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SURVEY

Dimension estimates in smooth dynamics: a survey of recent results

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Abstract. We survey a collection of results in the dimension theory of dynamical systems, with emphasis on the study of repellers and hyperbolic sets of smooth dynamics. We discuss the most preeminent results in the area as well as the main difficulties in developing a general theory. Despite many interesting and non-trivial developments, only the case of conformal dynamics is completely understood. The study of the dimension of invariant sets of non-conformal maps has unveiled several new phenomena, but it still lacks today a satisfactory general approach. Indeed, we have a complete understanding of only a few classes of invariant sets of non-conformal maps satisfying certain simplifying assumptions. For example, the assumptions may ensure that there is a clear separation between different Lyapunov directions or that number-theoretical properties do not influence the dimension.

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1. *Introduction*

During the last two decades, the dimension theory of dynamical systems has progressively developed into an independent field of research. Roughly speaking, its main objective is to measure the complexity from the dimensional point of view of the objects that remain invariant under the dynamics, such as the invariant sets and measures. The first monograph that clearly took this point of view was Pesin's book [**74**], which describes the state of

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the art up to 1997. We refer to [**4**] for a description of many of the more recent results in the area. In this survey we deliberately avoid discussing results that are not of dynamical nature, regardless of the importance that they may have in the general dimension theory.

Despite many interesting and non-trivial developments in the dimension theory of dynamical systems, only the case of *conformal* dynamics is completely understood. In the case of repellers it corresponds to assuming that the derivative of the map is a multiple of an isometry at every point. In particular, this results in there being no privileged directions, and that the elements of all Markov partitions (which are used to model the repeller) are essentially balls, in some precise sense. These crucial properties allowed Bowen [**16**] and Ruelle [**80**] to develop a complete theory for the dimension of repellers of conformal maps. We note that the thermodynamic formalism introduced by Ruelle in his seminal work [**78**] (see also [**79**]) played an important role in this development. Essentially, all known equations used to compute or estimate dimensions are appropriate versions of an equation introduced by Bowen in [**16**] in his study of quasi-circles, which involves the notion of topological pressure. Moreover, for any repeller of a conformal map there exists an ergodic invariant measure of full dimension, that is, a measure concentrated on the set having the same dimension as the repeller.

To a certain extent, the study of the dimension of hyperbolic sets is analogous. Indeed, assuming that along the stable and unstable directions the derivative of the map is a multiple of an isometry, starting with the work of McCluskey and Manning in [**66**] it was possible to develop a sufficiently complete corresponding theory. However, we note that there are non-trivial differences with respect to repellers. For example, unless some cohomology relations are satisfied, there are no invariant measures of full dimension concentrated on a given hyperbolic set.

On the other hand, the study of the dimension of invariant sets of *non-conformal* maps has uncovered several new phenomena, but it still lacks a satisfactory general approach, for both repellers and hyperbolic sets. Indeed, most authors make additional assumptions that essentially avoid two main types of difficulties. The first difficulty is the lack of a clear separation between different Lyapunov directions, together with a possible small regularity of the associated distributions. Typically these distributions are only Hölder continuous, which means that in general it is not possible to add the dimensions along various distributions (thinking of the tangent bundle as a product of them). This contrasts strongly with what happens for hyperbolic sets of a dynamics that is conformal along the stable and unstable directions, in which case the stable and unstable holonomies are Lipschitz (which thus means that the tangent bundle is taken by a Lipschitz map with Lipschitz inverse to the product of the stable and unstable distributions). The second difficulty is the existence of number-theoretical properties that may cause a variation of the Hausdorff dimension with respect to a certain typical value (such as that obtained by Falconer in [**25**]; see Theorem 3.7). Other authors have obtained results not for a specific invariant set, but instead for almost all invariant sets in a given parameterized family. However, sometimes it is quite difficult to determine or it is even unknown what happens for each specific value of the parameters.

As a result, in the case of non-conformal maps we are often only able to establish estimates instead of giving formulas for the dimension of the invariant sets.

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Thus, sometimes the emphasis is on how to obtain sharp dimension estimates, starting essentially with the seminal work of Douady and Oesterlé [**19**] who devised an approach to cover the invariant set in a more optimal manner. We have made an effort to describe all the preeminent results using this approach, in several directions. As a rule, sharp lower dimension estimates are more difficult to obtain. In some cases they remain unknown or may even be impossible to obtain for *all* sets in some specific classes of invariant sets of non-conformal dynamical systems.

We also discuss briefly some motivations for the study of dimension in the context of dynamical systems. In many situations the longtime behavior of dynamical systems such as those from delay differential equations or partial differential equations can essentially be described in terms of global attractors (see [**1, 40, 88**]). An important question, in particular in the context of infinite-dimensional systems, is how many degrees of freedom are actually necessary to specify the dynamics on the attractor. It turns out that in many situations the attractors have finite Hausdorff dimension or even finite box dimension, and hence the dynamics on the attractor is essentially finite-dimensional (see [**1, 40, 88**] for related discussions). In particular Mañé [**63**] obtained the following result.

THEOREM 1.1. *Let $f : E \to E$ be a C^1 map of a Banach space such that at every point its derivative is the sum of a compact map and a contraction. Then every compact f-invariant set in E has finite upper box dimension.*

An analogous statement for the Hausdorff dimension was given by Mallet-Paret [**62**] in Hilbert spaces. Moreover, particularly in the experimental study of attractors one often considers their projection into an Euclidean space. It is also possible to give conditions for the invertibility of the projection. In particular, the following result is due to Mañé [**63**].

THEOREM 1.2. *Let E be a Banach space and let $F \subset E$ be a p-dimensional subspace. For a residual set of the space of all continuous projections of E onto F (with respect to the operator norm topology) each projection is injective on a compact set $\Lambda \subset E$ provided that* $\dim_H(\Lambda \times \Lambda) + 1 < p < \infty$.

For an arbitrary projection of a compact subset of a Banach space Hunt and Kaloshin [**50**] showed that typically (in the sense of prevalence in [**51**]) the projection is injective and has Hölder continuous inverse. Earlier results concerning the Hölder continuity of the inverse are due to Ben-Artzi *et al* [**11**] in $\mathbb{R}^n$ and to Foias and Olson [**33**] in Hilbert spaces. These results estimate how much the dimension of the set may decrease under the projection.

2. *Dimension theory and thermodynamic formalism*

We briefly recall in a pragmatic manner the notions from dimension theory and the thermodynamic formalism that are needed in this survey. We refer to [**4, 15, 74, 93**] for more details.

2.1. *Hausdorff and box dimensions.*

Let X be a separable metric space. Given a set $Z \subset X$ and a number $\alpha > 0$, we define the *α-dimensional Hausdorff measure* of Z by

$$m(Z, \alpha) = \lim_{\varepsilon \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^{\alpha},$$

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where the infimum is taken over all finite or countable covers $\mathcal{U}$ of Z by open sets U with diameter diam $U \leq \varepsilon$. The *Hausdorff dimension* of Z is defined by

$$\dim_H Z = \inf\{\alpha > 0 : m(Z, \alpha) = 0\}.$$

Moreover, the *lower* and *upper box dimensions* of Z are defined respectively by

$$\underline{\dim}_B Z = \liminf_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon} \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\varepsilon \to 0} \frac{\log N(Z, \varepsilon)}{-\log \varepsilon},$$

where $N(Z, \varepsilon)$ denotes the number of balls of radius ε that are needed to cover Z. When these two numbers coincide we denote the common value by $\dim_B Z$, and we call it the *box dimension* of Z. One can easily show that

$$\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z.$$

We note that in general these inequalities may be strict.

Now let μ be a finite measure in X. The *Hausdorff dimension* and the *lower* and *upper box dimensions* of μ are defined respectively by

$$\begin{aligned}
\dim_H \mu &= \inf\{\dim_H Z : \mu(Z) = \mu(X)\},\\
\underline{\dim}_B \mu &= \lim_{\delta \to 0} \inf\{\underline{\dim}_B Z : \mu(Z) \geq \mu(X) - \delta\},\\
\overline{\dim}_B \mu &= \lim_{\delta \to 0} \inf\{\overline{\dim}_B Z : \mu(Z) \geq \mu(X) - \delta\}.
\end{aligned}$$

We emphasize that in general these three numbers do not respectively coincide with the Hausdorff dimension and the lower and upper box dimensions of the support of μ. Thus, they contain additional information about the distribution of the measure μ on its support. One can show that

$$\dim_H \mu \leq \underline{\dim}_B \mu \leq \overline{\dim}_B \mu.$$

Again, in general these inequalities may be strict. The following criterion for equality was established by Young in [**96**]: if there exists a constant $d \geq 0$ such that

$$\lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = d \tag{1}$$

for μ-almost every $x \in X$, then

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d.$$

The limit in (1), when it exists, is called the *pointwise dimension* of μ at x.

2.2. *Topological pressure and variational principle.* There is a close relation between dimension theory and the thermodynamic formalism.

We first introduce the notion of topological pressure. Let $f : X \to X$ be a continuous transformation of a compact metric space. For each $n \in \mathbb{N}$ we define a distance ρ_n in X by

$$\rho_n(x, y) = \max\{\rho(f^k(x), f^k(y)) : k = 0, \ldots, n-1\}.$$

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A set of points $\{x_i\}_{i\in I}$ in X is said to be (n, ε)-*separated* (with respect to f) if $\rho_n(x_i, x_j) \geq \varepsilon$ whenever $i \neq j$. The *topological pressure* of a continuous function $\varphi\colon X \to \mathbb{R}$ (with respect to f) is defined by

$$P(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup \sum_{i\in I} \exp \sum_{k=0}^{n-1} (\varphi \circ f^k)(x_i), \tag{2}$$

where the supremum is taken over all (n, ε)-separated sets $\{x_i\}_{i\in I}$. The *topological entropy* of f is defined by $h(f) = P(0)$. The notion of topological pressure was introduced by Ruelle in [**78**] for expansive transformations, and by Walters in [**92**] in the general case. We recall that f is said to be *expansive* if there exists $\delta > 0$ such that if

$$d(f^n(x),\ f^n(y)) < \delta \quad \text{for every } n \in \mathbb{N},$$

then $x = y$ (when f is invertible we replace $\mathbb{N}$ by $\mathbb{Z}$ in the definition). For more details and further references see [**15, 79, 93**].

