbb{Q}^d$ and radius $r \in \mathbb{Q}$ containing $\mathbf{x}$. Now the $V_{\mathbf{x}}$ form a countable open cover of $\mathcal{U}$, which henceforth we identify with the $\mathcal{U}_k$. Since balls are clopen, the $\mathcal{U}_k$ are compact, so we can find a strictly positive bound for $|\det M_{\mathbf{z}}(\mathbf{x})|_p$ in each $\mathcal{U}_k$. Now if the claim of the theorem holds for each of these open balls $\mathcal{U}_k$, then by sigma-additivity (taking

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the union of countably many Hausdorff f-measure 0 exceptional sets from each $\mathcal{U}_k$) the claim holds for $\mathcal{U}$ as well. This proves the claim when $S_{\mathcal{M}}$ is empty. Otherwise, if $S_{\mathcal{M}}$ is non-empty, then by condition (IIp) the closed set $\overline{S}_{\mathcal{M}}$ is of f-measure 0. Applying the above argument to the open set $\mathcal{U} \setminus \overline{S}_{\mathcal{M}}$ and adding the exceptional set $\overline{S}_{\mathcal{M}}$ in the final step (sigma-additivity argument) leads to the same conclusion.

Furthermore we may assume that $\mathcal{U} \subset \mathbb{Z}_p^d$ and $\mathbf{0} \in \mathcal{U}$.

For any $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1}$ consider the set

$$S(\mathbf{a}) := \{\mathbf{x} \in \mathcal{U} : |a_1 x_1 + \dots a_d x_d + a_{d+1} g_1(\mathbf{x}) + \dots + a_n g_{n-d}(\mathbf{x}) + a_0 + \theta|_p < \Psi(\mathbf{a})\}.$$

We prove the following key lemma.

Lemma 3.3. *For any $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{Z} \times (\mathbb{Z}^n \setminus \{\mathbf{0}\})$, $\boldsymbol{g} : \mathcal{U} \to \mathbb{Q}_p^n$ satisfying (IIp), and dimension function f satisfying (Ip) for some $d-1 \le s < 2(d-1)$, we have that*

$$\mathcal{H}^f(S(\mathbf{a})) \ll \|(a_1, \dots, a_n)\|_p^{(d-1)-s} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})),$$

with the implied constant independent of $\mathbf{a}$.

The lower bound $d-1 \le s$ is just for completeness, clearly if (Ip) holds for some s, then also for all larger values of s.

4. Proof of Lemma 3.3

By the argument in § 3.2, we may restrict to a compact subset K (a clopen ball) of $\mathcal{U}$ and assume $S_{\mathcal{M}} = \emptyset$. For $\mathbf{a} \in \mathbb{Z}^{n+1}$ write

$$\mathbf{a} = (a_0, \mathbf{a}_1, \mathbf{a}_2) = (a_0, a_1, \dots, a_d, a_{d+1} \dots, a_n) \in \mathbb{Z} \times \mathbb{Z}^d \times \mathbb{Z}^{n-d},$$

and let

$$\widetilde{a_2} = \begin{cases} 1 & \text{if } \mathbf{a}_2 = \mathbf{0}, \\ \|\mathbf{a}_2\|_p & \text{otherwise.} \end{cases}$$

Define $h_{\mathbf{a}} : \mathbb{Q}_p^d \to \mathbb{Q}_p$ by

$$h_{\mathbf{a}}(\mathbf{x}) = \widetilde{a_2} \mathbf{a}_1 \cdot \mathbf{x} + \widetilde{a_2} \mathbf{a}_2 \cdot \mathbf{g}(\mathbf{x}) + \widetilde{a_2}(a_0 + \theta),$$

where

$$\widetilde{a_2} \mathbf{a}_1 \cdot \mathbf{x} = \widetilde{a_2} a_1 x_1 + \dots + \widetilde{a_2} a_d x_d,$$

and similarly for $\widetilde{a_2} \mathbf{a}_2 \cdot \mathbf{g}(\mathbf{x})$. We may write

$$S(\mathbf{a}) = \left\{\mathbf{x} \in K : |h_{\mathbf{a}}(\mathbf{x})|_p < \widetilde{a_2}^{-1} \Psi(\mathbf{a})\right\}.$$

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Note that

$$\begin{aligned}\nabla h_{\mathbf{a}}(\mathbf{x}) &= \left(\frac{\partial h_{\mathbf{a}}(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial h_{\mathbf{a}}(\mathbf{x})}{\partial x_d}\right), \\ &= \left(\widetilde{a}_2 a_1 + \sum_{k=1}^{n-d} \widetilde{a}_2 a_{d+k} \frac{\partial g_k(\mathbf{x})}{\partial x_1}, \dots, \widetilde{a}_2 a_d + \sum_{k=1}^{n-d} \widetilde{a}_2 a_{d+k} \frac{\partial g_k(\mathbf{x})}{\partial x_d}\right), \\ &= \widetilde{a}_2 \mathbf{a}_1 + \widetilde{a}_2 \mathbf{a}_2 \cdot (\nabla \mathbf{g}(\mathbf{x})),\end{aligned}$$

for

$$\nabla \mathbf{g}(\mathbf{x}) = \left(\frac{\partial \mathbf{g}(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial \mathbf{g}(\mathbf{x})}{\partial x_d}\right) \quad \text{with} \quad \frac{\partial \mathbf{g}(\mathbf{x})}{\partial x_i} = \left(\frac{\partial g_1(\mathbf{x})}{\partial x_i}, \dots, \frac{\partial g_{n-d}(\mathbf{x})}{\partial x_i}\right)^t,$$

a matrix with $n-d$ rows and d columns and

$$\nabla^2 h_{\mathbf{a}}(\mathbf{x}) = \left(\sum_{k=1}^{n-d} \widetilde{a}_2 a_{d+k} \frac{\partial^2 g_k(\mathbf{x})}{\partial x_i \partial x_j}\right)_{1 \le i,j \le d}. \tag{6}$$

Note that if $\mathbf{a}_2 \neq \mathbf{0}$ then

$$\|\widetilde{a}_2 \mathbf{a}_2\|_p = |\widetilde{a}_2|_p \cdot \|\mathbf{a}_2\|_p = |\, \|\mathbf{a}_2\|_p \,|_p \cdot \|\mathbf{a}_2\|_p = \|\mathbf{a}_2\|_p^{-1} \cdot \|\mathbf{a}_2\|_p = 1, \tag{7}$$

where we employed the identity $|\, |y|_p \,|_p = |y|_p^{-1}$ for any $y \in \mathbb{Q}_p$ that is readily checked. So identifying $z_k = \widetilde{a}_2 a_{d+k}$ for $k = 1, 2, \dots, n-d$, by our assumption (IIp) and the comments in § 3.2, $\nabla^2 h_{\mathbf{a}}(\mathbf{x})$ has a non-zero determinant (with p-norm strictly bounded away from 0) on $\{\mathbf{z} \in \mathbb{Z}_p^{n-d} : \|\mathbf{z}\|_p = 1\}$. We next bound $\nabla^2 h_{\mathbf{a}}$ uniformly from above.

