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Generalized maximal surfaces in Lorentz–Minkowski space $\mathbb{L}^3$

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0. *Introduction*

In this paper we carry out a systematic study of generalized maximal surfaces in Lorentz–Minkowski space $\mathbb{L}^3$, with emphasis on their branch points. Roughly speaking, such a surface is given by a conformal mapping from a Riemann surface S in $\mathbb{L}^3$. In the last years, several authors [**1**, **2**, **5**, **6**] have dealt with regular maximal surfaces in $\mathbb{L}^3$, i.e. with isometric immersions, with zero mean curvature, of Riemannian 2-manifolds M in $\mathbb{L}^3$. So, the term 'regular' means free of branch points. As in the minimal case, a conformal structure is naturally induced on M, which becomes a Riemann surface S. The corresponding isometric immersion is then conformal on S, and it does not have any singular points on S (i.e. points on which the differential of the mapping is not one-to-one). This is the way in which generalized maximal surfaces include regular ones. Moreover, branch points are the singular points of the conformal mapping on S. Whereas branch points of generalized minimal surfaces are isolated, we shall show in Section 2 that, in addition to isolated branch points, a generalized maximal surface in $\mathbb{L}^3$ may have non-isolated ones, in fact they constitute a 1-dimensional submanifold in a certain open subset of S (see Section 2). So our purpose is two-fold, firstly to study and explain in detail the branch points, and secondly to state several geometric results involving prescribed behaviour of those points on the surface.

The content of this paper is organized as follows. In Section 1 we first adopt a more technical definition of a generalized maximal surface, but equivalent to the previous one, of course, tuning in with the classical notion of a generalized minimal surface in 3-dimensional Euclidean space (see [**7**]). We also recall in Lemma 1·1 the Enneper–Weierstrass representation. Using this, branch points are divided into two classes A and B, with non-empty intersection in general. B consists of all zeros of a certain holomorphic 1-form ω, and A consists of the points p such that $|g(p)| = 1$, where g is a meromorphic function obtained in Lemma 1·1. The Gauss map is next introduced as a mapping from the complement of A in the surface, in the two-sheet hyperbolic plane in $\mathbb{L}^3$, and it is represented by the meromorphic function g. Section 3 is devoted to constructing several examples of generalized maximal surfaces. Some open subsets of them were already known as regular maximal surfaces (see [**5**, **6**]). We exhibit here the whole surfaces, giving the branch points in each case. Moreover, in

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Example 3·2 we construct two correspondences between generalized maximal surfaces and generalized minimal ones, explaining their geometric meaning. These correspondences are important tools in the proof of Theorem 4·4. We prove, in Section 4, that the only complete generalized maximal surface with a finite number of branch points is the plane. This is an extension of a known result in the regular case [**1**, **5**]. A counter-example is then given to show that our assumption on branch points cannot be removed. In Section 5 we study the behaviour of the Gauss curvature near a branch point, obtaining in Theorem 5·1 the non-existence of non-isolated branch points from a certain boundedness condition on the Gauss curvature. Finally, we also study in Section 5 the total curvature of these surfaces proving in Theorem 5·2 that the only complete generalized maximal surface with $B = \varnothing$ and finite total curvature is the plane.

1. *Preliminaries*

Let us denote by (x_1, x_2, x_3) the usual coordinate system of the Lorentz–Minkowski space $\mathbb{L}^3$ and by $ds_L^2 = dx_1^2 + dx_2^2 - dx_3^2$ the Lorentz metric of $\mathbb{L}^3$. Let S be a Riemann surface and $X = (X_1, X_2, X_3): S \to \mathbb{L}^3$ a non-constant harmonic mapping. Let us suppose that at any point p of S there exists a (complex) coordinate neighbourhood (U, z) such that the complexified derivatives $\phi_i = \partial X_i/\partial z$, $i = 1, 2, 3$ satisfy

$$\phi_1^2 + \phi_2^2 - \phi_3^2 = 0, \tag{1·1}$$

$$|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 \not\equiv 0 \quad \text{(not identically zero)}. \tag{1·2}$$

Then (X, S) is said to be a generalized maximal surface. Note that the functions X_k are harmonic functions on S, hence S cannot be compact.