We illustrate with a simple example the relation between dimension theory and the thermodynamic formalism.

Example 2.1. Consider constants $\lambda_1, \ldots, \lambda_p \in (0, 1)$, and disjoint closed intervals $\Delta_1, \ldots, \Delta_p \subset \mathbb{R}$ of lengths $\lambda_1, \ldots, \lambda_p$. For each $k = 1, \ldots, p$, we again choose p disjoint closed intervals $\Delta_{k1}, \ldots, \Delta_{kp} \subset \Delta_k$ of lengths respectively $\lambda_k\lambda_1, \ldots, \lambda_k\lambda_p$. Iterating this procedure, for each $n \in \mathbb{N}$ we obtain p^n disjoint closed intervals $\Delta_{i_1\cdots i_n}$ of lengths $\prod_{k=1}^n \lambda_{i_k}$, and we consider the limit set

$$F = \bigcap_{n=1}^{\infty} \bigcup_{i_1\cdots i_n} \Delta_{i_1\cdots i_n}.$$

Moran showed in [**68**] that $\dim_H F = s$, where s is the unique real number such that

$$\sum_{k=1}^{p} \lambda_k^s = 1. \tag{3}$$

It is remarkable that the Hausdorff dimension of F does not depend on the location of the intervals $\Delta_{i_1\cdots i_n}$ but only on their lengths. Now we consider the continuous function $\varphi\colon F \to \mathbb{R}$ given by $\varphi(x) = \log \lambda_i$ when $x \in F \cap \Delta_i$. We obtain

$$\begin{aligned} P(s\varphi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp\left(s \sum_{k=1}^{n} \log \lambda_{i_k} \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \prod_{k=1}^{n} \lambda_{i_k}^s \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i=1}^{p} \lambda_i^s \right)^n = \log \sum_{i=1}^{p} \lambda_i^s. \end{aligned}$$

Therefore, equation (3) is equivalent to

$$P(s\varphi) = 0. \tag{4}$$

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Equation (4) is called Bowen's equation. It was introduced by Bowen in [**16**] in his study of quasi-circles. Virtually all known equations used to compute or estimate dimensions are appropriate versions of equation (4).

Now let μ be an f-invariant probability measure in X. This means that $\mu(f^{-1}A) = \mu(A)$ for every measurable set $A \subset X$. A finite or countable family ξ of measurable subsets of X is called a *partition* of X if $\mu(\bigcup_{C\in\xi} C) = 1$, and $\mu(C \cap D) = 0$ for every $C, D \in \xi$ with $C \neq D$. We write

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C) \log \mu(C),$$

with the convention that $0 \log 0 = 0$. The *Kolmogorov–Sinai entropy* of f with respect to μ is defined by

$$h_\mu(f) = \sup_{H_\mu(\xi)<\infty} \lim_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=0}^{n-1} f^{-k}\xi\Big), \tag{5}$$

where $\bigvee_{k=0}^{n-1} f^{-k}\xi$ is the partition of X into the sets

$$C_{i_1\cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} C_{i_{k+1}}$$

with $C_{i_1}, \ldots, C_{i_n} \in \xi$. Indeed it can be shown that the limit in (5) exists. The topological pressure satisfies the *variational principle*

$$P(\varphi) = \sup_\mu \Big\{ h_\mu(f) + \int_X \varphi \, d\mu \Big\}, \tag{6}$$

with the supremum taken over all f-invariant probability measures in X. This was established by Ruelle in [**78**] for expansive maps, and by Walters in [**92**] in the general case. An f-invariant probability measure μ is said to be an *equilibrium measure* of φ (with respect to f) if

$$P(\varphi) = h_\mu(f) + \int_X \varphi \, d\mu,$$

that is, if the supremum in (6) is attained at μ. It follows immediately from (6) that when the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, each continuous function φ has at least one equilibrium measure. For example, when f is expansive the entropy map is upper semi-continuous.

2.3. *Non-additive topological pressure.* We also consider the non-additive version of the topological pressure introduced by Barreira in [**2**].

Let $f : X \to X$ be a continuous transformation of a compact metric space. Let $\mathcal{U}$ be a finite open cover of X. Given $n \in \mathbb{N}$, we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of n-tuples $U = (U_1, \ldots, U_n)$ with $U_1, \ldots, U_n \in \mathcal{U}$. For each $U \in \mathcal{W}_n(\mathcal{U})$ we set $m(U) = n$, and we define the open set

$$X(U) = \{x \in X : f^{k-1}(x) \in U_k \text{ for } k = 1, \ldots, m(U)\}.$$

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We say that a collection $\Gamma \subset \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(\mathcal{U})$ *covers* a set $Z \subset X$ provided that $\bigcup_{U\in\Gamma} X(U) \supset Z$. Now let $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $\varphi_n : X \to \mathbb{R}$, and set

$$\gamma_n(\Phi, \mathcal{U}) = \sup\{|\varphi_n(x) - \varphi_n(y)| : x, y \in X(U) \text{ and } U \in \mathcal{W}_n(\mathcal{U})\}.$$

We assume that

$$\lim_{\operatorname{diam}\mathcal{U}\to 0} \limsup_{n\to\infty} \frac{1}{n}\gamma_n(\Phi, \mathcal{U}) = 0, \tag{7}$$

where $\operatorname{diam}\mathcal{U} = \sup_{U\in\mathcal{U}} \operatorname{diam} U$ is the *diameter* of the cover $\mathcal{U}$. We observe that (7) holds when Φ is an additive sequence, that is, when

$$\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k \tag{8}$$

for some continuous function $\varphi : X \to \mathbb{R}$ and all $n \in \mathbb{N}$. For each $U \in \mathcal{W}_n(\mathcal{U})$ we write

$$\varphi(U) = \begin{cases} \sup\limits_{X(U)} \varphi_n & \text{if } X(U) \neq \emptyset, \\ -\infty & \text{if } X(U) = \emptyset. \end{cases}$$

Given $Z \subset X$ and $\alpha \in \mathbb{R}$, we set

$$M_Z(\alpha, \Phi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp(-\alpha m(U) + \varphi(U)),$$

where the infimum is taken over all collections $\Gamma \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{U})$ covering Z. One can show that the function $\alpha \mapsto M_Z(\alpha, \Phi, \mathcal{U})$ jumps from $+\infty$ to 0 at a unique value of α, which we denote by

$$P_Z(\Phi, \mathcal{U}) = \inf\{\alpha \in \mathbb{R} : M_Z(\alpha, \Phi, \mathcal{U}) = 0\}.$$

Moreover, the limit

$$P_Z(\Phi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\Phi, \mathcal{U})$$

exists (see [**2**] for details), and we call it the *non-additive topological pressure* of the sequence Φ on the set Z (with respect to f). One can easily verify that if Φ is the (additive) sequence of functions in (8), then $P_X(\Phi)$ coincides with the topological pressure $P(\varphi)$ of the function φ (see (2)).

The non-additive thermodynamic formalism developed in [**2**] contains as a particular case a new formulation of the subadditive thermodynamic formalism introduced by Falconer in [**26**]. It also includes a variational principle for the topological pressure, although with a restrictive assumption on the sequence Φ. Namely, if there exists a continuous function $\varphi : X \to \mathbb{R}$ such that

$$\varphi_{n+1} - \varphi_n \circ f \to \varphi \text{ uniformly when } n \to \infty, \tag{9}$$

then

$$P_X(\Phi) = \sup_{\mu}\left(h_\mu(f) + \int_X \varphi \, d\mu\right),$$

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with the supremum taken over all f-invariant probability measures on X. The restrictive assumption in (9) meant that until recently there was no discussion of equilibrium and Gibbs measures in the general context of the non-additive thermodynamic formalism. On the other hand, it is well known that equilibrium and Gibbs measures play a prominent role in dimension theory and in particular in the multifractal analysis of dynamical systems. This justifies the interest in the more general class of almost additive sequences, for which it is possible not only to establish a variational principle, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures, among other properties. We recall that a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *almost additive* if there is a constant $C > 0$ such that

$$-C + \varphi_n + \varphi_m \circ f^n \leq \varphi_{n+m} \leq C + \varphi_n + \varphi_m \circ f^n$$

for every $n, m \in \mathbb{N}$. Non-trivial examples of almost additive sequences occur, for instance, in the multifractal analysis of Lyapunov exponents for non-conformal maps in [**6**]. We refer to [**5**] for a survey of the existing results of the almost additive thermodynamic formalism, including the discussion of some recent developments and applications.

3. *Dimension estimates for repellers*

We discuss in this section the dimension of repellers, which are invariant sets of a hyperbolic non-invertible dynamics. After describing how Markov partitions can be used to model repellers, we present the existing results concerning dimension estimates, including the well-established case of conformal dynamics, as well as the most recent results for non-conformal dynamics and non-uniformly hyperbolic dynamics, among several other topics.

3.1. *Notion of repeller and examples.*

Let $f\colon M \to M$ be a differentiable transformation of a smooth manifold, and let $J \subset M$ be a compact f-invariant set (this means that $f^{-1}J = J$). We say that J is a *repeller* of f and that f is *expanding* on J if there exist constants $c > 0$ and $\beta > 1$ such that

$$\|d_x f^n v\| \geq c\beta^n \|v\| \tag{10}$$

for every $n \in \mathbb{N}$, $x \in J$, and $v \in T_x M$. We note that by (10) the linear transformation $d_x f^n$ is invertible for every $x \in J$ and $n \in \mathbb{N}$. In particular, f is a local diffeomorphism, that is, each $x \in J$ has an open neighborhood U_x such that $f|U_x\colon U_x \to f(U_x)$ is invertible and has differentiable inverse.