Lemma 4.1. *For $\mathbf{a} \in \mathbb{Z}^{n+1}$ and $h_{\mathbf{a}}$ defined as above*

(a) If $\mathbf{a}_2 = \mathbf{0}$ then

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \asymp \|\mathbf{a}_1\|_p \quad \forall\, \mathbf{x} \in K.$$

(b) If $\mathbf{a}_2 \neq \mathbf{0}$ and $\|\mathbf{a}_1\|_p > \sup_{\mathbf{w} \in K} \|\mathbf{a}_2 \cdot \nabla \boldsymbol{g}(\mathbf{w})\|_p$ then

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \asymp \|\mathbf{a}_2\|_p^{-1} \|\mathbf{a}_1\|_p \quad \forall\, \mathbf{x} \in K.$$

(c) If there exists $\mathbf{v} \in K$ with $\nabla h_{\mathbf{a}}(\mathbf{v}) = \mathbf{0}$ then

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \asymp \|\mathbf{x} - \mathbf{v}\|_p \quad \forall\, \mathbf{x} \in K.$$

(d) If no such $\mathbf{v} \in K$ exists, and $\|\mathbf{a}_1\|_p \le \sup_{\mathbf{w} \in K} \|\mathbf{a}_2 \cdot \nabla \boldsymbol{g}(\mathbf{w})\|_p$ then

$$0 < \|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \le \sup_{\mathbf{w} \in K} \|\nabla \boldsymbol{g}(\mathbf{w})\|_p \quad \Longrightarrow \quad \|\nabla h_{\mathbf{a}}(\mathbf{x})\| \asymp 1 \quad \forall\, \mathbf{x} \in K.$$

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In all of the above cases $\|\nabla^2 h_{\mathbf{a}}(\mathbf{x})\|_p \ll 1$ *for all* $\mathbf{x} \in K$.

Proof of Lemma 4.1: Note that if $\mathbf{a}_2 = \mathbf{0}$ immediately we have that

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p = \|\mathbf{a}_1\|_p$$

for all $\mathbf{x} \in K$, giving (a).

If $\|\widetilde{a_2}\mathbf{a}_1\|_p > \sup_{\mathbf{w}\in K} \|\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{w})\|_p$, which follows by the assumption

$$\|\mathbf{a}_1\|_p > \sup_{\mathbf{w}\in K} \|\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{w})\|_p,$$

then by the strong triangle inequality

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p = \max\left\{\|\widetilde{a_2}\mathbf{a}_1\|_p, \|\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{x})\|_p\right\} = \|\widetilde{a_2}\mathbf{a}_1\|_p.$$

Henceforth assume $\|\widetilde{a_2}\mathbf{a}_1\|_p \le \sup_{\mathbf{w}\in K} \|\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{w})\|_p$.

Since $\mathbf{g}$ is analytic, and thus so is each g_i for $1 \le i \le n-d$, each partial derivative $\frac{\partial g_i}{\partial x_j}$ is analytic and so the function $\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g} : K \subset \mathbb{Q}_p^d \to \mathbb{Q}_p^d$ is analytic. Furthermore $\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}$ is strongly differentiable with the linear function $L\mathbf{x} = \nabla^2 h_{\mathbf{a}}(\mathbf{x})$ and non-zero determinant (thus invertible) on some small ball $B_{\mathbf{x}} \subset K$. Hence $\nabla^2 h_{\mathbf{a}}(\mathbf{x})$ is a linear isomorphism and so $\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}$ is a local isomorphism by Theorem 3.1. Hence by Theorem 3.2 for any $\mathbf{y} \in B_{\mathbf{x}}$

$$\|\nabla h_{\mathbf{a}}(\mathbf{x}) - \nabla h_{\mathbf{a}}(\mathbf{y})\|_p = \|\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{x}) - \widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{y})\|_p \asymp \|\mathbf{x} - \mathbf{y}\|_p. \tag{8}$$

The above implied constant depends continuously on $\widetilde{a_2}\mathbf{a}_2$, as clearly $F_A : \mathbf{r} \to \mathbf{r} \cdot A$ for fixed $A \in \mathbb{Q}_p^{(n-d)\times d}$ induces a continuous map $\mathbb{Q}_p^{n-d} \to \mathbb{Q}_p^d$, which we apply with $A = \nabla \mathbf{g}(\mathbf{x}) - \nabla \mathbf{g}(\mathbf{y})$. However, by (7) the numbers $\widetilde{a_2}\mathbf{a}_2 \in \{\mathbf{z} \in \mathbb{Z}^{n-d} : \|\mathbf{z}\|_p = 1\} \subseteq \{\mathbf{z} \in \mathbb{Z}_p^{n-d} : \|\mathbf{z}\|_p = 1\} \subseteq \mathbb{Z}_p^{n-d}$ are contained in the compact set $\mathbb{Z}_p^{n-d}$. So viewing (8) over $\mathbb{Z}_p^n$, uniform bounds for the implied constants exist, which are strictly positive. This settles (b).

Assume as in (c) there exists some $\mathbf{v} \in K$ such that $\nabla h_{\mathbf{a}}(\mathbf{v}) = 0$ (or $\|\nabla h_{\mathbf{a}}(\mathbf{v})\|_p < \|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p$ for all $\mathbf{x} \in K\backslash\{\mathbf{v}\}$)[3]. Then we can apply (8) identifying $\mathbf{v} = \mathbf{y}$ to get in a neighborhood $B_{\mathbf{v}}$ of $\mathbf{v}$ that

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p = \|\nabla h_{\mathbf{a}}(\mathbf{x}) - \nabla h_{\mathbf{a}}(\mathbf{v})\|_p \asymp \|\mathbf{x} - \mathbf{v}\|_p, \qquad \mathbf{x} \in B_{\mathbf{v}}.$$

So the claim holds within $B_{\mathbf{v}}$. In the complement of the sets $B_{\mathbf{v}}$ within K (in fact we may assume $\mathbf{v}$ is unique in K), the analogous argument and estimates as in case (d) carried out below will apply. So we can assume now (d) applies, we have $\nabla h_{\mathbf{a}}(\mathbf{x}) \neq \mathbf{0}$ for