On each neighbourhood (U, z), $z = u_1 + iu_2$, we have $\phi_k = (\partial X_k/\partial u_1) - i(\partial X_k/\partial u_2)$ and if we denote by $\langle\,,\rangle$ the induced metric on S, the components g_{jq} of $\langle\,,\rangle$ are obtained as $g_{jq} = \langle \partial X/\partial u_j, \partial X/\partial u_q \rangle$, $1 \leqslant j, q \leqslant 2$. Now condition (1·1) is expressed as $g_{11} - g_{22} - 2ig_{12} = 0$ and hence $g_{jq} = \lambda^2 . \delta_{jq}$, where

$$\lambda^2 = \langle \partial X/\partial u_i, \partial X/\partial u_i \rangle = \tfrac{1}{2}(|\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2) \quad (i = 1, 2), \tag{1·3}$$

and so u_1, u_2 are isothermal parameters on U.

LEMMA 1·1. *Let ω be a holomorphic 1-form on S and g a meromorphic function on S such that $|g| \not\equiv 1$ everywhere and the poles of g with order m are also zeros of ω with order at least $2m$. Assume that the holomorphic $1-$forms*

$$\alpha_1 = \tfrac{1}{2}i(1 - g^2)\,\omega, \quad \alpha_2 = \tfrac{1}{2}(1 + g^2)\,\omega, \quad \alpha_3 = g\omega \tag{1·4}$$

on S have no real periods. If $X: S \to \mathbb{L}^3$ is the mapping given by

$$X_k = \operatorname{Re} \int \alpha_k \quad (k = 1, 2, 3), \tag{1·5}$$

then (X, S) is a generalized maximal surface.

Conversely, let (X, S) be a generalized maximal surface. Then either $X(S)$ is contained in the plane $x_3 =$ constant or X is of the form (1·5) where $\alpha_1, \alpha_2, \alpha_3$ are given by (1·4).

On the other hand, since S cannot be compact, from the Uniformization Theorem, the universal covering of S is conformally equivalent either to the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$. If (X, S) is any generalized maximal surface, then we can write

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$\omega = fdz$ locally, f being a holomorphic function defined on a simply connected open subset of S. From (1·3) and (1·4) we have

$$ds^2 = \lambda^2|dz|^2, \quad \text{with} \quad \lambda = \tfrac{1}{2}|f||1-|g|^2| \tag{1·6}$$

locally. It should be noted that the condition on $|g|$ assumed in Lemma 1·1 is equivalent to condition (1·2) of the previous definition.

2. *Branch points*

A point of S on which the equation $|\phi_1|^2+|\phi_2|^2-|\phi_3|^2 = 0$ is satisfied is called a branch point of (X, S). A point which is not a branch point is called a regular point of (X, S). If the subset of S consisting of branch points is the empty set, (X, S) is called a regular maximal surface.

From (1·3) and (1·6) the function $|\phi_1|^2+|\phi_2|^2-|\phi_3|^2$ is non-negative and the induced metric on S is positive definite at every regular point and zero at every branch point. Moreover, the complement of the subset consisting of branch points in S is, via X, a space-like surface in $\mathbb{L}^3$. Then, a generalized maximal surface is space-like (at regular points). We recall that a regular maximal surface is a space-like surface of vanishing mean curvature (see [**5**]), and the mean curvature of a generalized maximal surface is zero at regular points.

The expression (1·6) permits us to classify branch points into two types: the points where $\partial X_1/\partial z = \partial X_2/\partial z = \partial X_3/\partial z = 0$ (this set of branch points will be denoted by B); and the set consisting of the points p of S such that $|g(p)| = 1$ (denoted by A). Note that $A \cap B \neq \varnothing$ in general. Branch points in $B\backslash(A \cap B)$ are isolated. Nevertheless, observe that from the Open Mapping Theorem branch points of A cannot be isolated. On the other hand, 1 is a regular value of the function $p \mapsto |g(p)|^2$ defined on $S\backslash\{\text{critical points of } g\}$. Therefore from the Implicit Function Theorem, $A\backslash\{\text{critical points of } g\}$ is a 1-dimensional embedded submanifold of $S\backslash\{\text{critical points of } g\}$ (and hence of S), which is not necessarily connected. This is an essential difference from generalized minimal surfaces in the Euclidean space $\mathbb{R}^3$. Recall that branch points were isolated points in this case (see [**7**], p. 47). In Section 3 we will construct several examples of generalized maximal surfaces such that the corresponding set A of branch points is not empty.