We describe two examples of expanding maps and repellers.

Example 3.1. (Rational maps) Consider a rational map $f\colon S \to S$ of degree at least two on the Riemann sphere S. A n-periodic point x (this means that $f^n(x) = x$) is said to be *repelling* if $|(f^n)'(x)| > 1$. The *Julia set* $J \subset S$ of f is the closure of the set of repelling periodic points of f. In particular, J is compact, non-empty, and f-invariant. For example, the map $z \mapsto z^2 + c$ is expanding on its Julia set provided that $|c| < 1/4$ (see, for example, [**30**] for details), in which case J is a repeller of f.

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Example 3.2. (One-dimensional Markov maps) Consider disjoint closed intervals $\Delta_1, \dots, \Delta_p \subset [0, 1]$. We say that $f : [0, 1] \to [0, 1]$ is a *Markov map* provided that:

(1) for every $j = 1, \dots, p$ there is a set of indices $I(j)$ such that $f(\Delta_j) = \bigcup_{i \in I(j)} \Delta_i$;

(2) at every point $x \in \Delta := \bigcup_{i=1}^{p} \operatorname{int} \Delta_i$ the derivative of f exists and satisfies $|f'(x)| \geq \gamma$ for some fixed $\gamma > 0$;

(3) there exist $\lambda > 1$ and $n \in \mathbb{N}$ such that $|(f^n)'(x)| \geq \lambda$ whenever $f^k(x) \in \Delta$ for $0 \leq k \leq n$.

Then the set

$$J = \bigcap_{k=1}^{\infty} \overline{f^{-k}\Delta}$$

is a repeller of f.

3.2. *Markov partitions and geometric constructions.* Let J be a repeller of a differentiable transformation $f : M \to M$. A finite cover of J by non-empty closed sets $R_1, \dots, R_p \subset J$ is called a *Markov partition* of J if:

(1) $\overline{\operatorname{int} R_i} = R_i$ for each i;

(2) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ whenever $i \neq j$;

(3) $f(R_i) \supset R_j$ whenever $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \emptyset$.

We note that the interior of each set R_i is computed with respect to the induced topology on J. Any repeller has Markov partitions with arbitrarily small diameter (see [**80**]).

We now describe how Markov partitions can be use to model repellers. Given a Markov partition, we define a $p \times p$ matrix $A = (a_{ij})$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \emptyset, \\ 0 & \text{if } f(\operatorname{int} R_i) \cap \operatorname{int} R_j = \emptyset. \end{cases} \tag{11}$$

We also consider the space of sequences $\Sigma_p^+ = \{1, \dots, p\}^{\mathbb{N}}$ equipped with the distance

$$d(i_1 i_2 \cdots, j_1 j_2 \cdots) = \sum_{k=1}^{\infty} e^{-k} |i_k - j_k|, \tag{12}$$

and the shift map $\sigma : \Sigma_p^+ \to \Sigma_p^+$ satisfying $(\sigma\omega)_n = \omega_{n+1}$ for all $n \in \mathbb{N}$. The restriction of σ to the σ-invariant set

$$\Sigma_A^+ = \{(i_1 i_2 \cdots) \in \Sigma_p^+ : a_{i_n i_{n+1}} = 1 \text{ for every } n \in \mathbb{N}\}$$

is called the *topological Markov chain* with *transition matrix* A.

It is easy to show that one can define a map $\chi : \Sigma_A^+ \to J$ by

$$\chi(i_1 i_2 \cdots) = \bigcap_{k=0}^{\infty} f^{-k} R_{i_{k+1}}.$$

This map is surjective, Hölder continuous (with the distance in Σ_p^+ given by (12)), and it satisfies

$$\chi \circ \sigma = f \circ \chi. \tag{13}$$

In general χ is not invertible, although $\operatorname{card} \chi^{-1} x \leq p^2$ for every $x \in \Sigma_A^+$. Nevertheless, identity (13) still allows one to see χ as a dictionary transferring the symbolic

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dynamics $\sigma|\Sigma_A^+$ (and often the results at this level) to the dynamics of f on J. For each $(i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$ we set

$$\Delta_{i_1 \cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} R_{i_{k+1}}. \tag{14}$$

Given a continuous function $\varphi\colon J \to \mathbb{R}$ we have

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1 \cdots i_n} \exp \sup_{\Delta_{i_1 \cdots i_n}} \sum_{k=0}^{n-1} (\varphi \circ f^k),$$

where the sum is taken over all n-tuples $i_1 \cdots i_n$ formed by the first n elements of some sequence in Σ_A^+.

More generally, one can consider constructions defined by an arbitrary symbolic dynamics. Namely, a *geometric construction* in $\mathbb{R}^m$ is defined by:

(1) a compact shift-invariant set $Q \subset \Sigma_p^+$ for some positive integer p;

(2) a decreasing sequence of closed sets $\Delta_{i_1 \cdots i_n} \subset \mathbb{R}^m$ for each $(i_1 i_2 \cdots) \in Q$ and $n \in \mathbb{N}$, with $\operatorname{diam} \Delta_{i_1 \cdots i_n} \to 0$ as $n \to \infty$.

We also say that the geometric construction is modeled by Q. The *limit set* of the construction is the compact set defined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{i_1 \cdots i_n} \Delta_{i_1 \cdots i_n},$$

where the union is taken over all Q-admissible n-tuples, that is, all n-tuples $(i_1 \cdots i_n)$ such that

$$(j_1 \cdots j_n) = (i_1 \cdots i_n) \quad \text{for some } (j_1 j_2 \cdots) \in Q.$$

In particular, each Markov partition of a repeller defines a geometric construction modeled by Σ_A^+, with the sets $\Delta_{i_1 \cdots i_n}$ given by (14).

We recall that a transformation f is said to be *topologically mixing* on a set J if, given open sets U and V with non-empty intersection with J, there exists $n \in \mathbb{N}$ such that

$$f^m U \cap V \cap J \neq \emptyset \quad \text{for every } m > n.$$

If f is topologically mixing on a repeller J, then for each transition matrix A associated with a Markov partition of J there exists $m \in \mathbb{N}$ such that A^m has only positive entries.

3.3. *A priori bounds for the dimension.* Our main aim in this section is to present the best possible dimension estimates of repellers using the least possible information about the dynamics. It is in this sense that we call them *a priori* bounds. When more geometric information is available one can often provide sharper estimates for the dimension or even compute its value (see §3.4 for a related discussion). However, this is done at the expense of a more elaborate approach that in particular may require the non-additive version of the thermodynamic formalism or some appropriate substitution.

Given a repeller J of a differentiable transformation f, we consider the functions $\underline{\varphi}, \overline{\varphi}\colon J \to \mathbb{R}$ given by

$$\underline{\varphi}(x) = -\log \|d_x f\| \quad \text{and} \quad \overline{\varphi}(x) = \log \|(d_x f)^{-1}\|. \tag{15}$$

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Since f is expanding, the functions

$$s \mapsto P(s\underline{\varphi}) \quad \text{and} \quad s \mapsto P(s\overline{\varphi})$$

are Lipschitz and strictly decreasing. This implies that there exist unique roots $\underline{s}$ and $\overline{s}$ of the equations

$$P(\underline{s}\underline{\varphi}) = 0 \quad \text{and} \quad P(\overline{s}\,\overline{\varphi}) = 0. \tag{16}$$

THEOREM 3.3. *If J is a repeller of a C^1 map, then*

$$\underline{s} \le \dim_H J \le \underline{\dim}_B J \le \overline{\dim}_B J \le \overline{s}. \tag{17}$$

Theorem 3.3 is a special case of results of Barreira in [**2**]. The upper bound was obtained in [**38**] in the more general case of volume expanding maps. By the variational principle in (6), it follows from (17) that

$$\frac{h(f)}{\log \max_{x\in J} \|d_x f\|} \le \underline{s} \le \overline{s} \le \frac{h(f)}{-\log \max_{x\in J} \|(d_x f)^{-1}\|}.$$

In a related direction, Shafikov and Wolf showed in [**82**] that for a repeller J of a C^2 map $f\colon M \to M$ we have

$$\overline{\dim}_B J \le \dim M + \frac{P(-\log|\det df|)}{\lambda},$$

where

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in J} \|d_x f^n\|.$$

In the particular case of conformal maps the inequalities in Theorem 3.3 become identities. We recall that f is said to be *conformal* on J if $d_x f$ is a multiple of an isometry for every $x \in J$. In this case the two functions $\underline{\varphi}$ and $\overline{\varphi}$ in (15) coincide, and thus $\underline{s} = \overline{s}$. This yields equalities in (17). For examples of repellers of conformal maps see Examples 3.1 and 3.2.

Let us formulate an explicit statement. Consider the function $\varphi\colon J \to \mathbb{R}$ defined by

$$\varphi(x) = -\log \|d_x f\|.$$

THEOREM 3.4. *If J is a repeller of a C^1 map f which is conformal on J, then*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s,$$

where s is the unique root of the equation $P(s\varphi) = 0$. Moreover, if f is of class $C^{1+\alpha}$ for some $\alpha \in (0, 1)$ and is topologically mixing on J, then the unique equilibrium measure μ of $s\varphi$ satisfies

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = s.$$

Ruelle [**80**] showed that $\dim_H J = s$ when f is of class $C^{1+\alpha}$, assuming that f is topologically mixing. His proof involves showing that the s-dimensional Hausdorff measure on J is equivalent to the equilibrium measure of $s\varphi$ (with Radon–Nikodym derivative bounded and bounded away from zero). The particular case of quasi-circles was considered earlier by Bowen in [**16**]. The coincidence between the Hausdorff and box dimensions was established by Falconer in [**27**]. The extension of these results to repellers of C^1 maps was obtained independently by Barreira [**2**] and Gatzouras and Peres [**36**], using different approaches. See [**89**] for a survey on the dimension theory of holomorphic endomorphisms.