[3] Observe that this case is not possible. To see this note that $\nabla h_{\mathbf{a}}$ is continuous, and so if $\nabla h_{\mathbf{a}}(\mathbf{v}) \neq 0$ there exists a neighbourhood V of $\mathbf{v}$ such that $\|\nabla h_{\mathbf{a}}(\mathbf{v})\|_p = \|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p$ for all $\mathbf{x} \in V$. See [24, Section 4.2]

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all $\mathbf{x} \in K$, and so $\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \gg 1$ uniformly by compactness of K and the continuity of $\nabla h_{\mathbf{a}}$. By the strong triangle inequality, we have that

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \leq \max\{\|\widetilde{a_2}\mathbf{a}_1\|_p, \|\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{x})\|_p\}. \tag{9}$$

By our assumption

$$0 < \|\widetilde{a_2}\mathbf{a}_1\|_p \leq \sup_{\mathbf{x}\in K} \|\widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{x})\|_p \leq \sup_{\mathbf{x}\in K} \max_{1\leq i\leq d} \left\| \frac{\partial \mathbf{g}(\mathbf{x})}{\partial x_i} \right\|_p \ll 1$$

and so by (9) we have that

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \ll 1.$$

Combined with $\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \gg 1$ we have

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \asymp 1.$$

The final claim follows from the compactness of K and the continuity of $\nabla^2 h_{\mathbf{a}} = \nabla^2 \mathbf{g}$.

We prove the following key lemma which is a p-adic analogue of [22, Lemma 2.4].

Lemma 4.2. *Let $\phi : U \subset \mathbb{Q}_p^d \to \mathbb{Q}_p$ be an analytic function, and fix $\alpha > 0$, $\delta > 0$, and $\mathbf{x} \in U$ such that $B_d(\mathbf{x}, \alpha) \subset U$. Assume $\nabla^2\phi$ does not vanish identically, i.e. ϕ is not affine. Then there exists a constant $C > 0$ depending only on d such that if*

$$\|\nabla\phi(\mathbf{x})\|_p \geq C\alpha \sup_{\mathbf{w}\in U} \|\nabla^2\phi(\mathbf{w})\|_p, \tag{10}$$

then

$$S(\phi, \delta) = \{\mathbf{y} \in B_d(\mathbf{x}, \alpha) : |\phi(\mathbf{y})|_p < \|\nabla\phi(\mathbf{x})\|_p \delta\}$$

can be covered by $\asymp (\alpha/\delta)^{d-1}$ balls in $\mathbb{Q}_p^d$ of radius δ.

The assumption of non-vanishing $\nabla^2\phi$ is just for convenience in the proof, as affine ϕ will not occur in our setting. Our proof will slightly deviate from the real case in [22], but we employ the same idea.

Proposition 4.3. *Let $d \in \mathbb{N}$. The invertible matrices form an open subset of $\mathbb{Q}_p^{d\times d}$.*

Proof. As in the real case it is readily checked that A is invertible iff $\det A \neq 0$. So the set of invertible matrices is the preimage of the open set $\mathbb{Q}_p \setminus \{0\}$ under $A \to \det A$ viewed as a continuous map $\mathbb{Q}_p^{d\times d} \to \mathbb{Q}_p$, therefore they form an open set. □

Proposition 4.4. *Let $d \in \mathbb{N}$. If $\Phi : \mathbb{Q}_p^d \to \mathbb{Q}_p^d$ is a Lipschitz map with Lipschitz constant L, and $U \subseteq \mathbb{Q}_p^d$ can be covered by k balls of radius $r > 0$, then $\Phi(U)$ can be covered by $\ll_{d,L} k$ balls of the same radius r.*

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Proof. Let $B_d(\mathbf{x}, r)$ be one of the k balls that cover U. For any $\mathbf{y} \in B_d(\mathbf{x}, r)$ since Φ is Lipschitz we have

$$\|\Phi(\mathbf{x}) - \Phi(\mathbf{y})\|_p \leq L\|\mathbf{x} - \mathbf{y}\|_p \leq Lr,$$

and so $\Phi(B_d(\mathbf{x}, r)) \subseteq B_d(\Phi(\mathbf{x}), Lr)$ and so $\Phi(U)$ can be covered by k balls of radius Lr. We now show the following claim:

For any $\mathbf{x} \in \mathbb{Q}_p^d$ and any $\rho, K > 0$ the p-adic ball $B_d(\mathbf{y}, K\rho)$ can be covered by $\ll \max\{1, K^d\}$ balls of radius $\rho > 0$.

Using this we can deduce that $\Phi(U)$ can be covered by $\ll L^d k$ balls as required. To prove the statement recall by the properties of the ultrametric norm that any two balls are either disjoint or one contains the other, that is, the intersection is either empty or full. If $1 \geq K > 0$ then trivially $B_d(\mathbf{y}, r)$ is a cover of $B_d(\mathbf{y}, K\rho)$ so the number of balls is $\ll 1$. So assume $K > 1$. Let $\{B_i\}$ be a collection of balls of radius ρ that cover $B_d(\mathbf{y}, K\rho)$. We can assume this collection is disjoint and finite by the properties of the ultrametric norm and since $B_d(\mathbf{y}, K\rho)$ is bounded. Furthermore we can assume

$$B_d(\mathbf{y}, K\rho) = \bigcup_i B_i.$$

Now by properties of the p-adic d-dimensional Haar measure $\mu_{p,d}$ we have that

$$(K\rho)^d \asymp \mu_{p,d}\left(B_d(\mathbf{y}, K\rho)\right) = \sum_i \mu_{p,d}\left(B_i\right) \asymp \sum_i \rho^d.$$

Hence the cardinality of the set of balls $\{B_i\}$ is $\asymp K^d$. □

Proof of Lemma 4.2 We proceed as in the proof of [22, Lemma 2.4]. By translation, without loss of generality, we may assume $\mathbf{x} = \mathbf{0}$. For simplicity let $\kappa := \|\nabla\phi(\mathbf{0})\|_p$. We can assume $\kappa > 0$ as otherwise by (10) the function $\nabla^2\phi$ vanishes on the entire open set U, which is prohibited in the lemma. Denote by $\mathbf{e}_d$ the d-th canonical base vector in $\mathbb{Q}_p^d$. We may assume $\nabla\phi(\mathbf{0}) = \kappa\mathbf{e}_d$, we comment on it shortly below. Now consider the map

$$\Phi : \mathscr{B} := B_d(\mathbf{0}, \alpha) \to \mathbb{Q}_p^d$$

defined by the formula

$$\Phi(\mathbf{y}) = (\kappa y_1, \ldots, \kappa y_{d-2}, \kappa y_{d-1}, \phi(\mathbf{y})),$$

for $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{Q}_p^d$. If the condition $\nabla\phi(\mathbf{0}) = \kappa\mathbf{e}_d$ fails, we accordingly alter Φ via a linear transformation (base change) which preserves the conclusion of the theorem upon altering the implied constant, we omit details. We have