Let (X, S) be a generalized maximal surface with $X = (X_1, X_2, X_3)$, p a point of S and (U, z) a coordinate neighbourhood in S with $p \in U$. It is clear that X is not regular at the point p (i.e. the differential of X at the point p is not a $1-1$ map) if and only if $\partial(X_i, X_j)/\partial(u, v) = 0$, for $1 \leqslant i < j \leqslant 3$. On the other hand, it is easy to see from (1·3) that this happens if and only if $(|f|(|g|^4-1))(p) = 0$. Hence we conclude that branch points of (X, S) are exactly those points of S where X is not regular.

Consider now the non-connected space-like surface H^2 in the Lorentz–Minkowski space $\mathbb{L}^3$ given by $H^2 = \{(x_1, x_2, x_3) \in \mathbb{L}^3 : x_1^2+x_2^2-x_3^2 = -1\}$. The Gauss map N of a generalized maximal surface (X, S) is defined by assigning to each point p of $S\backslash(A \cup B)$ a unit time-like normal vector at p in a differentiable way. By using the Enneper–Weierstrass representation stated in Lemma 1·1, the Gauss map may be expressed as follows

$$N = (1/(|g|^2-1))(2\,\mathrm{Im}\{g\}, 2\,\mathrm{Re}\{g\}, |g|^2+1). \tag{2·1}$$

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Clearly N extends analytically to each isolated branch point. Let $\Pi : H^2\backslash\{(0,0,1)\} \to \mathbb{C}$ be the stereographic projection from the point $(0,0,1)$. From (2·1) we obtain $\Pi \circ N = -(\mathrm{Im}\{g\}, \mathrm{Re}\{g\})$ on the complement of poles of g in $S\backslash A$. If we extend Π to H^2 by putting $\Pi(0,0,1) = \infty$, then a mapping $\tilde{\Pi} : H^2 \to \hat{\mathbb{C}}$ is obtained, where $\hat{\mathbb{C}}$ represents the Riemann sphere $\mathbb{C} \cup \{\infty\}$, in such a way that $\tau \circ \tilde{\Pi} \circ N = g_{|S-A}$, the map $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ being given by $\tau(x_1, x_2) = -(x_2, x_1)$. Therefore the restriction of the meromorphic function g to $S\backslash A$ can be geometrically interpreted as the Gauss map of (X, S). If (X, S) is regular, then either $|g| < 1$ everywhere or $|g| > 1$ everywhere, and the image of N is contained in exactly one of the two connected components of H^2.

The Gauss curvature K of a generalized maximal surface is given by $K(p) = -\det(dN_p)$ at each regular point p of S. From (1·3) it is easy to obtain

$$K = 16|g'|^2/(|f|(|g|^2-1)^2)^2 \tag{2·2}$$

on the corresponding coordinate neighbourhood. Accordingly, the Gauss curvature of a generalized maximal surface is non-negative, and unless the surface is a plane, it can only have isolated zeros.

A generalized maximal surface is said to be complete if it satisfies the following two conditions:

(i) the set of branch points A is compact;

(ii) $\int_c \lambda |dz| = \infty$, for every divergent curve c in S.

Note that if A is not compact, there could exist divergent curves in S, contained in A, on which condition (ii) is not satisfied (in fact curves of zero length).

3. *Examples*

1. Let $(\tilde{X}, S)$ be a generalized minimal surface in the Euclidean space $\mathbb{R}^3$ with (classical) Enneper–Weierstrass representation (ω, g) such that $|g|$ is not identically equal to 1. If the holomorphic 1-form $i\omega g$ has no real periods on S, then $(ig^2\omega, 1/g)$ is the Enneper–Weierstrass representation of a generalized maximal surface (X, S). Moreover, if $\tilde{X} = (\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ and $X = (X_1, X_2, X_3)$, then from (1·5) we have $X_k = \tilde{X}_k$ for $k = 1, 2$, and $\tilde{X}_3 - iX_3$ is a holomorphic function globally defined on S. On the other hand, given such a $(\tilde{X}, S)$ as above, if the holomorphic 1-forms $i(1-g^2)\omega$ and $(1+g^2)\omega$ have no real periods on S, then (ω, g) defines a generalized minimal surface (Y, S), $Y = (Y_1, Y_2, Y_3)$, and this case we have $Y_3 = \tilde{X}_3$. It is clear that two generalized minimal surfaces can be also constructed from each given generalized maximal surface. Moreover, these constructions are the converse ones of those given. A generalized maximal surface can be obtained from the universal covering surface of each generalized minimal surface by using one of the two constructions. Thus two correspondences are obtained between the class consisting of all generalized minimal surfaces and the class of all generalized maximal surfaces. This permits us to transfer some geometric properties from one family of surfaces to the other one. In this way, it is easy to see that *if T represents any of the two previous correspondences, then (X, S) is a plane if and only if $T(X, S)$ is a plane.*

