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Some formulas for the Bernoulli and Euler polynomials at rational arguments

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Abstract

In a recent paper [**5**], the classical Bernoulli and Euler polynomials were expressed as finite sums involving the Hurwitz zeta function. The object of this sequel is first to give several remarkably shorter proofs of each of these summation formulas. Various generalizations and analogues, which are relevant to the present investigation, are also considered.

1. *Introduction*

In the usual notations, let $B_n(x)$ and $E_n(x)$ denote, respectively, the classical Bernoulli and Euler polynomials of degree n in x, defined by the generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi)\tag{1·1}$$

and

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad(|t|<\pi).\tag{1·2}$$

Also let

$$B_n:=B_n(0)\quad\text{and}\quad E_n:=E_n(0),\tag{1·3}$$

where B_n and E_n are, respectively, the Bernoulli and Euler numbers of order n.

The Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s,a)$ are defined (for $\Re(s)>1$) by

$$\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s}\quad\text{and}\quad\zeta(s,a):=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\quad(a\neq 0,-1,-2,\ldots)\tag{1·4}$$

and (for $\Re(s)\leqslant 1; s\neq 1$) by their meromorphic continuations (for details see [**11**]).

By employing a number of properties and characteristics of certain Dirichlet and trigonometric series, Cvijović and Klinowski [**5**] evaluated the Bernoulli polynomials $B_n(x)$ with $n=2,3,4,\ldots$, and the Euler polynomials $E_n(x)$ with $n=1,2,3,\ldots$, for $0\leqslant x\leqslant 1$ when x is a rational number. For the sake of ready reference, we recall

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here the *main* results of Cvijović and Klinowski [**5**] as Theorem A and Theorem B below.

THEOREM A. *In terms of the Hurwitz zeta function* $\zeta(s,a)$ *defined by* (1·4), *the Bernoulli polynomials* $B_n(x)$ *at rational arguments are given by*

$$B_{2n-1}\left(\frac{p}{q}\right) = (-1)^n \frac{2(2n-1)!}{(2q\pi)^{2n-1}} \sum_{j=1}^{q} \zeta\left(2n-1, \frac{j}{q}\right) \sin\left(\frac{2jp\pi}{q}\right)$$
$$\left(n \in \mathbb{N}\backslash\{1\}; \quad \mathbb{N} := \{1,2,3,\ldots\}; \quad p \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}; \quad q \in \mathbb{N}; \quad 0 \leqslant p \leqslant q\right) \quad (1{\cdot}5)$$

and

$$B_{2n}\left(\frac{p}{q}\right) = (-1)^{n-1} \frac{2(2n)!}{(2q\pi)^{2n}} \sum_{j=1}^{q} \zeta\left(2n, \frac{j}{q}\right) \cos\left(\frac{2jp\pi}{q}\right)$$
$$(n \in \mathbb{N}; \quad p \in \mathbb{N}_0; \quad q \in \mathbb{N}; \quad 0 \leqslant p \leqslant q). \quad (1{\cdot}6)$$

THEOREM B. *In terms of the Hurwitz zeta function* $\zeta(s,a)$ *defined by* (1·4), *the Euler polynomials* $E_n(x)$ *at rational arguments are given by*

$$E_{2n-1}\left(\frac{p}{q}\right) = (-1)^n \frac{4(2n-1)!}{(2q\pi)^{2n}} \sum_{j=1}^{q} \zeta\left(2n, \frac{2j-1}{q}\right) \cos\left(\frac{(2j-1)p\pi}{q}\right)$$
$$(n \in \mathbb{N}; \quad p \in \mathbb{N}_0; \quad q \in \mathbb{N}; \quad 0 \leqslant p \leqslant q) \quad (1{\cdot}7)$$

and

$$E_{2n}\left(\frac{p}{q}\right) = (-1)^n \frac{4(2n)!}{(2q\pi)^{2n+1}} \sum_{j=1}^{q} \zeta\left(2n+1, \frac{2j-1}{2q}\right) \sin\left(\frac{(2j-1)p\pi}{q}\right)$$
$$(n \in \mathbb{N}; \quad p \in \mathbb{N}_0; \quad q \in \mathbb{N}; \quad 0 \leqslant p \leqslant q). \quad (1{\cdot}8)$$

As pointed out in [**5**, p. 1535], the formula (1·5) was derived in a completely different way by Almkvist and Meurman [**1**, p. 107, proposition 10], who applied this formula to show *eventually* that

$$q^{2n-1} B_{2n-1}\left(\frac{p}{q}\right) \quad (n \in \mathbb{N}\backslash\{1\})$$

is an integer for *all* positive integers p and q. The main object of this paper is to present some remarkably shorter proofs of Theorem A and Theorem B. Indeed we show that Theorem B is actually a simple consequence of Theorem A. We also consider a number of other results which are relevant to our present investigation.