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$$\nabla\Phi(\mathbf{0}) = \kappa I_d$$

with I_d the $d \times d$ identity matrix. On the other hand

$$\sup_{\mathbf{w}\in\mathscr{B}} \|\nabla^2\Phi(\mathbf{w})\|_p = \sup_{\mathbf{w}\in\mathscr{B}} \|\nabla^2\phi(\mathbf{w})\|_p \leq \frac{\|\nabla\Phi(\mathbf{0})\|_p}{C\alpha} = \frac{\kappa}{C\alpha}.$$

Denote $B_{d,d}(\kappa I_d, \kappa/(C\alpha))$ the $d \times d$ matrices with distance at most $\kappa/(C\alpha)$ from κI_d, in terms of the maximum norm on $\mathbb{Q}_p^{d\times d}$ (maximum norm $|.|_p$ of entries). Since $\mathscr{B}$ has diameter 2α, it follows from the Mean Value Inequality (Theorem 3.2) applied to the gradient $\nabla\Phi$ that $\nabla\Phi(\mathbf{w}) \in B_{d,d}(\kappa I_d, 2\alpha \cdot \kappa/(C\alpha)) = B_{d,d}(\kappa I_d, 2\kappa/C)$ for all $\mathbf{w} \in \mathscr{B}$. Thus by the Mean Value Inequality applied to Φ, and the strong triangle inequality, for all $\mathbf{y}, \mathbf{w} \in \mathscr{B}$

$$\|\Phi(\mathbf{w}) - \Phi(\mathbf{y})\|_p \leq \max\{\|\kappa I_d\|_p, 2\kappa/C\}\|\mathbf{w} - \mathbf{y}\|_p = \max\{\kappa, 2\kappa/C\}\|\mathbf{w} - \mathbf{y}\|_p.$$

The invertible $d \times d$ matrices form an open subset of $\mathbb{Q}_p^{d\times d}$ by Proposition 4.3 and $\kappa > 0$, so we see from the p-adic Inverse Function Theorem 3.1 that if $C > 2$ is sufficiently large, then $B_{d,d}(\kappa I_d, 2\kappa/C)$ consists of invertible matrices. Hence Φ is bi-Lipschitz with a uniform bi-Lipschitz constant. Note

$$S(\phi, \delta) = \Phi^{-1}\big(B_{d-1}(\mathbf{0}, \kappa\alpha) \times B_1(0, \delta\kappa)\big),$$

and it is clear (via a very special case of Proposition 4.4 for linear maps induced by scalar multiplication) that the latter set $B_{d-1}(\mathbf{0}, \kappa\alpha) \times B_1(0, \delta\kappa)$ can be covered by

$$\asymp \kappa^d(\delta/\alpha)^{-(d-1)} = \kappa^d(\alpha/\delta)^{d-1}$$

balls of radius δ. Since κ is fixed the proof is finished via Proposition 4.4, when we allow the implied constant to depend only on the bi-Lipschitz constant. □

We now use Lemma 4.2 in each of the possible cases outlined by Lemma 4.1 in order to finish the proof of Lemma 3.3.

Case (a)

In this case observe that $S(\mathbf{a})$ is a p-adic $\Psi(\mathbf{a})$-thickened hyperplane (of $\mathbb{Q}_p^d$) so can be covered by $\asymp \Psi(\mathbf{a})^{-(d-1)}$ balls of radius $\Psi(\mathbf{a})$. Hence by $s \geq d-1$ we infer

$$\mathcal{H}^f(S(\mathbf{a})) \ll \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})) \leq \|(a_1, \ldots, a_n)\|_p^{(d-1)-s} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})).$$

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Case (b)

By the assumption and conclusion of the case (b) we have

$$\begin{aligned}\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p &\gg \|\mathbf{a}_2\|_p^{-1}\|\mathbf{a}_1\|_p\\ &\geq \sup_{\mathbf{w}\in K}\|\widetilde{a_2}\mathbf{a}_2\cdot\nabla\mathbf{g}(\mathbf{w})\|_p\\ &\gg_{\mathbf{a}} 1\\ &\gg \sup_{\mathbf{w}\in K}\|\nabla^2 h_{\mathbf{a}}(\mathbf{w})\|_p.\end{aligned}$$

The final inequality is due to the observation from Lemma 4.1 that $\|\nabla^2 h_{\mathbf{a}}(\mathbf{x})\|_p \ll 1$ for all $\mathbf{x}\in K$, and the constant in the penultimate inequality is dependent on

$$\widetilde{a_2}\mathbf{a}_2 \in \{\mathbf{z}\in\mathbb{Z}^{n-d} : \|\mathbf{z}\|_p = 1\} \subseteq \{\mathbf{z}\in\mathbb{Z}_p^{n-d} : \|\mathbf{z}\|_p = 1\},$$

by (7). Since the supremum is taken over a precompact set of $\widetilde{a_2}\mathbf{a}_2$ and the continuous dependence of implied factors above, via viewing the above over the compact set $\mathbb{Z}_p$, we can choose a uniform lower bound. Note that this constant is strictly positive since $\widetilde{a_2}\mathbf{a}_2\cdot\nabla\mathbf{g}(\mathbf{w}) = 0$ for all $\mathbf{w}\in K$ implies $\nabla^2 h_{\mathbf{a}}(\mathbf{w}) = \mathbf{0}$, which we have removed. Thus, for suitably chosen $\epsilon > 0$ dependent on the implied constants in the inequalities above and $C > 0$ appearing in Lemma 4.2, let

$$\alpha = \epsilon, \quad \delta = \frac{\Psi(\mathbf{a})}{\widetilde{a_2}\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p} \asymp \frac{\Psi(\mathbf{a})}{\|\mathbf{a}_1\|_p}, \quad \phi = h_{\mathbf{a}}.$$