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3.4. *Sharp upper dimension estimates.* We describe in this section sharp upper dimension estimates for a repeller when more geometric information is available.

Given a linear map $L\colon \mathbb{R}^m \to \mathbb{R}^m$, let

$$\sigma_1(L) \geq \cdots \geq \sigma_n(L) \geq 0$$

be the *singular values* of L, that is, the eigenvalues of $(L^*L)^{1/2}$, counted with their multiplicities, where L^* denotes the transpose of L. These numbers coincide with the semiaxes of the ellipsoid which is the image of the unit ball in $\mathbb{R}^m$ under the map L. For each $s \in [0, n]$ we set

$$\omega_s(L) = \sigma_1(L) \cdots \sigma_{\lfloor s \rfloor}(L)\sigma_{\lfloor s \rfloor+1}(L)^{s-\lfloor s \rfloor}, \tag{18}$$

where $\lfloor s \rfloor$ denotes the integer part of s.

In their seminal paper [**19**] Douady and Oesterlé obtained an upper bound for the Hausdorff dimension. Given a C^1 map $f\colon M \to M$, we consider the function $\psi^s\colon M \to \mathbb{R}$ defined by

$$\psi^s(x) = \log \omega_s(d_x f).$$

Moreover, given a subset $J \subset M$ we let

$$\dim_L(f, J) = \inf\left\{ s \in (0, \dim M] : \sup_{x\in J} \psi^s(x) < 0 \right\}.$$

THEOREM 3.5. [**19**] *If f is a C^1 map and J is a compact f-invariant set, then*

$$\dim_H J \leq \dim_L(f, J). \tag{19}$$

Since J is f^m-invariant for every $m \in \mathbb{N}$, we also have

$$\dim_H J \leq \inf_{m\in\mathbb{N}} \dim_L(f^m, J), \tag{20}$$

which may sometimes give a better estimate than that in (19). It was shown by Hunt in [**49**] that we can replace $\dim_H J$ by $\overline{\dim}_B J$ in (19) for maps in $\mathbb{R}^n$. See [**37**] for a proof including the case of maps on manifolds. Leonov [**57**] established estimates for the Hausdorff dimension under weaker assumptions using Lyapunov-type functions. Namely, to show that $\dim_H J \leq s$ it suffices to show that

$$\sup_{x\in J}\left(\frac{\psi(f(x))}{\psi(x)}\omega_s(d_x f) \right) < 1,$$

where $\psi\colon J \to \mathbb{R}^+$ is some continuous function. This approach can be interpreted as a change of metric on the manifold, as studied for example by Noack and Reitmann in [**70**]. It may sometimes improve the estimate in Theorem 3.5. Indeed, while the Hausdorff dimension is invariant under smooth changes of the metric, the singular values may change, and thus the function corresponding to the new metric may be strictly smaller than φ_s.

Now we consider the sequences of functions $\Phi^s = (\varphi_n^s)_{n\in\mathbb{N}}$ defined by

$$\varphi_n^s(x) = \log \omega_s((d_x f^n)^{-1})$$

with ω_s as in (18). Using these functions Falconer [**29**] computed the Hausdorff dimension of a class of repellers of non-conformal transformations (building on his earlier work [**26**]). His main result can be reformulated as follows.

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THEOREM 3.6. *If J is a repeller of a C^2 map f which is topologically mixing on J, and*

$$\|(d_x f)^{-1}\|^2 \|d_x f\| < 1 \quad \textit{for every } x \in J,$$

then

$$\underline{\dim}_B J = \overline{\dim}_B J \le s, \tag{21}$$

where s is the unique root of the equation $P_J(\Phi^s) = 0$.

Clearly, $\omega_s(L) \le \sigma_1(L)^s$, and thus,

$$\begin{aligned}\varphi_n^s(x) &\le s \log \sigma_1((d_x f^n)^{-1}) \\ &\le s \log \|(d_x f^n)^{-1}\| \le s \sum_{k=0}^{n-1} \overline{\varphi}(f^k(x)),\end{aligned}$$

with $\overline{\varphi}$ as in (15). This shows that $s \le \bar{s}$ (see (16)), and thus in general (21) may give a sharper estimate than that given by the upper bound in (17).

Under an additional geometric assumption, satisfied for example when J contains a non-differentiable arc, the number s in Theorem 3.6 is equal to $\dim_H J$ (see [**29**]). Related results were obtained by Zhang in [**97**], and in the case of volume expanding maps by Gelfert in [**38**]. In another direction, Hu [**47**] computed the box dimension of a class of repellers of non-conformal transformations that leave invariant a strong unstable foliation. His formula for the box dimension is also expressed in terms of the topological pressure. Related results were obtained earlier by Bedford in [**9**] (see also [**10**]), for a class of self-similar sets that are graphs of continuous functions. We note that Manning and Simon [**65**] found a mistake in the related paper [**3**], which discusses dimension estimates using the sequences Φ^s. As a result, some statements in that paper do not hold (we refer to [**65**] for full details).

3.5. *Self-affine repellers and nonlinear generalizations.* In another direction, Falconer [**25, 28**] studied a class of limit sets of geometric constructions obtained from the composition of affine transformations that are not necessarily conformal. Consider affine transformations $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, p$, given by $f_i(x) = A_i x + b_i$ for some linear contraction A_i and some vector $b_i \in \mathbb{R}^n$. Then there is a unique non-empty compact set $J \subset \mathbb{R}^n$ such that

$$J = \bigcup_{i=1}^{p} f_i(J) \tag{22}$$

(see [**52**]), and for every non-empty compact set $R \subset \mathbb{R}^n$ such that $f_i(R) \subset R$ for $i = 1, \ldots, p$ we have

$$J = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k} (f_{i_1} \circ \cdots \circ f_{i_k})(R).$$

Set

$$s = \inf\left\{ d \in [0, n] : \sum_{k=1}^{\infty} \sum_{i_1 \cdots i_k} \omega_d(A_{i_1} \circ \cdots \circ A_{i_k}) < \infty \right\}$$

with ω_d as in (18). We emphasize that the number s does not depend on the vectors $b_1, \ldots, b_p$.

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THEOREM 3.7. *We have* $\overline{\dim}_B J \le s$. *In addition, if* $\|A_i\| < 1/2$ *for* $i = 1, \ldots, p$, *then for Lebesgue almost every* $(b_1, \ldots, b_p) \in (\mathbb{R}^n)^p$ *we have*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s. \tag{23}$$

The statement in Theorem 3.7 is due to Falconer [**25**] when $\|A_i\| < 1/3$ for $i = 1, \ldots, p$, and to Solomyak [**86**] in the general case.

The class of self-affine repellers can also be used to illustrate that to determine or even estimate the dimension of the limit set J, sometimes it is not sufficient to know the geometric shape of the sets $\Delta_{i_1 \cdots i_n}$, in strong contrast to what happens in Theorem 3.4. For example, the dimension can be affected by certain number-theoretical properties. Namely, consider a geometric construction in $\mathbb{R}^2$ modeled by Σ_2^+ such that the sets

$$\Delta_{i_1 \cdots i_n} = (f_{i_1} \circ \cdots \circ f_{i_n})([0, 1] \times [0, 1])$$

are rectangles with sides of length a^n and b^n, obtained from the composition of the functions

$$f_1(x, y) = (ax, by) \quad \text{and} \quad f_2(x, y) = (ax - a + 1, by - b + 1),$$

for some fixed constants $a \in (0, 1)$ and $b \in (0, 1/2)$. In particular, the projection of $\Delta_{i_1 \cdots i_n}$ on the horizontal axis is an interval with right endpoint given by

$$a^n + \sum_{k=0}^{n-1} j_k a^k, \tag{24}$$

where

$$j_k = \begin{cases} 0 & \text{if } i_k = 1, \\ 1 - a & \text{if } i_k = 2. \end{cases}$$

Now we assume that $a = (\sqrt{5} - 1)/2$. In this case we have $a^2 + a = 1$, and thus for each $n > 2$ there is more than one vector $(i_1 \cdots i_n)$ for which we obtain the same value in (24). This causes a larger concentration of the sets $\Delta_{i_1 \cdots i_n}$ in certain regions of the limit set J. Therefore, to compute the Hausdorff dimension, when we take an open cover of J it may happen that it is possible to replace, in the regions of larger concentration of the sets $\Delta_{i_1 \cdots i_n}$, several elements of the cover by a single element. This may cause J to have a Hausdorff dimension strictly smaller than the box dimension. It was established by Neunhäuserer in [**69**] that this does indeed happen when $a = (\sqrt{5} - 1)/2$. See also [**76**] for earlier related results of Przytycki and Urbański. We note that the constant $(\sqrt{5} - 1)/2$ is only an example among many other possible values that lead to a similar phenomenon. McMullen [**67**] and Gatzouras and Lalley [**35**] also obtained different Hausdorff and box dimensions for particular classes of self-affine sets. Moreover, the s-dimensional Hausdorff measure with $s = \dim_H J$ need be neither positive nor finite [**35, 73**]. Finally, for a smoothly parameterized family of self-affine sets the Hausdorff dimension may not vary continuously with the parameters [**25, 35, 76**].

It should be noted that Theorem 3.6 gives no indication of which sets J actually have dimension equal to s. In [**48**] Hueter and Lalley gave sufficient conditions for a given set $J \subset \mathbb{R}^2$ to satisfy (23). Let $f_1, \ldots, f_p \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformations in Theorem 3.6, which we continue to write in the form $f_i(x) = A_i x + b_i$. Then the following statement holds.