2. Put $S = \mathbb{C}$ and let $X : \mathbb{C} \to \mathbb{L}^3$ be given by $X(u, v) = (u, v, 0)$. $(X, \mathbb{C})$ is clearly a regular maximal surface and $X(\mathbb{C})$ is the space-like plane $x_3 = 0$ in $\mathbb{L}^3$. Observe that $(X, \mathbb{C})$ is also a regular minimal surface in the Euclidean space $\mathbb{R}^3$ which is a space-like surface when considered as a mapping into $\mathbb{L}^3$. Conversely we have:

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PROPOSITION 3·1. *The plane is the only complete generalized minimal surface in* $\mathbb{R}^3$, *with a finite number of branch points, which is space-like when considered as a surface in* $\mathbb{L}^3$.

Proof. Let $\bar{N} = (\bar{N}_1, \bar{N}_2, \bar{N}_3)$ be the Euclidean normal vector field of the minimal surface S in $\mathbb{R}^3$ and let N_L be the normal vector field of the surface in $\mathbb{L}^3$. It is easy to see that $N_L = (\bar{N}_1, \bar{N}_2, -\bar{N}_3)$, then $\langle N_L, N_L \rangle = \bar{N}_1^2 + \bar{N}_2^2 - \bar{N}_3^2$. From this and our assumption, the meromorphic mapping g associated to the generalized minimal surface satisfies $-|g|^4 + 6|g|^2 - 1 < 0$. Thus the Gauss map of this generalized minimal surface has non-dense image in the Riemann sphere. By using [**8**], proposition 1·3, we have that g is a constant function. This completes the proof.

In this result the assumption on the branch set cannot be omitted. In fact, there exist complete generalized minimal surfaces with infinitely many branch points, non-flat and space-like in $\mathbb{L}^3$ (see for instance [**7**], p. 73, choosing $0 < \epsilon < 1/\sqrt{6}$).

3. (*a*) *Maximal Enneper surface.* Put $S = \mathbb{C}$, $\omega = dz$ and $g(z) = z$. The holomorphic 1-forms

$$\alpha_1 = \tfrac{1}{2}i(1 - z^2)\,dz, \quad \alpha_2 = \tfrac{1}{2}(1 + z^2)\,dz, \quad \alpha_3 = z\,dz,$$

define the maximal Enneper surface $(X, \mathbb{C})$ with

$$X(u, v) = (u^2 v - v - \tfrac{1}{3}v^3, -uv^2 + u + \tfrac{1}{3}u^3, u^2 - v^2) \tag{3·1}$$

for any (u, v) in $\mathbb{R}^2$. The set of branch points of (X, S) is $\{(u, v) : u^2 + v^2 = 1\}$. It is easy to see that $(X, \mathbb{C})$ is complete.

(*b*) *Maximal helicoid surface.* Consider the holomorphic 1-forms

$$\alpha_1 = \tfrac{1}{2}ie^{-z}(1 + e^{2z})\,dz, \quad \alpha_2 = \tfrac{1}{2}e^{-z}(1 - e^{-2z})\,dz, \quad \alpha_3 = -i\,dz,$$

defined on $\mathbb{C}$. Thus $(X, \mathbb{C})$ is a generalized maximal surface with

$$X(u, v) = (\sin v \operatorname{ch} u, \cos v \operatorname{ch} u, v) \tag{3·2}$$

for any $(u, v) \in \mathbb{R}^2$. The set of branch points of the surface is $\{(u, v) : u = 0\}$ and the image of them via X is the null helix $\{(\sin v, \cos v, v) : v \in \mathbb{R}\}$ in $\mathbb{L}^3$. Note that $X(\mathbb{C})$ is a boundary subset of the classical helicoid of $\mathbb{R}^3$. Since A is not compact, this surface is not complete.