Hereby for the equivalence in δ we used that $\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \ll 1$ uniformly as well since $\nabla h_{\mathbf{a}}$ is continuous on the compact set K. Then by Lemma 4.2 and as we can consider ϵ fixed now

$$S(\mathbf{a})\cap B(\mathbf{x},\epsilon),$$

can be covered by

$$\asymp \epsilon^{(d-1)}\|\mathbf{a}_1\|_p^{(d-1)}\Psi(\mathbf{a})^{-(d-1)} \asymp \|\mathbf{a}_1\|_p^{(d-1)}\Psi(\mathbf{a})^{-(d-1)}$$

balls of radius $\asymp \Psi(\mathbf{a})\|\mathbf{a}_1\|_p^{-1}$. Thus

$$\begin{aligned}\mathcal{H}^f(S(\mathbf{a})) &\ll \|\mathbf{a}_1\|_p^{(d-1)}\Psi(\mathbf{a})^{-(d-1)} f\left(\frac{\Psi(\mathbf{a})}{\|\mathbf{a}_1\|_p}\right)\\ &\overset{\text{condition (Ip)}}{\ll} \|\mathbf{a}_1\|_p^{(d-1)-s}\Psi(\mathbf{a})^{-(d-1)} f\left(\Psi(\mathbf{a})\right) \qquad (\text{since } \|\mathbf{a}_1\|_p^{-1}\geq 1).\end{aligned}$$

Since

$$\|\mathbf{a}_2\|_p^{-1}\|\mathbf{a}_1\|_p > \sup_{\mathbf{w}\in K}\|\widetilde{a_2}\mathbf{a}_2\cdot\nabla\mathbf{g}(\mathbf{w})\|_p \gg 1$$

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with the implied constant independent of $\mathbf{a}$ as noticed above, we have $\|\mathbf{a}_1\|_p \gg \max\{\|\mathbf{a}_1\|_p, \|\mathbf{a}_2\|_p\} = \|(\mathbf{a}_1, \mathbf{a}_2)\|_p$ and as the reverse inequality $\|\mathbf{a}_1\|_p \leq \|(\mathbf{a}_1, \mathbf{a}_2)\|_p$ is trivial, we have that $\|\mathbf{a}_1\|_p \asymp \|(\mathbf{a}_1, \mathbf{a}_2)\|_p$. So

$$\mathcal{H}^f(S(\mathbf{a})) \ll \|(\mathbf{a}_1, \mathbf{a}_2)\|_p^{(d-1)-s} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})).$$

Case (c)

Let $\mathbf{v}$ be as in Lemma 4.1, (c). Fix $k \in \mathbb{Z}$ and consider the p-adic annulus

$$A_k = K \cap B_d(\mathbf{v}, p^{-k}) \backslash B_d(\mathbf{v}, p^{-(k+1)}).$$

Note that $\{A_k\}_{k\in\mathbb{Z}}$ partitions $\mathbb{Q}_p^d \backslash \{\mathbf{v}\}$ and since K is bounded there exists some $k_0 \in \mathbb{Z}$ such that

$$\bigcup_{k \geq k_0} A_k \supseteq K \backslash \{\mathbf{v}\}. \tag{11}$$

Observe that $\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \asymp \|\mathbf{x} - \mathbf{v}\|_p \asymp p^{-k}$ for all $\mathbf{x} \in A_k$. Note that by Lemma 4.1

$$\|\nabla^2 h_{\mathbf{a}}(\mathbf{x})\|_p \ll 1, \qquad \mathbf{x} \in A_k.$$

Choose suitable $\epsilon > 0$ such that

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \geq C(\epsilon p^{-k}) \sup_{\mathbf{w}\in K} \|\nabla^2 h_{\mathbf{a}}(\mathbf{w})\|_p,$$

where $C > 0$ comes from Lemma 4.2. Letting

$$\alpha = \epsilon p^{-k}, \quad \delta = \frac{\Psi(\mathbf{a})}{\widetilde{a_2} \|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p} \asymp p^k \frac{\Psi(\mathbf{a})}{\widetilde{a_2}}, \quad \phi = h_{\mathbf{a}},$$

then noticing that $\nabla^2 \phi = \nabla^2 h_{\mathbf{a}}$ is not constant 0 by condition (IIp) as noticed below (6), we may apply Lemma 4.2, which yields

$$S(h_{\mathbf{a}}, \delta) = S(\mathbf{a}) \cap B_d(\mathbf{x}, \epsilon p^{-k})$$

can be covered by

$$\asymp \left(\frac{\alpha}{\delta}\right)^{d-1} \asymp \epsilon^{d-1} \widetilde{a_2}^{d-1} p^{-2k(d-1)} \Psi(\mathbf{a})^{-(d-1)} \asymp \widetilde{a_2}^{d-1} p^{-2k(d-1)} \Psi(\mathbf{a})^{-(d-1)},$$

balls of radius $\asymp p^k \frac{\Psi(\mathbf{a})}{\widetilde{a_2}}$. Observe that $B_d(\mathbf{v}, p^{-k}) \supseteq A_k$ can be covered by p^d disjoint balls of radius $p^{-(k+1)}$. To see this, first take $\widetilde{\mathbf{v}} \in \mathbb{Q}^d$ to be the rational vector obtained from cutting off in any component the Hensel digits from position $k+1$ onwards (which is of the form $\widetilde{\mathbf{v}} = N/p^a$ for some $N \in \mathbb{Z}^d, a \in \mathbb{N}_0$, with $p^a = \|\mathbf{v}\|$, thus $a = 0$ and

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$\widetilde{\mathbf{v}} \in \mathbb{Z}^d$ if $\mathbf{v} \in \mathbb{Z}_p^d$). Then consider the p^d balls of radius $p^{-(k+1)}$ with centres $\widetilde{\mathbf{v}} + tp^{k+1}$ for $t \in \{0,1,\dots,p-1\}^d$. We remark that one of these balls is $B(\mathbf{v}, p^{-(k+1)})$, as $\|\mathbf{v} - \widetilde{\mathbf{v}} + t\| \leq p^{-(k+1)}$ for some $t \in \{0,1,\dots,p-1\}^d$ and in p-adic space if $x \in B(y,r)$ then $B(y,r) = B(x,r)$. So in fact we only require $p^d - 1$ balls to cover A_k. Thus

$$\mathcal{H}^f(S(\mathbf{a}) \cap A_k) \ll (p^d - 1)p^{-2k(d-1)}\widetilde{a_2}^{(d-1)}\Psi(\mathbf{a})^{-(d-1)} f\left(p^k \frac{\Psi(\mathbf{a})}{\widetilde{a_2}}\right).$$