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THEOREM 3.8. [**48**] *Assume that:*

(1) $\|A_i\| < 1$ *and* $\sigma_1(A_i)^2 < \sigma_2(A_i)$ *for* $i = 1, \ldots, p$;

(2) *the sets* $A_1^{-1}Q, \ldots, A_p^{-1}Q$ *are disjoint, where* Q *is the closed second quadrant;*

(3) *the sets* $A_1 J, \ldots, A_p J$ *are disjoint.*

Then (23) holds. Moreover, the s*-dimensional Hausdorff measure of* J *is finite, and there exists a unique ergodic measure of full dimension in* J.

We note that all the hypotheses in the theorem persist under sufficiently small perturbations of the entries of the matrices $A_1, \ldots, A_p$.

A nonlinear extension of the work of Hueter and Lalley was obtained by Luzia in [**59**]. Let $f_1, \ldots, f_p : \mathbb{R}^2 \to \mathbb{R}^2$ be C^2 diffeomorphisms such that:

(1) $\sup_{x \in \mathbb{R}^2} \|d_x f_i\| < 1$ for $i = 1, \ldots, p$;

(2) there is a convex bounded open set U such that $f_1(\overline{U}), \ldots, f_p(\overline{U})$ are pairwise disjoint subsets of U;

(3) $d_x f_i P \subset \operatorname{int} P$ for every $x \in U$, where P is the union of the closed first and third quadrants;

(4) $\|d_x f_i v\|^3 / |\det d_x f_i| < 1$ for every $x \in U$ and $v \in P$ with $\|v\| = 1$.

Then there exists a unique non-empty compact set satisfying (22). We also consider the function $\varphi : \{1, \ldots, p\}^{\mathbb{N}} \to \mathbb{R}$ defined by

$$\varphi(i_1 i_2 \cdots) = \log \|d_{\pi(i_2 i_3 \cdots)} f_{i_1} | V\|,$$

where

$$\pi(i_2 i_3 \cdots) = \lim_{n \to \infty} (f_{i_2} \circ \cdots \circ f_{i_n})(\overline{U})$$

and

$$V = \lim_{n \to \infty} d_{(f_{i_1} \circ \cdots \circ f_{i_n})^{-1} \pi(i_1 i_2 \cdots)} (f_{i_1} \circ \cdots \circ f_{i_n}) P.$$

Then the following statement holds.

THEOREM 3.9. [**59**] *We have*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s,$$

where s *is the unique root of the equation* $P(s\varphi) = 0$. *Moreover, the* s*-dimensional Hausdorff measure of* Λ *is finite, and there is a unique ergodic measure of full dimension in* Λ.

3.6. *Measures of full dimension.* We now consider the related problem of the existence of measures of full dimension.

We say that an invariant measure μ in a repeller J is of *full dimension* if $\dim_H \mu = \dim_H J$. It was shown by Ruelle in [**80**] that for a map of class $C^{1+\alpha}$ which is conformal and topologically mixing on J, if μ is the unique equilibrium measure in Theorem 3.4, then μ is of full dimension. This follows from the equivalence between μ and the s-dimensional Hausdorff measure in J. The existence of an ergodic measure of full dimension for any repeller of a C^1 map was established by Gatzouras and Peres in [**36**].

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The situation is much more complicated in the case of non-conformal dynamics, and there exist only some partial results. In particular, Gatzouras and Peres [**36**] considered maps of the form

$$f(x_1, x_2) = (f_1(x_1),\ f_2(x_2)),$$

where f_1 and f_2 are C^1 maps with repellers respectively J_1 and J_2 such that $f_1|J_1$ and $f_2|J_2$ are conformal. We note that $f|(J_1 \times J_2)$ is a factor of a topological Markov chain, and we denote the factor map by π. They obtained the following result.

THEOREM 3.10. *If*

$$\min_{x_1 \in J_1} \|d_{x_1} f_1\| \geq \max_{x_2 \in J_2} \|d_{x_2} f_2\|,$$

then for any compact f-invariant set $J \subset J_1 \times J_2$ such that $\pi^{-1}J$ satisfies specification we have

$$\dim_H J = \sup_{\mu} \dim_H \mu,$$

where the supremum is taken over all ergodic f-invariant probability measures in J.

For piecewise linear maps f_i, Gatzouras and Lalley [**35**] proved earlier that certain invariant sets, corresponding to full shifts in the symbolic dynamics, carry an ergodic measure of full dimension. Kenyon and Peres [**54**] obtained the same result for linear maps f_i and arbitrary compact invariant sets. The proof of Theorem 3.10 uses the result in [**35**] and the idea from [**54**] of approximating arbitrary invariant sets by self-affine sets (corresponding to full shifts in their symbolic dynamics). Bedford and Urbański considered a particular class of self-affine sets in [**10**] and obtained conditions for the existence of a measure of full dimension.

Earlier related ideas appeared in work by Bedford [**8**] and McMullen [**67**]. We briefly describe their results. Given integers $m \leq n$ and a set $D \subset \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$, we define

$$J = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} n^{-k} & 0 \\ 0 & m^{-k} \end{pmatrix} d_k : d_k \in D \text{ for each } k \in \mathbb{N} \right\}.$$

Then J is called a *general Sierpiński carpet*. We note that, viewed as a subset of the 2-torus $\mathbb{T}^2$, the set J is invariant under the map $\left(\begin{smallmatrix} n & 0 \\ 0 & m \end{smallmatrix}\right)$. Bedford and McMullen obtained the following result independently.

THEOREM 3.11. *There exists an ergodic measure of full dimension in J.*

The following higher-dimensional generalization was obtained by Kenyon and Peres [**54**].

THEOREM 3.12. *If $J \subset \mathbb{T}^m$ is invariant under a toral endomorphism whose eigenvalues are roots of integers, then there exists an ergodic measure of full dimension in J.*

More recently, Yayama [**95**] considered general Sierpiński carpets modeled by arbitrary topological Markov chains, and she gave conditions for the existence of a unique measure of full dimension. In another direction, Luzia [**60**] considered expanding maps of the

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2-torus of the form $f(x, y) = (a(x, y), b(y))$ that are C^2-perturbations of linear maps. He showed that if f is sufficiently C^2-close to a general Sierpiński carpet, then

$$\dim_H J = \sup_{\mu} \dim_H \mu, \tag{25}$$

where the supremum is taken over all ergodic f-invariant probability measures in J. He also showed in [**61**] that the supremum in (25) is attained.

3.7. *Non-uniformly expanding repellers.* Only a few related results exist in the case of transformations that are not uniformly expanding.

We first formulate a result established by Gelfert in [**38**]. For each $s \in [0, \dim M]$, let

$$\varphi^s(x) = \log \omega_s((d_x f)^{-1})$$

with ω_s as in (18).

THEOREM 3.13. *If J is a compact invariant set of an expansive C^1 local diffeomorphism f, then*

$$\overline{\dim}_B J \le \inf\left\{ s \in (0, \dim M] : \sup_{x \in J} \varphi^s(x) < 1 \text{ and } P(\varphi^s) < 0 \right\}.$$

The proof is analogous to that of the upper bound in Theorem 3.3, and uses the characterization of the singular values as semiaxes of ellipsoids as well as covering techniques in the spirit of [**19**]. We note that the proof in [**38**] contains a mistake, although the statement remains true under the additional hypothesis of expansivity.

In a different direction, Horita and Viana [**45**] and Dysman [**20**] studied abstract models, called maps with holes, which include examples of non-uniform repellers. Namely, let $f : M \to M$ be a map such that there exist domains (that is, compact path-connected sets) $R_1, \ldots, R_p \subset M$ with pairwise disjoint interiors, and such that $\overline{\dim}_B \partial R_i < \dim M$ for $i = 1, \ldots, p$. We also assume that each restriction $f|R_i$ is a $C^{1+\alpha}$ diffeomorphism onto some domain W_i containing $R_1 \cup \cdots \cup R_p$. Moreover, we require that the holes

$$H_i = W_i \setminus (R_1 \cup \cdots \cup R_p)$$

have non-empty interior, and that the inner diameter (the supremum of the inner distances between any two points in the same connected component) of each set R_i is finite. The *repeller* of a map f with holes is the set of points whose forward orbit never falls into the holes, that is,

$$J = \{x \in M : f^k(x) \in R_1 \cup \cdots \cup R_p \text{ for } k \in \mathbb{N}\}.$$

Given $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, p\}$, we set

$$C_{i_1 \cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} R_{i_{k+1}}$$

and

$$\varphi_n(i_1 \cdots i_n) = \frac{1}{n} \sum_{k=1}^{n} \inf_{x \in C_{i_1 \cdots i_k}} \log \|d_{f^k(x)} f^{-1}\|^{-1}.$$

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Now we assume that there exist constants $\beta, \gamma > 0$ such that

$$\sum_{i_1 \cdots i_n} m(C_{i_1 \cdots i_n}) \leq e^{-\beta n}$$

for any sufficiently large n, where the sum is taken over all n-tuples $i_1 \cdots i_n$ such that $\varphi_n(i_1 \cdots i_n) \leq \gamma$, and where m denotes the Lebesgue measure. That is, although the expansion in J need not be uniform, the Lebesgue measure of the set of points remaining within a small neighborhood of J decreases exponentially fast with time. Under these assumptions, Dysman [**20**] showed that $\overline{\dim}_B J < \dim M$. This inequality generalizes a corresponding result for the Hausdorff dimension by Horita and Viana in [**45**] obtained under the same assumptions.

In [**46**] Horita and Viana investigate non-uniformly expanding repellers emerging from a perturbation of an Anosov diffeomorphism f of the three-dimensional torus through a Hopf bifurcation. A saddle fixed point becomes an attractor, and the complement of its basin of attraction can be considered a repeller for the new diffeomorphism. We note that since the repeller contains an invariant circle produced by the bifurcation it is not uniformly expanding. Based on the above result of Dysman it is shown in [**46**] that the box dimension of the repeller is strictly less than three for all diffeomorphisms sufficiently C^5-close to f but different from f.