2. *Shorter proofs of Theorem* A *and Theorem* B

First Proof of Theorem A. We begin by recalling the following known result (cf., e.g., [**10**, p. 27]):

$$B_n(x) = -\frac{2 \cdot n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{1}{k^n} \cos\left(2k\pi x - \frac{n\pi}{2}\right)$$
$$(n \in \mathbb{N}\backslash\{1\} \quad \text{and} \quad 0 \leqslant x \leqslant 1; \quad n = 1 \quad \text{and} \quad 0 < x < 1), \quad (2{\cdot}1)$$

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which, in view of the definition in (1·4) *and* the elementary series identity:

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q} \sum_{k=0}^{\infty} f(qk+j) \quad (q \in \mathbb{N}), \tag{2·2}$$

immediately yields

$$B_n\left(\frac{p}{q}\right) = -\frac{2 \cdot n!}{(2q\pi)^n} \sum_{j=1}^{q} \zeta\left(n, \frac{j}{q}\right) \cos\left(\frac{2jp\pi}{q} - \frac{n\pi}{2}\right)$$
$$(n \in \mathbb{N}\backslash\{1\}; \quad p \in \mathbb{N}_0; \quad q \in \mathbb{N}; \quad 0 \leqslant p \leqslant q), \tag{2·3}$$

where we have also set

$$x = \frac{p}{q} \quad (p \in \mathbb{N}_0; \quad q \in \mathbb{N}; \quad 0 \leqslant p \leqslant q).$$

Upon replacing n in (2·3) by $2n-1$ and $2n$, respectively, we obtain the assertions (1·5) and (1·6) of Theorem A.

Second Proof of Theorem A. The series identity (2·2), when applied to the classical Hurwitz formula (cf., e.g., [**12**, p. 269]):

$$\zeta(s,a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ \sin\left(\frac{s\pi}{2}\right) \sum_{k=1}^{\infty} \frac{\cos(2ka\pi)}{k^{1-s}} + \cos\left(\frac{s\pi}{2}\right) \sum_{k=1}^{\infty} \frac{\sin(2ka\pi)}{k^{1-s}} \right\} \tag{2·4}$$

with

$$s \longmapsto 1-s \quad \text{and} \quad a = \frac{p}{q} \quad (p, q \in \mathbb{N}),$$

gives us Rademacher's formula (cf. [**10**, p. 23]; see also [**3**, p. 261]):

$$\zeta\left(1-s, \frac{p}{q}\right) = \frac{2\Gamma(s)}{(2q\pi)^s} \sum_{j=1}^{q} \zeta\left(s, \frac{j}{q}\right) \cos\left(\frac{2jp\pi}{q} - \frac{s\pi}{2}\right) \quad (p, q \in \mathbb{N}). \tag{2·5}$$

Upon setting $s = n$ ($n \in \mathbb{N}\backslash\{1\}$) in (2·5), if we apply the familiar relationship [**10**, p. 23]:

$$\zeta(1-n, a) = -\frac{B_n(a)}{n} \quad (n \in \mathbb{N}), \tag{2·6}$$

we arrive once again at the formula (2·3) which unifies the assertions (1·5) and (1·6) of Theorem A, the missing case $p = 0$ for the Bernoulli numbers B_n being easily verified *directly* by means of the well-known identities:

$$B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) \quad (n \in \mathbb{N}_0) \tag{2·7}$$

and

$$\zeta(s) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{j}{q}\right) \quad (q \in \mathbb{N}). \tag{2·8}$$

First Proof of Theorem B. We now show that Theorem B is actually a simple consequence of Theorem A. Indeed, by appealing to the known relationship [**10**, p. 29]:

$$E_n(x) = \frac{2}{n+1} \left[B_{n+1}(x) - 2^{n+1} B_{n+1}\left(\frac{x}{2}\right) \right] \quad (n \in \mathbb{N}_0), \tag{2·9}$$