So, by (11),

$$\begin{aligned}\mathcal{H}^f(S(\mathbf{a})) &\ll \widetilde{a_2}^{(d-1)}\Psi(\mathbf{a})^{-(d-1)} \sum_{k \geq k_0} p^{-2k(d-1)} f\left(p^k \frac{\Psi(\mathbf{a})}{\widetilde{a_2}}\right) \\ &\overset{\text{condition (Ip)}}{\ll} \widetilde{a_2}^{(d-1)-s}\Psi(\mathbf{a})^{-(d-1)} f\left(\Psi(\mathbf{a})\right) \sum_{k \geq k_0} p^{-2k(d-1)+ks} \\ &\overset{s<2(d-1)}{\ll} \widetilde{a_2}^{(d-1)-s}\Psi(\mathbf{a})^{-(d-1)} f\left(\Psi(\mathbf{a})\right).\end{aligned}$$

Note that in order for such $\mathbf{v} \in K$ appearing in case (c) to exist we have that

$$\nabla h_{\mathbf{a}}(\mathbf{v}) = 0 \quad \Longrightarrow \quad \widetilde{a_2}\mathbf{a}_1 = \widetilde{a_2}\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{v}).$$

By the strong triangle inequality, we can deduce that

$$\|\mathbf{a}_1\|_p = \|\mathbf{a}_2 \cdot \nabla \mathbf{g}(\mathbf{v})\|_p \leq \|\mathbf{a}_2\|_p \cdot \|\nabla \mathbf{g}(\mathbf{v})\|_p \ll \|\mathbf{a}_2\|_p,$$

and so $\widetilde{a_2} \overset{\text{def}}{=} \|\mathbf{a}_2\|_p \asymp \|(\mathbf{a}_1, \mathbf{a}_2)\|_p$. Hence

$$\mathcal{H}^f(S(\mathbf{a})) \ll \|(\mathbf{a}_1, \mathbf{a}_2)\|_p^{(d-1)-s}\Psi(\mathbf{a})^{-(d-1)} f\left(\Psi(\mathbf{a})\right).$$

Case (d)

Since $\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \asymp 1$ we have that, for suitably chosen $C, \epsilon > 0$,

$$\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p \geq C\epsilon \sup_{\mathbf{w} \in K} \|\nabla^2 h_{\mathbf{a}}(\mathbf{w})\|_p.$$

Applying Lemma 4.2 with

$$\alpha = \epsilon, \quad \delta = \frac{\Psi(\mathbf{a})}{\widetilde{a_2}\|\nabla h_{\mathbf{a}}(\mathbf{x})\|_p} \asymp \frac{\Psi(\mathbf{a})}{\widetilde{a_2}}, \quad \phi = h_{\mathbf{a}}$$

we have that $S(\mathbf{a})$ can be covered by

$$\asymp \epsilon^{(d-1)}\widetilde{a_2}^{(d-1)}\Psi(\mathbf{a})^{-(d-1)} \asymp \widetilde{a_2}^{(d-1)}\Psi(\mathbf{a})^{-(d-1)},$$

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balls of radius $\asymp \frac{\Psi(\mathbf{a})}{\widetilde{a_2}}$. Thus

$$\mathcal{H}^f(S(\mathbf{a})) \ll \widetilde{a_2}^{(d-1)}\Psi(\mathbf{a})^{-(d-1)} f\left(\frac{\Psi(\mathbf{a})}{\widetilde{a_2}}\right) \overset{\text{condition (Ip)}}{\ll} \widetilde{a_2}^{(d-1)-s}\Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})).$$

Note that in this case we have that

$$\|\mathbf{a}_1\|_p \leq \sup_{\mathbf{w}\in K} \|\mathbf{a}_2 \cdot \nabla g(\mathbf{w})\|_p \ll \|\mathbf{a}_2\|_p \overset{\text{def}}{=} \widetilde{a_2}$$

and so $\widetilde{a_2} \asymp \|(\mathbf{a}_1, \mathbf{a}_2)\|_p$. Thus

$$\mathcal{H}^f(S(\mathbf{a})) \ll \|(\mathbf{a}_1, \mathbf{a}_2)\|_p^{(d-1)-s}\Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})).$$

Combining the outcomes of the above four cases, we have that for any $\mathbf{a} = (a_0, a_1, \ldots, a_n) = (a_0, \mathbf{a}_1, \mathbf{a}_2) \in \mathbb{Z} \times (\mathbb{Z}^n \backslash \{\mathbf{0}\})$,

$$\mathcal{H}^f(S(\mathbf{a})) \ll \|(\mathbf{a}_1, \mathbf{a}_2)\|_p^{(d-1)-s}\Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})).$$

The proof of Lemma 3.3 is complete.

5. Completing the proof of Theorems 2.3 and 2.5

Note that for any subset of integers $Z \subseteq \mathbb{Z}^{n+1}\backslash\{\mathbf{0}\}$ we have

$$E = E(Z) = \{\mathbf{x} \in K : (\mathbf{x}, g(\mathbf{x})) \in D^{\theta}_{n,p}(\Psi, Z)\} = \limsup_{\mathbf{a}\in Z} S(\mathbf{a}).$$

The map $G : \mathbf{x} \mapsto (\mathbf{x}, g(\mathbf{x}))$ is bi-Lipschitz since K is bounded, and so

$$\mathcal{H}^f(G(E)) \asymp \mathcal{H}^f(D^{\theta}_{n,p}(\Psi, Z) \cap \{(\mathbf{x}, \mathbf{g}(\mathbf{x})) : \mathbf{x} \in K\}) \asymp \mathcal{H}^f(E).$$

Now, by the Hausdorff-Cantelli Lemma, we have that

$$\mathcal{H}^f(E) = 0 \quad \text{if} \quad \sum_{\mathbf{a}\in Z} \mathcal{H}^f(S(\mathbf{a})) < \infty,$$

and so

$$\mathcal{H}^f(D^{\theta}_{n,p}(\Psi, Z) \cap \mathcal{M}) = 0 \quad \text{if} \quad \sum_{\mathbf{a}\in Z} \mathcal{H}^f(S(\mathbf{a})) < \infty.$$

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By Lemma 3.3 the above summation can be written as

$$\sum_{\mathbf{a}\in\{Z:(\mathbf{a}_1,\mathbf{a}_2)\neq\mathbf{0}\}} \|(\mathbf{a}_1,\mathbf{a}_2)\|_p^{(d-1)-s}\Psi(\mathbf{a})^{-(d-1)}f(\Psi(\mathbf{a})) + \sum_{\mathbf{a}\in\{Z:(\mathbf{a}_1,\mathbf{a}_2)=\mathbf{0}\}} \Psi(\mathbf{a})^{-(d-1)}f(\Psi(\mathbf{a})). \tag{12}$$