3.8. *Upper bounds involving Lyapunov exponents.* We now briefly describe some upper bounds involving Lyapunov exponents. Related ideas go back to Kaplan and Yorke in [**53**]. The *local Lyapunov exponents* $\nu_1(x) \geq \cdots \geq \nu_n(x)$ of a smooth map $f\colon M \to M$ of a n-dimensional manifold are defined recursively by

$$\nu_1(x) + \cdots + \nu_j(x) = \limsup_{m \to \infty} \frac{1}{m} \log \omega_j(d_x f^m), \quad j = 1, \ldots, n.$$

We consider an ergodic f-invariant probability measure μ in M. Then the local Lyapunov exponents are constant μ-almost everywhere, and we denote them by $\nu_1(\mu) \geq \cdots \geq \nu_n(\mu)$. The *Lyapunov dimension* with respect to μ is defined by

$$\dim_L(f, \mu) = k + \frac{\nu_1(\mu) + \cdots + \nu_k(\mu)}{|\nu_{k+1}(\mu)|},$$

where $k < n$ is the smallest integer for which $\nu_1(\mu) + \cdots + \nu_{k+1}(\mu) < 0$. Ledrappier established the following statement in [**55**].

THEOREM 3.14. *If J is a compact f-invariant set, then*

$$\sup_{\mu} \dim_L(f, \mu) = \inf_{m \in \mathbb{N}} \dim_L(f^m, J),$$

where the supremum is taken over all ergodic f-invariant probability measures in J.

Therefore, it follows from (20) that

$$\dim_H J \leq \sup_{\mu} \dim_L(f, \mu).$$

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There are examples where the supremum coincides with the Hausdorff dimension. Let us consider a C^2 diffeomorphism f on a compact manifold and an ergodic f-invariant probability measure μ in M. It follows from work of Ledrappier and Young in [**56**] that if μ is an SRB measure, then

$$\dim_H J = \sup_{\mu} \dim_L(f, \mu). \tag{26}$$

However, as illustrated by Example 3.16 below, in general (26) does not hold.

We now consider the so-called *uniform Lyapunov exponents* $\nu_1^u \geq \cdots \geq \nu_n^u$ defined recursively by

$$\nu_1^u + \cdots + \nu_j^u = \lim_{m\to\infty} \frac{1}{m} \log \max_{x\in J} \omega_j(d_x f^m), \quad j = 1, \ldots, n,$$

and we define

$$d^u(f, J) = k + \frac{\nu_1^u + \cdots + \nu_k^u}{|\nu_{k+1}^u|},$$

where $k < n$ is the smallest integer for which $\nu_1^u + \cdots + \nu_{k+1}^u < 0$. We note that

$$d^u(f, J) \leq \inf_{m\in\mathbb{N}} \dim_L(f^m, J).$$

Temam [**88**] obtained the following result.

THEOREM 3.15. *If f is a C^1 map, and J is a compact f-invariant set, then*

$$\overline{\dim}_B \Lambda \leq d^u(f, J).$$

We also refer to [**88**] for generalizations to the infinite-dimensional setting, and for applications to attractors of many physical systems. See [**21–24**] for further developments. Related problems were considered by Blinchevskaya and Ilyashenko [**12**] and Chapyzhov and Ilyin [**17**].

Leonov [**58**] developed techniques to compute the Lyapunov dimension involving Lyapunov functions. These can be interpreted as changes of Riemannian metrics. For more details see [**37, 58**]. We give an example.

Example 3.16. Given constants $\alpha > 0$, $\beta \in (0, 1)$, we consider the Hénon map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(x, y) \mapsto (\alpha + \beta y - x^2, x).$$

This is conjugate to the map $(x, y) \mapsto (1 + y - \alpha x^2, \beta x)$ by the linear transformation $L(x, y) = (\alpha x, \alpha\beta^{-1} y)$. We consider the function $\psi(x, y) = \gamma(x + \beta y)$, where

$$\gamma = \frac{1 - s}{(\beta - 1 - 2x_0)\sqrt{x_0^2 + \beta}}, \quad x_0 = \frac{\beta - 1 - \sqrt{(\beta - 1)^2 + 4\alpha}}{2}$$

and $s \in (0, 1)$ is a parameter. We introduce a metric g in $\mathbb{R}^2$ by

$$g_{(x,y)}(v, w) = \exp\left(\frac{1}{1 + s}\psi(x, y)\right)\langle Av, w\rangle, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix},$$

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where the inner product $\langle \cdot, \cdot \rangle$ is induced by the Euclidean metric. The singular values of $d_{(x,y)}f$ with respect to the Riemannian structure are

$$\sigma_i(d_{(x,y)}f) = \exp\left(\frac{2}{1+s}(\psi(f(x,\,y)) - \psi(x,\,y))\right)\zeta_i(x,\,y),$$

where $\zeta_i(x,\,y)$ are the singular values of the matrix $\sqrt{A}d_{(x,y)}f\sqrt{A^{-1}}$. For

$$s = \frac{1}{1 - \log\beta/\log\zeta_1(x_0,\,x_0)}$$

one can show (see [**58**]) that

$$1 = \omega_{1+s}(d_{p_0}f) = \sup_{(x,y)\in\mathbb{R}^2} \omega_{1+s}(d_{(x,y)}f), \tag{27}$$

where $p_0 = (x_0,\,x_0)$ is a fixed point of f. For this fixed point we obtain

$$1 + s = \dim_L(f,\,p_0) \le d^u(f,\,J)$$

for any f-invariant compact set $J \subset \mathbb{R}^2$ containing p_0. Hence, it follows from (27) that

$$1 + s = d^u(f,\,J) = \lim_{m\to\infty} \dim_L(f^m,\,J) = \dim_L(f,\,J).$$

4. *Dimension estimates for hyperbolic sets*

We consider in this section the dimension of hyperbolic sets, and we describe a theory corresponding to the one in §3 in the case of repellers. In particular, besides describing how Markov partitions can be used to model hyperbolic sets and considering the well-understood case of conformal dynamics, we discuss the existing results for non-conformal and non-uniformly hyperbolic dynamics. We also consider the problem of the existence of measures of maximal dimension.

4.1. *Notion of hyperbolic set and examples.*

Let $f\colon M \to M$ be a diffeomorphism, and let $\Lambda \subset M$ be a compact f-invariant set. We say that Λ is a *hyperbolic set* for f if for every point $x \in \Lambda$ there exists a decomposition of the tangent space

$$T_xM = E^s(x) \oplus E^u(x)$$

such that

$$d_xf\,E^s(x) = E^s(f(x)) \quad \text{and} \quad d_xf\,E^u(x) = E^u(f(x)),$$

and there exist constants $\lambda \in (0,\,1)$ and $c > 0$ such that

$$\|d_xf^n|E^s(x)\| \le c\lambda^n \quad \text{and} \quad \|d_xf^{-n}|E^u(x)\| \le c\lambda^n$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$.

Example 4.1. (Linear horseshoes in $\mathbb{R}^2$) Consider a C^1 map contracting the unit square $Q = [0,\,1]^2 \subset \mathbb{R}^2$ horizontally by a factor $\lambda < 1/2$, expanding it vertically by a factor $1/\lambda$, and folding the resulting rectangle into a horseshoe. We assume that the set $f(Q) \cap Q$

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has two vertical components V_1 and V_2, and that the set $Q \cap f^{-1}(Q)$ has two horizontal components H_1 and H_2. The compact f-invariant set $\Lambda \subset Q$ given by

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n Q$$

is called a *Smale horseshoe*, and it is a hyperbolic set for f with respect to the decomposition of the tangent space into horizontal and vertical lines $T_x\mathbb{R}^2 = E^s(x) \oplus E^u(x)$, where $E^s(x) = \mathbb{R}$ is the horizontal line and $E^u(x) = \mathbb{R}$ is the vertical line, for each $x \in \Lambda$.

Example 4.2. (Solenoids) Let D be the unit disc in $\mathbb{R}^2$ centered at the origin. Consider the map $f\colon S^1 \times D \to S^1 \times D$ given by

$$f(\theta, x, y) = (2\theta \bmod 1, \lambda x + \varepsilon \cos(2\pi\theta), \mu y + \varepsilon \sin(2\pi\theta)),$$

for some constants $\lambda, \mu < \min\{\varepsilon, 1/2\}$. The limit set

$$\Lambda = \bigcap_{n\in\mathbb{N}} f^n(S^1 \times D)$$

is called a *solenoid*. One can easily verify that Λ is a hyperbolic set for f. Furthermore, Λ is an attractor, in the sense that there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(U)$.

4.2. *Additional structure.* Any hyperbolic set Λ has a very rich structure. Given $\varepsilon > 0$, for each $x \in \Lambda$ we consider the sets

$$V^s(x) = \{y \in B(x, \varepsilon) : d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \geq 0\}$$

and

$$V^u(x) = \{y \in B(x, \varepsilon) : d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \leq 0\},$$

where d is the distance in M and $B(x, \varepsilon) \subset M$ is the open ball of radius ε centered at x. If Λ is a hyperbolic set for a C^1 diffeomorphism, then there exists $\varepsilon > 0$ such that for each $x \in \Lambda$ the sets $V^s(x)$ and $V^u(x)$ are smooth manifolds containing x, and satisfying

$$T_x V^s(x) = E^s(x) \quad \text{and} \quad T_x V^u(x) = E^u(x).$$

$V^s(x)$ and $V^u(x)$ are respectively called the *local stable manifold* and the *local unstable manifold* at x (of size ε). Furthermore, there exists $\delta = \delta(\varepsilon) > 0$ such that if $d(x, y) < \delta$ for some $x, y \in \Lambda$, then the intersection $V^s(x) \cap V^u(y)$ has exactly one point. The function

$$[\cdot, \cdot]\colon \{(x, y) \in \Lambda \times \Lambda : d(x, y) < \delta\} \to M$$

defined by $[x, y] = V^s(x) \cap V^u(y)$ is called a *product structure*.