(*c*) *Maximal catenoid surface.* Let us consider $S = \mathbb{C}\backslash\{0\}$, $\omega = (1/z^2)\,dz$, $g(z) = z$. The holomorphic 1-forms

$$\alpha_1 = \tfrac{1}{2}i(1/z^2 - 1)\,dz, \quad \alpha_2 = \tfrac{1}{2}(1/z^2 + 1)\,dz, \quad \alpha_3 = (1/z)\,dz$$

have no real periods on $\mathbb{C}\backslash\{0\}$. Therefore they define a generalized maximal surface $(X, \mathbb{C}\backslash\{0\})$ with

$$X(u, v) = (\tfrac{1}{2}v(1 - 1/(u^2 + v^2)), \tfrac{1}{2}u(1 - 1/(u^2 + v^2)), \tfrac{1}{2}\log(u^2 + v^2)) \tag{3·3}$$

for all $(u, v) \in \mathbb{R}^2\backslash\{(0, 0)\}$. The branch point set is $\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1\}$ and its image under X is the point $(0, 0, 0)$. This surface is complete: let $\gamma : [0, b) \to \mathbb{C}\backslash\{0\}$ be a divergent curve with endpoint 0 and suppose that $|\gamma'(t)| = 1$ for all t. We have

$$L(\gamma) = \int_0^b \frac{|1 - |\gamma(t)|^2|}{2|\gamma(t)|^2}\,dt \geqslant \int_a^b \frac{dt}{4|\gamma(t)|^2} = \infty,$$

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where $0 < a < b$. If γ has endpoint ∞, reasoning with the divergent curve $\gamma_1 = 1/\gamma$ with endpoint 0 proves that $L(\gamma) = \infty$. On the other hand, if we put $\rho = \frac{1}{2}\log(u^2+v^2)$ and $\theta = \operatorname{arctg}(v/u)+\pi$, then, after a translation if necessary, from (3·3) we can write

$$X_1(u,v) = -\sin\theta \operatorname{sh}\rho, \quad X_2(u,v) = -\cos\theta \operatorname{sh}\rho, \quad X_3(u,v) = \rho,$$

and hence $X(\mathbb{C}\backslash\{0\})$ coincides with $\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2 = \frac{1}{2}(\operatorname{ch} 2x_3 - 1)\}$.

(*d*) *Maximal Scherk surface.* Consider on $S = \mathbb{C}\backslash\{1,-1,i,-i\}$ the holomorphic 1-forms

$$\alpha_1 = (2i/(1+z^2))\,dz, \quad \alpha_2 = (2/(1-z^2))\,dz, \quad \alpha_3 = (4z/(1-z^4))\,dz.$$

These 1-forms have no real periods and define a generalized maximal surface (X,S) where

$$X(u,v) = \left(\log\left|\frac{z-i}{z+i}\right|, \log\left|\frac{z+1}{z-1}\right|, \log\left|\frac{z^2+1}{z^2-1}\right|\right), \tag{3·4}$$

and where $z = u+iv$. The set of branch points is the non-connected subset $\{z\in S : |z| = 1\}$ of S. This surface is not complete. Now we consider the function h given by $h(x_1,x_2) = \log(\operatorname{ch} x_2/\operatorname{ch} x_1)$ defined on the subset $\{(x_1,x_2) : \operatorname{th}^2 x_1 + \operatorname{th}^2 x_2 \leqslant 1\}$. It is easy to see that the restriction $\tilde{h}$ of h to the open set given by $\operatorname{th}^2 x_1 + \operatorname{th}^2 x_2 < 1$ is the only solution of the maximal surface differential equation (see [2], p. 174) which satisfies $h(x_1,x_2) = \tilde{h}_1(x_1)+\tilde{h}_2(x_2)$. Moreover, the graph of $\tilde{h}$ is a regular maximal surface in $\mathbb{L}^3$ whose boundary $\{(x_1,x_2,h(x_1,x_2)) : \operatorname{th}^2 x_1 + \operatorname{th}^2 x_2 = 1\}$ is the branch point set of the surface.