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we find from (2·3) [with n replaced by $n+1$] that

$$E_n\left(\frac{p}{q}\right) = \frac{2}{n+1}\left[-\frac{2(n+1)!}{(2q\pi)^{n+1}}\sum_{j=1}^{q}\zeta\left(n+1,\frac{j}{q}\right)\sin\left(\frac{2jp\pi}{q}-\frac{n\pi}{2}\right)\right.$$
$$\left.+2^{n+1}\frac{2(n+1)!}{(4q\pi)^{n+1}}\sum_{j=1}^{2q}\zeta\left(n+1,\frac{j}{2q}\right)\sin\left(\frac{jp\pi}{q}-\frac{n\pi}{2}\right)\right]$$
$$=\frac{4\cdot n!}{(2q\pi)^{n+1}}\left[\sum_{j=1}^{2q}\zeta\left(n+1,\frac{j}{2q}\right)\sin\left(\frac{jp\pi}{q}-\frac{n\pi}{2}\right)\right.$$
$$\left.-\sum_{j=1}^{q}\zeta\left(n+1,\frac{j}{p}\right)\sin\left(\frac{2jp\pi}{q}-\frac{n\pi}{2}\right)\right]$$
$$(n\in\mathbb{N};\quad p\in\mathbb{N}_0;\quad q\in\mathbb{N};\quad 0\leqslant p\leqslant q). \tag{2·10}$$

Obviously, since

$$\sum_{j=1}^{2q}\zeta\left(n+1,\frac{j}{2q}\right)\sin\left(\frac{jp\pi}{q}-\frac{n\pi}{2}\right)=\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-1}{2q}\right)$$
$$\times\sin\left(\frac{(2j-1)p\pi}{q}-\frac{n\pi}{2}\right)+\sum_{j=1}^{q}\zeta\left(n+1,\frac{j}{q}\right)\sin\left(\frac{2jp\pi}{q}-\frac{n\pi}{2}\right), \tag{2·11}$$

upon separating the even and odd terms, (2·10) leads us immediately to the formula:

$$E_n\left(\frac{p}{q}\right)=\frac{4\cdot n!}{(2q\pi)^{n+1}}\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-1}{2q}\right)\sin\left(\frac{(2j-1)p\pi}{q}-\frac{n\pi}{2}\right)$$
$$(n\in\mathbb{N};\quad p\in\mathbb{N}_0;\quad q\in\mathbb{N};\quad 0\leqslant p\leqslant q), \tag{2·12}$$

which yields the assertions (1·7) and (1·8) of Theorem B when n is replaced by $2n-1$ and $2n$, respectively.

Second Proof of Theorem B. Just as in our first proof of Theorem A, the unification (2·12) of the assertions (1·7) and (1·8) of Theorem B can be proven *directly* by merely applying the series identity (2·2) in the following known result (cf., e.g., [**10**, p. 30]):

$$E_n(x)=\frac{4\cdot n!}{\pi^{n+1}}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{n+1}}\sin\left((2k-1)\pi x-\frac{n\pi}{2}\right)$$
$$(n\in\mathbb{N}\quad\text{and}\quad 0\leqslant x\leqslant 1;\quad n=0\quad\text{and}\quad 0<x<1), \tag{2·13}$$

with, of course,

$$x=\frac{p}{q}\quad(p\in\mathbb{N};\quad q\in\mathbb{N};\quad 0\leqslant p\leqslant q).$$

3. *Formulas involving the Hurwitz–Lerch zeta function*

Since

$$\zeta(n+1,z)=\frac{(-1)^{n+1}}{n!}\psi^{(n)}(z)\quad(n\in\mathbb{N};\quad z\neq 0,-1,-2,\ldots), \tag{3·1}$$

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where $\psi^{(n)}(z)\,(n \in \mathbb{N}_0)$ denotes the polygamma function defined by

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}}\{\log \Gamma(z)\} \quad (n \in \mathbb{N}_0), \tag{3·2}$$

each of the formulas (2·3) and (2·12), and their even and odd integer versions given by Theorem A and Theorem B, can easily be restated in terms of the polygamma function.