We say that a set $\Lambda \subset M$ is *locally maximal* if there is an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U.$$

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In this case there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in \Lambda$ with $d(x, y) < \delta$ we have card $[x, y] = 1$ and $[x, y] \in \Lambda$. A non-empty closed set $R \subset \Lambda$ is called a *rectangle* if diam $R < \delta$, $R = \overline{\operatorname{int} R}$ (with the interior computed with respect to the induced topology on Λ), and $[x, y] \in R$ whenever $x, y \in R$. A finite cover of Λ by rectangles $R_1, \ldots, R_p$ is called a *Markov partition* of Λ (with respect to f) if:

(1) $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ whenever $i \neq j$;

(2) if $x \in \operatorname{int} R_i \cap f^{-1}(\operatorname{int} R_j)$, then

$$f(V^s(x) \cap R_i) \subset V^s(f(x)) \cap R_j$$

and

$$V^u(f(x)) \cap R_j \subset f(V^u(x) \cap R_i).$$

Any locally maximal hyperbolic set has Markov partitions with arbitrarily small diameter (see [**15**]).

Markov partitions yield symbolic models for the hyperbolic set. Let Λ be a locally maximal hyperbolic set, and let $R_1, \ldots, R_p$ be a Markov partition of Λ. We define a $p \times p$ matrix $A = (a_{ij})$ with entries a_{ij} as in (11). We also consider the space of two-sided sequences $\Sigma_p = \{1, \ldots, p\}^{\mathbb{Z}}$, and the corresponding shift map $\sigma : \Sigma_p \to \Sigma_p$. The restriction of σ to the σ-invariant set

$$\Sigma_A = \{(\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_p : a_{i_n i_{n+1}} = 1 \text{ for every } n \in \mathbb{Z}\}$$

is called the *(two-sided) topological Markov chain* with *transition matrix* A. In a similar manner to that in the case of repellers, one can define a map $\chi : \Sigma_A \to \Lambda$ by

$$\chi(\cdots i_{-1} i_0 i_1 \cdots) = \bigcap_{k \in \mathbb{Z}} f^{-k} R_{i_k}.$$

The map χ is surjective, Hölder continuous with respect to the distance

$$d(\cdots i_0 \cdots, \cdots j_0 \cdots) = \sum_{k \in \mathbb{Z}} e^{-|k|} |i_k - j_k|,$$

and it satisfies $\chi \circ \sigma = f \circ \chi$.

4.3. *A priori bounds for the dimension.* The following upper bound for the box dimension of a hyperbolic set was obtained by Fathi in [**32**]. Compared to the sharper dimension estimates discussed below it has the advantage of not using the topological pressure.

THEOREM 4.3. [**32**] *If Λ is a hyperbolic set for a C^1 diffeomorphism f, then $\overline{\dim}_B \Lambda \leq -2h(f)/\lambda$, where*

$$\lambda = \max\left\{ \lim_{n \to \infty} \frac{1}{n} \log \max_{x \in \Lambda} \|d_x f^n | E^s\|, \lim_{n \to \infty} \frac{1}{n} \log \max_{x \in \Lambda} \|(d_x f^n)^{-1} | E^u\| \right\}.$$

Now we describe sharper upper as well as lower dimension estimates. Given a locally maximal hyperbolic set Λ for a C^1 diffeomorphism f, we define functions $\underline{\varphi}_s$, $\overline{\varphi}_s : \Lambda \to \mathbb{R}$ by

$$\underline{\varphi}_s(x) = -\log \|(d_x f)^{-1} | E^s\| \quad \text{and} \quad \overline{\varphi}_s(x) = \log \|d_x f | E^s\|, \tag{28}$$

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and $\underline{\varphi}_u, \overline{\varphi}_u \colon \Lambda \to \mathbb{R}$ by

$$\underline{\varphi}_u(x) = -\log \|d_x f|E^u\| \quad \text{and} \quad \overline{\varphi}_u(x) = \log \|(d_x f)^{-1}|E^u\|. \tag{29}$$

Let $R_1, \ldots, R_p$ be a Markov partition of Λ. Given a point $x \in \Lambda$ and an element R_{i_0} of the Markov partition containing x, we set

$$A^s(x) = V^s(x) \cap R_{i_0} \quad \text{and} \quad A^u(x) = V^u(x) \cap R_{i_0}.$$

We denote by $\underline{t}_s(x)$ and $\overline{t}_s(x)$ the unique roots of the equations

$$P_{f|A^s(x)}(t\underline{\varphi}_s) = 0 \quad \text{and} \quad P_{f|A^s(x)}(t\overline{\varphi}_s) = 0,$$

and by $\underline{t}_u(x)$ and $\overline{t}_u(x)$ the unique roots of the equations

$$P_{f|A^u(x)}(t\underline{\varphi}_u) = 0 \quad \text{and} \quad P_{f|A^u(x)}(t\overline{\varphi}_u) = 0.$$

Here $P_Z(\varphi)$ denotes the non-additive topological pressure on the set Z of the additive sequence of functions $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$. The following result was established by Barreira in [**2**].

THEOREM 4.4. *If Λ is a locally maximal hyperbolic set of a topologically transitive C^1 diffeomorphism, then for every $x \in \Lambda$ we have*

$$\underline{t}_s(x) \le \dim_H(V^s(x) \cap \Lambda) \le \underline{\dim}_B(V^s(x) \cap \Lambda) \le \overline{\dim}_B(V^s(x) \cap \Lambda) \le \overline{t}_s(x)$$

and

$$\underline{t}_u(x) \le \dim_H(V^u(x) \cap \Lambda) \le \underline{\dim}_B(V^u(x) \cap \Lambda) \le \overline{\dim}_B(V^u(x) \cap \Lambda) \le \overline{t}_u(x).$$

The proof of Theorem 4.4 is an elaboration of the proof of Theorem 3.3.

We also describe a result of Franz in [**34**]. Let $\Lambda \subset M$ be a compact f-invariant set for a C^1 diffeomorphism f with an equivariant splitting $T_\Lambda M = E^1 \oplus E^2$. This means that

$$d_x f E^i(x) = E^i(f(x)), \quad i = 1, 2,$$

for every $x \in \Lambda$. We note that Λ is not necessarily hyperbolic. We relabel the singular values

$$\sigma_1(d_x f|E^1), \ldots, \sigma_{\dim E^1}(d_x f|E^1), \sigma_1((d_y f|E^2)^{-1}), \ldots, \sigma_{\dim E^2}((d_y f|E^2)^{-1})$$

as $\sigma_1(x, y) \ge \cdots \ge \sigma_{\dim M}(x, y)$, and for each $s \in [0, \dim M]$ we write

$$\overline{\omega}_s(x, y) = \sigma_1(x, y) \cdots \sigma_{\lfloor s \rfloor}(x, y)\sigma_{\lfloor s \rfloor + 1}(x, y)^{s - \lfloor s \rfloor}.$$

Franz showed in [**34**] that

$$\dim_H \Lambda \le \inf\left\{ s \in [0, \dim M] : \log \max_{x, y \in \Lambda} \overline{\omega}_s(x, y) < -2h(f|\Lambda) \right\},$$

where $h(f|\Lambda)$ denotes the topological entropy on Λ. This result generalizes work of Gu in [**39**] for hyperbolic sets of C^2 maps satisfying a certain pinching condition.

The following result requires the existence of an equivariant subbundle with respect to which f is volume-expanding. It was established by Gelfert in [**38**]. We recall the function ω_s in (18).

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THEOREM 4.5. *If Λ is a compact f-invariant set of an expansive C^1 local diffeomorphism f, and $E \subset T_\Lambda M$ is an equivariant bundle, then*

$$\overline{\dim}_B \Lambda \le \operatorname{codim} E + \inf\Big\{ s \in [0, \operatorname{rank} E] : \max_{x\in\Lambda} \omega_s((d_x f|E)^{-1}) < 1 \text{ and } P(\log \omega_s((d_x f|E)^{-1})) < 0 \Big\}$$

(with the convention $\inf \emptyset = \operatorname{rank} E$*).*

Shafikov and Wolf showed in [**82**] that if Λ is a hyperbolic set for a C^2 diffeomorphism $f\colon M \to M$, then

$$\dim_H \Lambda \le \dim M + \frac{P(-\log|\det df|E^u|)}{\lambda}, \tag{30}$$

where

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in\Lambda} \|d_x f^n\|.$$

See [**90**] for the case of Axiom A endomorphisms. We note that (30) follows from Theorem 4.5.

4.4. *The conformal case.* We discuss in this section the particular case of hyperbolic sets of conformal maps, where the inequalities in Theorem 4.4 become identities.

We say that $f\colon M \to M$ is *conformal* on a hyperbolic set $\Lambda \subset M$ if $d_x f|E^s$ and $d_x f|E^u$ are multiples of isometries for every $x \in \Lambda$. For example, if M is a surface and $\dim E^s(x) = \dim E^u(x) = 1$ for every $x \in \Lambda$, then f is conformal on Λ. We note that in this case the functions in (28) and (29) satisfy

$$\varphi_s(x) := \underline{\varphi}_s(x) = \overline{\varphi}_s(x) = \log \|d_x f|E^s\| \tag{31}$$

and

$$\varphi_u(x) := \underline{\varphi}_u(x) = \overline{\varphi}_u(x) = -\log \|d_x f|E^u\|. \tag{32}$$

The following result is thus a consequence of Theorem 4.4.