4. *The Gauss map*

In 1968, Calabi [1] proved that a complete regular maximal surface in $\mathbb{L}^3$ is a plane. So it is natural to ask if this result holds if regularity is removed from the hypotheses. We shall prove

THEOREM 4·1. *Let (X,S) be a complete generalized maximal surface with a finite number of branch points. Then $X(S)$ is a plane.*

Proof. Let (ω,g) be the Enneper–Weierstrass representation of (X,S). Since branch points of the surface are isolated points, either $|g(p)| < 1$ at every $p\in S$ or $|g(p)| > 1$ at every $p\in S$. Now define the function h by

$$h(z) = \begin{cases} f(z) & \text{if } |g| < 1 \\ f(z)\,g^2(z) & \text{if } |g| > 1 \end{cases}$$

with z belonging to the neighbourhood where $\omega = f(z)\,dz$. Consider the branched metric ds_1^2 on S given by $ds_1^2 = |h(z)|^2|dz|^2$ locally. Note that the branch points of ds_1^2 are precisely the branch points of (X,S). Let c be a divergent curve in S. If $|g| < 1$ everywhere, then we have

$$\int_c |f|\,(1-|g|^2)\,|dz| \leqslant \int_c |f|\,|dz| = \int_c |h|\,|dz|.$$

If $|g| > 1$ everywhere, then we have

$$\int_c |f|\,(|g|^2-1)\,|dz| = \int_c |fg^2|\,(1-1/|g|^2)\,|dz| \leqslant \int_c |h|\,|dz|.$$

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In both cases the branched metric ds_1^2 is complete because the original branched metric ds^2 on S is assumed to be complete. The Gauss curvature K_1 of ds_1^2 is zero everywhere. Now let M be a Riemannian manifold obtained from S by removing a small disc around each branch point and attaching another disc to obtain a smooth Riemannian surface, i.e. the new metric $d\tilde{s}_1^2$ has no singularities. This can be done so that M is complete and has finite total curvature (see [**8**], p. 117). Thus from [**4**], S is conformally equivalent to a compact Riemann surface punctured at a finite number of points, $\tilde{S}\backslash\{p_1, \ldots, p_k\}$. Taking into account that either $|g| < 1$ or $|g| > 1$ happens, we see that each point p_j must be a pole of g. Therefore g extends meromorphically to the compact surface $\tilde{S}$. Since $|g| < 1$ or $|g| > 1$, g must be constant. This concludes the proof.

If the generalized maximal surface is assumed to be defined on the entire complex plane $\mathbb{C}$ then we have

THEOREM 4·2. *Let $(X, \mathbb{C})$ be a generalized maximal surface. If all the branch points of $(X, \mathbb{C})$ are isolated then $X(\mathbb{C})$ is a open subset of a plane.*

Proof. Since the branch points are isolated, it follows that $|g(p)| \neq 1$ at each point p of S. Thus we have a meromorphic function g on $\mathbb{C}$ which omits every value of the unit circle, and then by Picard's Theorem g must be constant, completing the proof of the theorem.

PROPOSITION 4·3. *Let $(X, \mathbb{D})$ be a generalized maximal surface with $A = \varnothing$ and with an infinite number of branch points $\{z_n\}$ such that $\sum(1-|z_n|) < \infty$. Then $(X, \mathbb{D})$ is not complete.*

Proof. Arguing as in [**7**] p. 73, there exists a divergent curve c in $\mathbb{D}$ such that $\int_c |f(z)|\,|dz| < \infty$. If $|g| < 1$ we have

$$L(c) = \int_c \lambda |dz| \leqslant \frac{1}{2}\int_c |f|\,|dz| < \infty,$$

which implies that the surface is not complete. On the other hand, if $|g| > 1$, the function fg^2 is holomorphic on $\mathbb{D}$ and its zeros are also zeros of f, and hence there exists a divergent curve c_1 in $\mathbb{D}$ such that $\int_{c_1} |fg^2|\,|dz| < \infty$. Thus

$$L(c_1) \leqslant \tfrac{1}{2}\int_{c_1} |fg^2|\,|dz| < \infty,$$

and the surface is not complete. This completes the proof.

Remark. There are generalized maximal surfaces which have an infinite number of branch points $\{z_n\}$ satisfying the condition $\sum(1-|z_n|) = \infty$ and which are complete surfaces. See the Remark after Theorem 4·4.