A general Hurwitz–Lerch zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [**7**, p. 27 *et seq.*])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$
$$(a \neq 0, -1, -2, \ldots; \quad s \in \mathbb{C} \quad \text{when} \quad |z| < 1; \quad \Re(s) > 1 \quad \text{when} \quad |z| = 1) \tag{3·3}$$

contains, as its special cases, not only the Riemann and Hurwitz zeta functions:

$$\zeta(s) = \Phi(1, s, 1) \quad \text{and} \quad \zeta(s, a) = \Phi(1, s, a) \tag{3·4}$$

and the Lerch zeta function:

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\Phi\left(e^{2\pi i\xi}, s, 1\right) \quad (\xi \in \mathbb{R}; \Re(s) > 1), \tag{3·5}$$

but also other functions such as the polylogarithmic function:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1)$$
$$(s \in \mathbb{C} \quad \text{when} \quad |z| < 1; \quad \Re(s) > 1 \quad \text{when} \quad |z| = 1) \tag{3·6}$$

and the generalized zeta function (cf. [**12**, p. 280, example 8]);

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi}, s, a\right)$$
$$(a \neq 0, -1, -2, \ldots; \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z}; \quad \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}), \tag{3·7}$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions.

For the general Hurwitz–Lerch zeta function $\Phi(z, s, a)$ defined by (3·3), it is easily seen by using the series identity (2·2) that

$$\Phi(z, s, a) = q^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{a+j-1}{q}\right) z^{j-1}, \tag{3·8}$$

which, in the special case when

$$z = \exp\left(\frac{2p\pi i}{q}\right) \quad (p \in \mathbb{Z}; \quad q \in \mathbb{N}),$$

yields the summation formula:

$$\phi\left(\frac{p}{q}, a, s\right) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a+j-1}{q}\right) \exp\left(\frac{2(j-1)p\pi i}{q}\right) \tag{3·9}$$

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for the generalized zeta function $\phi(\xi, a, s)$ defined by (3·7) in terms of the Hurwitz zeta function $\zeta(s, a)$.

For $z = 1$, (3·8) reduces at once to the identity [**8**, p. 360, entry (54·13·1)]:

$$\zeta(s, a) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a+j-1}{q}\right), \tag{3·10}$$

which, for $a = 1$, yields the well-known result (2·8). On the other hand, by setting $a = \frac{1}{2}$ in (3·8) and (3·9), we have

$$\sum_{n=1}^{\infty} \frac{z^n}{(2n-1)^s} = (2q)^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{2j-1}{2q}\right) z^{j-1} \tag{3·11}$$

and

$$\sum_{n=0}^{\infty} \frac{e^{(2n+1)p\pi i/q}}{(2n+1)^s} = (2q)^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{2j-1}{2q}\right) \exp\left(\frac{(2j-1)p\pi i}{q}\right), \tag{3·12}$$

respectively. Lastly, in their special cases when $a = 1$, (3·8) and (3·9) yield the following companions of the summation formulas (3·11) and (3·12), respectively.

$$\sum_{n=1}^{\infty} \frac{z^n}{n^s} =: \mathrm{Li}_s(z) = q^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{j}{q}\right) z^j \tag{3·13}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{e^{2np\pi i/q}}{n^s} &=: \ell_s\left(\frac{p}{q}\right) \\ &= q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{j}{q}\right) \exp\left(\frac{2jp\pi i}{q}\right). \end{aligned} \tag{3·14}$$

In particular, when $s = \nu \quad (\nu > 1)$, by simply equating the real and imaginary parts in (3·12) and (3·14), we immediately obtain the following summation formulas involving trigonometric series:

$$\sum_{n=0}^{\infty} \frac{\cos[(2n+1)p\pi/q]}{(2n+1)^\nu} = (2q)^{-\nu} \sum_{j=1}^{q} \zeta\left(\nu, \frac{2j-1}{2q}\right) \cos\left(\frac{(2j-1)p\pi}{q}\right); \tag{3·15}$$

$$\sum_{n=0}^{\infty} \frac{\sin[(2n+1)p\pi/q]}{(2n+1)^\nu} = (2q)^{-\nu} \sum_{j=1}^{q} \zeta\left(\nu, \frac{2j-1}{2q}\right) \sin\left(\frac{(2j-1)p\pi}{q}\right); \tag{3·16}$$

$$\sum_{n=1}^{\infty} \frac{\cos(2np\pi/q)}{n^\nu} = q^{-\nu} \sum_{j=1}^{q} \zeta\left(\nu, \frac{j}{q}\right) \cos\left(\frac{2jp\pi}{q}\right); \tag{3·17}$$

$$\sum_{n=1}^{\infty} \frac{\sin(2np\pi/q)}{n^\nu} = q^{-\nu} \sum_{j=1}^{q} \zeta\left(\nu, \frac{j}{q}\right) \sin\left(\frac{2jp\pi}{q}\right). \tag{3·18}$$

The assertions made by the lemma of [**5**, p. 1530] are *precisely* the special summation formulas (3·12) and (3·14) *with* $s = \nu\,(\nu > 1)$. Formula (3·14) with $s = \nu\,(\nu > 1)$ is,

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in fact, also one of the three *main* results in another paper by Cvijović and Klinowski [**6**, p. 207, equation (7)]; the other two *main* results of [**6**, p. 207, equations (8*a*) and (8*b*)] are essentially the same as (3·16) and (3·15), respectively, *with* p replaced by $2p$. Cvijović and Klinowski [**6**, p. 208, equations (9*a*) and (9*b*)] also gave the special summation formulas (3·16) and (3·15), respectively, *with* $q = Q$.