THEOREM 4.6. *If Λ is a locally maximal hyperbolic set of a topologically transitive C^1 diffeomorphism which is conformal on Λ, then for every $x \in \Lambda$ we have*

$$\dim_H(V^s(x)\cap\Lambda) = \underline{\dim}_B(V^s(x)\cap\Lambda) = \overline{\dim}_B(V^s(x)\cap\Lambda) = t_s \tag{33}$$

and

$$\dim_H(V^u(x)\cap\Lambda) = \underline{\dim}_B(V^u(x)\cap\Lambda) = \overline{\dim}_B(V^u(x)\cap\Lambda) = t_u, \tag{34}$$

where t_s and t_u are the unique real numbers such that

$$P(t_s\varphi_s) = P(t_u\varphi_u) = 0.$$

McCluskey and Manning showed in [**66**] that

$$\dim_H(V^s(x)\cap\Lambda) = t_s \quad \text{and} \quad \dim_H(V^u(x)\cap\Lambda) = t_u$$

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for every $x \in \Lambda$. The equality between the Hausdorff dimension and the lower and upper box dimensions is due to Takens in [**87**] for C^2 diffeomorphisms (see also [**71**]) and to Palis and Viana in [**72**] in the general case. Barreira in [**2**] and Pesin in [**74**] presented alternative proofs based on the thermodynamic formalism. We emphasize that in higher dimensions the Hausdorff and box dimensions of a locally maximal hyperbolic set may not coincide.

Moreover we have the following statement.

THEOREM 4.7. *If Λ is a locally maximal hyperbolic set of a topologically transitive C^1 diffeomorphism which is conformal on Λ, then*

$$\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t_s + t_u.$$

The first complete argument establishing Theorem 4.7 in the case of surface diffeomorphisms was given by Barreira in [**2**]. The proof of Theorem 4.7 given by Pesin in [**74**] includes the general case of conformal diffeomorphisms on manifolds of arbitrary dimension (the statement can also be obtained from results in [**2**]). Under the assumptions in the theorem the product structure is a Lipschitz homeomorphism with Lipschitz inverse (see [**41**] for details), and thus Theorem 4.7 follows easily from Theorem 4.6 by adding the dimensions in the stable and unstable directions, which is possible due to the coincidence between the Hausdorff and box dimensions in (33) and (34) (otherwise the Lipschitz property would be of little help).

Palis and Viana in [**72**] established the continuous dependence of the dimension on the diffeomorphism. Let Λ be a locally maximal hyperbolic set of a C^1 surface diffeomorphism f with $\dim E^s = \dim E^u = 1$. Then there is a neighborhood $\mathcal{U}$ of f in the space of C^1 diffeomorphisms and a continuous map $\mathcal{U} \ni g \mapsto h_g$ to the space of all continuous embeddings $\Lambda_f \to M$ satisfying $h_f = \mathrm{id}$, such that $\Lambda_g = h_g(\Lambda_f)$ is a locally maximal hyperbolic set for g, with $f|\Lambda_f$ and $g|\Lambda_g$ topologically conjugate.

THEOREM 4.8. [**72**] *The function $g \mapsto \dim_H \Lambda_g$ is continuous.*

Mañé [**64**] obtained even higher regularity for the dimension.

THEOREM 4.9. *Let Λ be a locally maximal and totally disconnected hyperbolic set of a C^r surface diffeomorphism f, $r \geq 2$ with $\dim E^s = \dim E^u = 1$. Then there is a C^r neighborhood $\mathcal{U}$ of f and a C^r map $\mathcal{U} \ni g \mapsto h_g$ to the space of continuous embeddings $\Lambda_f \to M$ such that $g \mapsto \dim_H h_g(\Lambda_f)$ is of class C^{r-1}.*

In higher-dimensional manifolds (and thus in the non-conformal case) the Hausdorff dimension of hyperbolic sets may vary discontinuously. Examples were given by Pollicott and Weiss in [**75**], followed by Bonatti *et al* in [**13**]. Díaz and Viana [**18**] studied a one-parameter family of diffeomorphisms on the 2-torus bifurcating from an Anosov map to a DA (derived from Anosov) map. They showed that for an open set of these families the Hausdorff and box dimensions of the non-wandering set are discontinuous across the bifurcation.

4.5. *Measures of maximal dimension.* We also address the question of the existence of measures of full dimension for hyperbolic sets. It turns out that the answer is almost

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always negative. More precisely, by results of McCluskey and Manning in [**66**] there exists an invariant measure of full dimension if and only if there exists a continuous function $\psi : \Lambda \to \mathbb{R}$ such that

$$t_s\varphi_s - t_u\varphi_u = \psi \circ f - f$$

on Λ, with φ_s and φ_u as in (31) and (32). By Livschitz's theorem, this happens if and only if

$$\|d_x f|E^s(x)\|^{t_s}\|d_x f|E^u(x)\|^{t_u} = 1$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$ such that $f^n(x) = x$.

Instead one can ask whether the supremum

$$\delta(f) = \sup\{\dim_H \mu : \mu \text{ is an } f\text{-invariant measure on } \Lambda\}$$

is attained. Any f-invariant measure on Λ with $\dim_H \nu = \delta(f)$ is called a measure of *maximal dimension*. The main difficulty of this problem is that the map $\mu \mapsto \dim_H \mu$ is not upper semi-continuous: simply consider the sequence $(\mu + (n-1)\delta)/n$ where $\dim_H \mu > 0$ and δ is an atomic measure. The following result was obtained by Barreira and Wolf in [**7**].

THEOREM 4.10. *Let Λ be a locally maximal hyperbolic set for a $C^{1+\alpha}$ surface diffeomorphism which is topologically mixing on Λ. Then there exists an ergodic measure of maximal dimension.*

See [**94**] for a related result of Wolf in the particular case of polynomial automorphisms of $\mathbb{C}^2$. It was shown by Rams in [**77**] that a unique ergodic measure of maximal dimension does not exist in general, even in the case of linear horseshoes (more precisely, he showed that there exists a one-parameter family of Bernoulli measures of maximal dimension).

4.6. *Further developments.* Related ideas to those in the proof of Theorem 3.7 were applied by Simon and Solomyak in [**85**] to compute the Hausdorff dimension of a class of horseshoes in $\mathbb{R}^3$, obtained from $C^{1+\alpha}$ transformations of the form

$$(x, y, z) \mapsto (\varphi(x, z) + a_k, \psi(y, z) + b_k, \zeta(z)). \tag{35}$$

Assuming that $|\partial\varphi/\partial x|, |\partial\psi/\partial y| < 1/2$ and $|\zeta'| > 1$, it is shown in [**85**] how to compute the Hausdorff dimension of the horseshoe for Lebesgue almost every $(a, b) \in \mathbb{R}^2$. Again, the dimension is expressed implicitly in terms of the topological pressure.

More precisely, we assume that there are disjoint closed intervals $I_1, \ldots, I_p$ whose union is a proper subset of $[0, 1]$ such that, letting $\Delta_k = [0, 1]^2 \times I_k$, we have

$$f\left(\bigcup_{k=1}^{p} \Delta_k\right) \subset (0, 1)^2 \times [0, 1],$$

with f given by (35) in Δ_k. The limit set

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n([0, 1]^3)$$

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is the horseshoe. Now let s_1, s_2, r_1, and r_2 be the unique solutions of the equations

$$P\left(s_1 \log\left|\frac{\partial\varphi}{\partial x}\right|\right) = 0, \quad P\left(s_2 \log\left|\frac{\partial\psi}{\partial y}\right|\right) = 0,$$
$$P\left(\log\left(\left|\frac{\partial\varphi}{\partial x}\right| \cdot \left|\frac{\partial\psi}{\partial y}\right|^{r_1}\right)\right) = 0, \quad P\left(\log\left(\left|\frac{\partial\psi}{\partial y}\right| \cdot \left|\frac{\partial\varphi}{\partial x}\right|^{r_2}\right)\right) = 0.$$

We also set $s = \max\{s_1, s_2\}$ and $r = \max\{r_1, r_2\}$. Finally, let t be the unique solution of the equation

$$P(-t \log|\zeta'|) = 0.$$

Then we have the following statement.

THEOREM 4.11. [**85**] *For Lebesgue almost every vector* $(a_1, b_1, \ldots, a_p, b_p)$ *we have:*

(i) *if* $s \leq 1$ *then*

$$\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t + s;$$

(ii) *if* $s > 1$ *then*

$$\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = t + 1 + \min\{1, r\}.$$

In particular, the dimension does not depend on the vector.

Finally, we consider more general solenoids than those in Example 4.2. We recall that a *solenoid* is a hyperbolic set $\Lambda = \bigcap_{n=1}^{\infty} f^n T$, where $T \subset \mathbb{R}^3$ is diffeomorphic to a solid torus $S^1 \times D$ for some closed disk $D \subset \mathbb{R}^2$, and $f : T \to T$ extends to a diffeomorphism in some open neighborhood of T such that for each $x \in S^1$ the *section*

$$\Lambda_x = f(T) \cap (\{x\} \times D)$$

is a disjoint union of a fixed number of sets homeomorphic to a closed disk. Bothe [**14**] and then Simon [**84**] (also using his methods in [**83**] for non-invertible transformations) studied the dimension of solenoids (see [**74, 81**] for a related discussion). In particular, it is shown in [**14**] that under certain conditions on the diffeomorphism the map $x \mapsto \dim_H \Lambda_x$ is constant (even though the holonomies are typically not Lipschitz). More recently, Hasselblatt and Schmeling conjectured in [**43**] (see also [**42**]) that, in spite of the difficulties due to the possible low regularity of the holonomies, the Hausdorff dimension of hyperbolic sets can be computed by adding the dimensions of the stable and unstable sections. They prove this conjecture for a class of solenoids, by showing that the Hausdorff dimension $\dim_H \Lambda_x$ of the sections is in fact independent of x.

For some related results in the case of non-uniformly hyperbolic invariant sets we mention the works of Hirayama [**44**] with an upper bound for the Hausdorff dimension of the stable set of the set of typical points for a hyperbolic measure, Fan *et al* [**31**] with the study of the dimension of the maximal invariant set of an asymptotically non-hyperbolic family, and Urbánski and Wolf [**91**] who considered horseshoe maps that are uniformly hyperbolic except at a parabolic point, in particular establishing a dimension formula for the horseshoe.

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