Our purpose now is to study the behaviour of the Gauss map of a generalized maximal surface with non-isolated branch points.

THEOREM 4·4. *Let (X, S) be a complete generalized maximal surface.*

(a) Suppose that the subset B of branch points of S is empty. If the Gauss map of (X, S) omits more than four points of H^2, then $X(S)$ is a plane.

(b) Suppose that B is finite and that S is a simply connected surface. Then either the Gauss map of S has dense image in H^2 or $X(S)$ is a plane.

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Proof. (*a*) Let (ω, g) be the Enneper–Weierstrass representation of the surface S and $\tilde{S}$ the universal covering surface of S, with projection Π. Consider $\tilde{\omega} = \Pi^* w$ and put $\tilde{g} = g \circ \Pi$, which is a meromorphic function on $\tilde{S}$. A surface $(\tilde{X}, \tilde{S})$ in Euclidean space $\mathbb{R}^3$ is then constructed by

$$\tilde{X} = \mathrm{Re}\left(\frac{1}{2}\int(1-\tilde{g}^2)\,\tilde{\omega}, \frac{1}{2}\int i(1+\tilde{g}^2)\,\tilde{\omega}, \int \tilde{g}\tilde{\omega}\right). \tag{4·1}$$

Thus $(\tilde{X}, \tilde{S})$ is a regular minimal surface (because $B = \varnothing$) with Gauss map $\tilde{g}$ and metric $d\tilde{s}_1^2 = \frac{1}{4}|\tilde{\omega}|^2(1+|\tilde{g}|^2)^2$. We shall see that this surface is complete. Let $ds^2 = \frac{1}{4}|\omega|^2|1-|g|^2|^2$ be the branched metric of the generalized maximal surface (X, S) and let us consider the metric on S given by $ds_1^2 = \frac{1}{4}|\omega|^2(1+|g|^2)^2$. Now we denote by L and by L_1 the respective lengths associated to ds^2 and ds_1^2. We have

$$L_1(c) = \int_c \tfrac{1}{2}|\omega|\,(1+|g|^2) \geqslant \int_c \tfrac{1}{2}|\omega|\,|1-|g|^2| = L(c) \tag{4·2}$$

for any curve c on S. If c is a divergent curve on S, then $L(c) = \infty$ because (X, S) is a complete surface. From (4·2) we get $L_1(c) = \infty$. Therefore (S, ds_1^2) is a complete Riemannian manifold. Now the universal covering surface $\tilde{S}$ of S, with the metric $d\tilde{s}_1^2$, is complete. Assume that the Gauss map g of our generalized maximal surface omits on H^2 more than 4 points. In that case $\tilde{g}$ restricted to $\tilde{S}\backslash\Pi^{-1}(A)$ omits on $\hat{\mathbb{C}}\backslash\{z : |z| = 1\}$ more than 4 points and therefore the mapping $\tilde{g}: \tilde{S} \to \hat{\mathbb{C}}$ omits more than 4 points. Note that the complete minimal surface $(\tilde{X}, \tilde{S})$ satisfies the assumptions in Fujimoto's Theorem [**3**]. Hence $\tilde{g}$ is a constant function and therefore g is also constant. Thus $N = N_0$ is a constant vector on $S\backslash A$. Consider now the differentiable function on S given by $\psi(p) = \langle X(p), N_0\rangle$. Since $d\psi = 0$ on $S\backslash A$ and ψ is continuous on S, ψ is constant on $S\backslash A$. Hence $\langle X(p) - X(p_0), N_0\rangle = 0$, at every point p of $S\backslash A$, and the conclusion follows because $S\backslash A$ is dense in S, concluding the proof. (*b*) is proved in the same way as (*a*), using proposition 1·3 of [**8**].

Remark. The condition imposed on B in (*b*) of Theorem 4·4 cannot be removed. In fact, if B has infinite points the same conclusion as in previous result cannot be obtained. To see this, consider on the unit disc $\mathbb{D}$ a holomorphic function f whose zeros $\{z_n\}$ satisfy the conditions $\Sigma(1-|z_n|) = \infty$ and $\int_c |f(z)|\,|dz| = \infty$ for any divergent curve c in $\mathbb{D}$ (see [**7**], p. 73). Put $g(z) = az$, with $a \in \mathbb{R}$ such that $0 < a < 1$. The generalized maximal surface defined on $\mathbb{D}$ by (f, g) has infinite branch points and it is complete because

$$L(c) = \frac{1}{2}\int_c |f|\,|1-|g|^2|\,|dz| \geqslant \tfrac{1}{2}(1-a^2)\int_c |f|\,|dz| = \infty,$$

for any divergent curve c on $\mathbb{D}$. On the other hand, the fibre of the universal covering map is not a finite set, and therefore the set $\tilde{B}$ in the covering surface corresponding to B could be infinite even if B is finite.