Each of the trigonometric sums (3·15) to (3·18), and indeed also various *further* special cases of many of the summation formulas considered here, can be found to be derived in [**5**].

4. *An application of Lerch's functional equation*

For the generalized zeta function $\phi(\xi, a, s)$ defined by (3·7), the functional equation:

$$\begin{aligned}\phi(\xi, a, 1-s) = \frac{\Gamma(s)}{(2\pi)^s}\left\{\exp\left[\left(\tfrac{1}{2}s - 2a\xi\right)\pi i\right]\phi(-a, \xi, s)\right.\\ \left.+\exp\left[\left\{-\tfrac{1}{2}s + 2a(1-\xi)\right\}\pi i\right]\phi(a, 1-\xi, s)\right\}\\ (s \in \mathbb{C};\quad 0 < \xi < 1)\end{aligned} \tag{4·1}$$

was first given in 1887 by Lerch [**9**], whose proof of (4·1) follows the lines of the first Riemann proof of the functional equation for $\zeta(s)$ [Equation (2·4) with $a = 1$] and makes use of a certain contour integral (which provides the analytic continuation of $\phi(\xi, a, s)$ as an entire function of s for $\xi \in \mathbb{R} \setminus \mathbb{Z}$). Several other interesting proofs of Lerch's functional equation (4·1) can be found in the works of (for example) Apostol [**2**] and Berndt [**4**]. Moreover, in terms of the general Hurwitz–Lerch zeta function $\Phi(z, s, a)$ defined by (3·3), the functional equation (4·1) was derived in (among other places) [**7**, p. 29, equation 1·11(7)].

In view of the summation formula (3·9), Lerch's functional equation (4·1) with

$$a = \frac{p}{q} \quad (p \in \mathbb{Z};\quad q \in \mathbb{N})$$

yields

$$\begin{aligned}\phi\left(\xi, \frac{p}{q}, 1-s\right) = \frac{\Gamma(s)}{(2p\pi)^s}\left\{\sum_{j=1}^{q}\zeta\left(s, \frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{s}{2} - \frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right.\\ \left.+\sum_{j=1}^{q}\zeta\left(s, \frac{j-\xi}{q}\right)\exp\left[\left(-\frac{s}{2} + \frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}.\end{aligned} \tag{4·2}$$

Finally, in terms of a *mild* generalization of the classical Bernoulli polynomials $B_n(x)$, which was defined by Apostol by means of the generating function [**2**, p. 165, equation (3·1)]:

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty}\mathscr{B}_n(x; \lambda)\frac{t^n}{n!} \quad (|t + \log\lambda| < 2\pi), \tag{4·3}$$

so that [cf. equation (1·1)]

$$\mathscr{B}_n(x; 1) = B_n(x) \quad (n \in \mathbb{N}_0), \tag{4·4}$$

it is known that [**2**, p. 164]

$$\phi(\xi, a, 1-n) = -\frac{\mathscr{B}_n\left(a; e^{2\pi i\xi}\right)}{n} \quad (n \in \mathbb{N}). \tag{4·5}$$

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Thus, setting $s = n\,(n \in \mathbb{N})$ in (4·2) and using (4·5), we obtain the following (presumably new) analogue of (2·3) for the generalized Bernoulli polynomials $\mathscr{B}_n(x;\lambda)$ defined by (4·3):

$$\mathscr{B}_n\left(\frac{p}{q};e^{2\pi i\xi}\right) = -\frac{n!}{(2q\pi)^n}\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{n}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right.$$
$$\left.+\sum_{j=1}^{q}\zeta\left(n,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}, \qquad (4\cdot6)$$

which holds true whenever each side exists. Indeed, in its *special* case when $\xi \in \mathbb{Z}$, the summation formula (4·6) can easily be shown to reduce to the known result (2·3).

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