Since

$$\sum_{\mathbf{a}\in Z} \Psi(\mathbf{a})^{-(d-1)}f(\Psi(\mathbf{a})) < \infty$$

by assumption (in whichever case of Z is chosen) the second summation is convergent, so it remains to prove convergence of the first sum. This summation can be rewritten as

$$\sum_{k\in\mathbb{N}_0}\sum_{\mathbf{a}\in Z(p,k)} p^{k(s-(d-1))}\Psi(\mathbf{a})^{-(d-1)}f(\Psi(\mathbf{a})),$$

where

$$Z(p,k) = \{(a_0,\dots,a_n)\in Z : p^k|a_i\,\forall\, 1\le i\le n\ \&\ \exists\, 1\le j\le n\ p^{k+1}\not| a_j\}.$$

Note that the index 0 is excluded from the divisibility conditions, so in other words

$$Z(p,k) = \{(a_0,\dots,a_n)\in Z : \|(\mathbf{a}_1,\mathbf{a}_2)\|_p = p^{-k}\}.$$

We will see that we can restrict ourselves to finite partial sums in all three cases. In the following arguments we will frequently implicitly use the well-known fact that

$$|a+b|_p = \max\{|a|_p,|b|_p\}, \qquad \text{if } |a|_p\neq|b|_p.$$

Now let us consider the three cases for the set Z occurring in our theorems one by one:

(i) $Z = Z(1)$ (pairwise coprime) and inhomogeneous: Then $\|(\mathbf{a}_1,\mathbf{a}_2)\|_p = 1$, and so (12) becomes the usual summation as in (i), which is convergent by assumption.

(ii) $Z = Z(2)$ (coprime): If $p|a_0$ then there exists some $1\le i\le n$ such that $p\not| a_i$, and so $\|(\mathbf{a}_1,\mathbf{a}_2)\|_p = 1$. Thus convergence of the summation is immediate by assumption. Henceforth assume $|a_0|_p = 1$. We may assume there exists $\ell\in\mathbb{N}_0$ such that

$$\|(\mathbf{x},\mathbf{g}(\mathbf{x}))\|_p \le p^{\ell} \quad \forall\, \mathbf{x}\in K, \tag{13}$$

since this is true for any compact subset of K and by sigma-additivity of measures. First assume $\theta = 0$. The strong triangle inequality, and the fact that $\Psi(\mathbf{a}) < 1$ for

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sufficiently large $\|\mathbf{a}\|$, implies that for $S(\mathbf{a})$ to be non-empty we at least require

$$|\mathbf{a}_1 \cdot \mathbf{x} + \mathbf{a}_2 \cdot \mathbf{g}(\mathbf{x})|_p = |a_0|_p = 1.$$

If $\|(\mathbf{a}_1, \mathbf{a}_2)\|_p \leq p^{-\ell-1}$ then this cannot be true for any $\mathbf{x} \in K$ since

$$|\mathbf{a}_1 \cdot \mathbf{x} + \mathbf{a}_2 \cdot \mathbf{g}(\mathbf{x})|_p \leq \|(\mathbf{a}_1, \mathbf{a}_2)\|_p \cdot \|(\mathbf{x}, \mathbf{g}(\mathbf{x}))\|_p \leq p^{-1} < 1$$

so we must have that $\|(\mathbf{a}_1, \mathbf{a}_2)\|_p > p^{-\ell-1}$, hence $\|(\mathbf{a}_1, \mathbf{a}_2)\|_p \geq p^{-\ell}$. Thus, in the cover of E we only need to consider $S(\mathbf{a})$ with $\|(\mathbf{a}_1, \mathbf{a}_2)\|_p = p^{-k}$ for $k = 0, \ldots, \ell$. That is, we only need to show convergence of the summation

$$\sum_{k=0}^{\ell} \sum_{\mathbf{a} \in Z(p,k)} p^{k(s-(d-1))} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})). \tag{14}$$

Note that for each $k = 0, \ldots, \ell$ the inner sum is convergent (by assumption), so since the outer sum is finite we indeed have convergence.

Now assume $\theta \neq 0$ and $|\theta|_p \neq 1$. Let $p^y := |\theta|_p$ for an integer $y < 0$ (note we assume $\theta \in \mathbb{Q}_p$. We can assume (13) so that if $\|(\mathbf{a}_1, \mathbf{a}_2)\|_p \leq p^{-\ell-1}$ we have

$$\|(\mathbf{a}_1, \mathbf{a}_2) \cdot (\mathbf{x}, \mathbf{g}(\mathbf{x})) + a_0 + \theta\|_p \leq \max\{\|(\mathbf{a}_1, \mathbf{a}_2)\|_p \cdot \|(\mathbf{x}, \mathbf{g}(\mathbf{x}))\|_p, |a_0|_p, |\theta|_p\} \tag{15}$$

and there is in fact equality. To see this observe that

$$\|(\mathbf{a}_1, \mathbf{a}_2)\|_p \cdot \|(\mathbf{x}, \mathbf{g}(\mathbf{x}))\|_p \leq p^{-\ell-1} p^{\ell} = p^{-1} < 1,$$

moreover $p^y < 1$ as we noticed $y < 0$, and $|a_0|_p = 1$. Hence

$$\|(\mathbf{a}_1, \mathbf{a}_2) \cdot (\mathbf{x}, \mathbf{g}(\mathbf{x})) + a_0 + \theta\|_p = \max\{1, p^y\} = 1,$$

is absolutely bounded from below uniformly on K. Since $\Psi \to 0$ there are only finitely many solutions (if any) $\mathbf{a}$ to $\|(\mathbf{a}_1, \mathbf{a}_2) \cdot (\mathbf{x}, \mathbf{g}(\mathbf{x})) + a_0 + \theta\|_p < \Psi(\mathbf{a})$. This argument again shows we may restrict to $\|(\mathbf{a}_1, \mathbf{a}_2)\|_p \geq p^{-\ell}$ leading to a finite sum

$$\sum_{k=0}^{\ell} \sum_{\mathbf{a} \in Z(p,k)} p^{k(s-(d-1))} \Psi(\mathbf{a})^{-(d-1)} f(\Psi(\mathbf{a})),$$

which converges for similar reasons as the sum for $\theta = 0$. Theorem 2.5 is proved.

(iii) $Z = \mathbb{Z}^{n+1} \backslash \{\mathbf{0}\}$, $\theta = 0$, and Ψ satisfies $p\Psi(p\mathbf{a}) \leq \Psi(\mathbf{a})$: Again, we may assume that there exists $\ell \in \mathbb{N}$ such that (13) holds. Suppose that $0 \neq \|(\mathbf{a}_1, \mathbf{a}_2)\|_p \leq p^{-\ell-1}$. We claim that in order for $S(\mathbf{a})$ to be non-empty we need $p|a_0$. From this we will deduce that $S(p^{-1}\mathbf{a}) \supseteq S(\mathbf{a})$.