5. *Gauss curvature and total curvature*

This last section is primarily devoted to the study of the behaviour of branch points under a boundedness condition on the Gauss curvature.

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THEOREM 5·1. *Let (X, S) be a generalized maximal surface with Gauss curvature K and $\phi: S \to \mathbb{R}$ a continuous function. If K satisfies the condition*

$$K(p) \leqslant \phi(p) \tag{5·1}$$

at every regular point p of S, then branch points of (X, S) are isolated. Moreover, if the universal covering space of S is the complex plane $\mathbb{C}$, then $X(S)$ is an open subset of a plane.

Proof. Suppose that the set A of branch points of the surface is not empty. Let $(\tilde{X}, \tilde{S})$ be the universal covering surface of (X, S) with $\tilde{X} = X \circ \Pi$, where Π denotes the corresponding covering projection. Let us consider the Enneper–Weierstrass representation (f, g) of $(\tilde{X}, \tilde{S})$. Let $\tilde{R} = \pi^{-1}(A \cup B)$ be the set of branch points of $(\tilde{X}, \tilde{S})$ and put $\tilde{A} = \pi^{-1}(A)$. From (5·1), $\tilde{K}$ is bounded from above by the continuous function $\tilde{\phi} = \phi \circ \Pi$ and therefore, using (2·2), we have

$$4|g'| \leqslant (\tilde{\phi})^{\frac{1}{2}} |f| (|g|^2 - 1)^2 \tag{5·2}$$

on $\tilde{S} \backslash \tilde{R}$. The functions f and g are defined on the whole of $\tilde{S}$ and the subset $\tilde{S} \backslash \tilde{R}$ is dense in $\tilde{S}$, hence inequality (5·2) is satisfied on the whole of $\tilde{S}$. From (5·2), we get $|g'| \equiv 0$ on $\tilde{A}$ because $|g| = 1$ on $\tilde{A}$ and $\tilde{\phi}$ is a continuous function on $\tilde{S}$. Since $\tilde{A}$ has non-empty accumulation we have $g' \equiv 0$ on S and therefore g is a complex number with modulus one. But this contradicts condition (1·2) of the definition of generalized maximal surfaces. Thus $\tilde{A} = \varnothing$ and $A = \pi(\tilde{A}) = \varnothing$. Finally, if in addition to (5·1) the universal covering space of S is $\mathbb{C}$, we obtain the conclusion from Theorem 4·2.

The *total curvature* of a generalized maximal surface is defined as the (improper) integral $C(S) = \int_S K dS$, which is geometrically interpreted as the area in H^2 of the image of $S \backslash A$ under the Gauss map. In contrast to the case of minimal surfaces in $\mathbb{R}^3$ we now have

THEOREM 5·2. *Let (X, S) be a complete generalized maximal surface. If the set B is empty and the total curvature $C(S)$ is finite, then $X(S)$ is a plane.*

Proof. If $X(S)$ is not a plane, from (a) of Theorem 4·4, the Gauss map of (X, S) omits less than 4 points of H^2. Therefore the total curvature $C(S)$ satisfies $C(S) = \text{Area}\,(H^2) = \infty$. This contradiction concludes the proof of the theorem.

Remarks. (1) If S is assumed to be simply connected, the above result is also true even if B is a finite set. This follows from Theorem 4·4 (b) and an argument as in Theorem 5·2.

(2) The complete generalized maximal surface given in the Remark after Theorem 4·4 has infinitely many branch points and its total curvature satisfies

$$C(S) = \iint_{\mathbb{D}} (4a^2/(1 - a^2(u^2 + v^2))^2)\, du\, dv \leqslant (4a^2\pi)/(1 - a^2)^2 < \infty.$$

So the assumption on branch points in Theorem 5·2 cannot be removed.

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