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By (13) we have

$$\begin{aligned}|\mathbf{a}_1\cdot\mathbf{x}+\mathbf{a}_2\cdot\mathbf{g}(\mathbf{x})+a_0|_p &\le \max\{|\mathbf{a}_1\cdot\mathbf{x}+\mathbf{a}_2\cdot\mathbf{g}(\mathbf{x})|_p,|a_0|_p\}\\ &\le \max\{\|(\mathbf{a}_1,\mathbf{a}_2)\|_p\cdot\|(\mathbf{x},\boldsymbol{g}(\mathbf{x}))\|_p,|a_0|_p\}\\ &\le \max\{p^{-\ell-1}p^{\ell},|a_0|_p\}\\ &\le \max\{p^{-1},|a_0|_p\}.\end{aligned}$$

There is equality in the above inequalities (apart from possibly the third) if $|a_0|_p = 1 > p^{-1}$, leading us in any case to conclude that $|\mathbf{a}_1\cdot\mathbf{x}+\mathbf{a}_2\cdot\mathbf{g}(\mathbf{x})+a_0|_p = 1 < \Psi(\mathbf{a})$ which cannot be true for sufficiently large $\|\mathbf{a}\|$. Thus $|a_0|_p < 1$ and so $p|a_0$. Hence $p^{-1}\mathbf{a} = (\mathbf{a}_1', \mathbf{a}_2', a_0') \in \mathbb{Z}^{n+1}\backslash\{\mathbf{0}\}$ and if $\mathbf{x}\in S(\mathbf{a})$ then

$$|\mathbf{a}_1'\cdot\mathbf{x}+\mathbf{a}_2'\cdot\mathbf{g}(\mathbf{x})+a_0'|_p < p\Psi(\mathbf{a}) \le \Psi(p^{-1}\mathbf{a})$$

thus $\mathbf{x}\in S(p^{-1}\mathbf{a})$, and so $S(\mathbf{a})\subseteq S(p^{-1}\mathbf{a})$. Since $D_{n,p}(\Psi)$ is defined as the limsup set of $S(\mathbf{a})$ over integer vectors $\mathbf{a}$, this argument implies that we only need to consider integer points $\mathbf{a}$ with $\|(\mathbf{a}_1,\mathbf{a}_2)\|_p \ge p^{-\ell}$. This again leads us to the convergent sum (14).

Lastly, if $(\mathbf{a}_1,\mathbf{a}_2)=\mathbf{0}$ then for $S(\mathbf{a})$ to be non-empty we have that

$$|a_0|_p < \Psi(\mathbf{a}) < |a_0|^{-1},$$

which cannot be true, so in this case $S(\mathbf{a})=\emptyset$. Theorem 2.3 is proved as well.

6. Final remarks on condition (IIp)

Obstruction 2 of [20, Section 2.2] still holds by an analogous proof (any row of $M_{\mathbf{z}}(\mathbf{x})$ is annihilated on a non-trivial subspace of $\mathbf{z}$ in $\mathbb{Q}_p^{n-d}$), which stated that we require $n\ge 2(n-d)$ or $d\ge n/2$ for (IIp) to hold for some manifold. On the other hand, the argument for Obstruction 1, which stated d must be even, does not work for the p-adic case. However other algebraic restrictions occur. In particular, some of the examples for (IIp) from [20, Section 3] do not work in the p-adic case. The reason is roughly speaking that, in contrast to the real case, for a multi-variate polynomial to vanish over $\mathbb{Q}_p$ only at the origin, it is insufficient to write it as a sum of squares. Indeed the problems get a more algebraic flavor rather than analytic flavor about zeros of forms in several variables, for $d=2$ (quadratic forms) this is related to isotropic vectors in $\mathbb{Q}_p$ (and $\mathbb{F}_p$). We do not intend to go in as much detail as in [20], but only provide examples of a p-adic manifold satisfying (IIp) for the special case $n-d=2$ with $n\ge 4, d\ge 2$ even. First assume $d=2, n=4$. Let

$$\mathbf{g}(x_1,x_2)=\left(\frac{x_1^2+px_2^2}{2},\ x_1x_2\right),$$

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so that

$$M_{\mathbf{z}}(\mathbf{x}) = M_{\mathbf{z}} = \begin{pmatrix} z_1 & z_2 \\ z_2 & pz_1 \end{pmatrix}.$$

Then $\det M_{\mathbf{z}} = pz_1^2 - z_2^2$. We claim this is non-zero for any $(z_1, z_2) \in \mathbb{Q}_p^2 \setminus \{(0,0)\}$. Firstly if $pz_1^2 = z_2^2$ then $|pz_1^2|_p = |z_2^2|_p$. Recall p-norms only take the discrete values $p^k, k \in \mathbb{Z}$, and 0. Now if $z_1 \neq 0$ then $|pz_1^2|_p = |p|_p \cdot |z_1|_p^2 = p^{2t-1}$ for some $t \in \mathbb{Z}$. Similarly if $z_2 \neq 0$ then $|z_2^2|_p = |z_2|_p^2 = p^{2\ell}$ for some $\ell \in \mathbb{Z}$. Consequently the identity $|pz_1^2|_p = |z_2^2|_p$ can only hold if $z_1 = z_2 = 0$. So condition (IIp) applies. An extension of this example analogously to [20, Example 3.2] (see also [20, Proposition 2.2 (ii)]) to manifolds for general pairs $n = d + 2$ with even $d \geq 2$ works similarly by taking

$$\mathbf{g}(x_1, \ldots, x_d) = \left(\frac{x_1^2 + px_2^2}{2} + \cdots + \frac{x_{d-1}^2 + px_d^2}{2} \, , \, x_1x_2 + \cdots + x_{d-1}x_d \right).$$

Then again $M_{\mathbf{z}}$ consists of 2×2 diagonal blocks as above so $\det M_{\mathbf{z}} = (pz_1^2 - z_2^2)^{d/2} \neq 0$.

Remark 6.1. It is well-known that any quadratic forms over $\mathbb{Q}_p$, see [15] in at least 5 variables represents 0 non-trivially, see Cassels' book [15]. Since $\det M_{\mathbf{z}}(\mathbf{x})$ is a form of degree d in $n-d$ variables $z_1, \ldots, z_{n-d}$ at any point $\mathbf{x}$, this means that when $d = 2, n \geq 7$ we cannot find a manifold satisfying (IIp). However, this is already covered by Obstruction 2 recalled above, which does not permit even $n \geq 5$ when $d = 2$. For larger d similar restrictions are expected.

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