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NOTES AND PROBLEMS

ON THE RELATION BETWEEN THE VEC AND BEKK MULTIVARIATE GARCH MODELS

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The question of which multivariate generalized autoregressive conditionally heteroskedastic (GARCH) models in the vec form are representable in the BEKK form is addressed. Using results from linear algebra, it is established that all vec models not representable in the simplest BEKK form contain matrices as parameters that map the vectorized positive semidefinite matrices into a strict subset of themselves. Moreover, a general result from linear algebra is presented implying that in dimension two the models are equivalent, and in dimension three a simple analytically tractable example for a vec model having no BEKK representation is given.

1. INTRODUCTION

Multivariate generalized autoregressive conditionally heteroskedastic (GARCH) models have been studied intensively in recent years, and many different specifications have been used in the literature (cf., for a comprehensive overview, Bauwens, Laurent, and Rombouts, 2006; for a detailed discussion on strict stationarity and geometric ergodicity, Boussama, 1998, 2006). In this paper we present some results on the relationship between the vec and BEKK models (the acronym BEKK comes from Baba, Engle, Kraft, and Kroner). These models have been presented and analyzed in detail in Engle and Kroner (1995). In that paper it has been noted that all BEKK models are representable as vec models but regarding the converse it has only been shown that all diagonal vec models are representable as diagonal BEKK models and stated that the BEKK parametrization "eliminates very few if any interesting models allowed by the vec representation." However, apart from the recent paper by Scherrer and Ribarits (2007), which came to our attention only after finishing the work on

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the present paper, no further results on the relationship between the two models seem to have been obtained since then, nor are there simple and tractable examples of vec models that are not representable in the BEKK form to be found in the literature.

Applying long-known results from linear algebra in a straightforward manner, we show in this paper that in dimension two the models are actually equivalent and that all vec models not representable in the simplest BEKK form with invertible parameter matrices necessarily exhibit some degeneracy, namely, that one of the matrices appearing in the vec model is degenerated in the sense that it maps the vectorized positive semidefinite matrices to a strict subset of themselves. Finally, we present an example of a vec model with no BEKK representation in dimension three. Scherrer and Ribarits (2007) have shown the equivalence in dimension two using semidefinite programming, whereas we note that it is an immediate consequence of a long-known result in linear algebra. The linear algebra literature we are referring to seems not to have been used in connection with GARCH models before, but it is obviously intimately connected to multivariate GARCH models and should be useful for obtaining other results also. For an example of a vec model having no BEKK representation Scherrer and Ribarits (2007) refer to Ribarits (2006). The example presented in that thesis on p. 61 (stated in a transformed way only) is of a rather complicated structure, and it is argued by numerical optimization and not an analytical proof that it gives an admissible vec term that cannot be represented in the BEKK form. In contrast to this we present a very simple example with interesting properties that is analyzed completely analytically.

The remainder of this paper is organized as follows. We briefly state the necessary definitions of multivariate GARCH models in the next section and then present our results in Section 3.

Regarding notation we denote the set of real $d \times d$ matrices by $M_d(\mathbb{R})$, the group of invertible $d \times d$ matrices by $GL_d(\mathbb{R})$, the linear subspace of symmetric matrices by $\mathbb{S}_d$, and the positive semidefinite cone by $\mathbb{S}_d^+$. Finally, A^T is the transpose of a matrix $A \in M_d(\mathbb{R})$.

2. MULTIVARIATE GARCH PROCESSES

The well-known single-dimensional GARCH(p,q) model introduced in Bollerslev (1986) is defined via an independent and identically distributed (i.i.d.) sequence $(\epsilon_n)_{n\in\mathbb{N}}$ and the equations

$$X_n = \sqrt{\sigma_n^2}\epsilon_n, \tag{1}$$

$$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i X_{n-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{n-j}^2 \tag{2}$$

for $n \in \mathbb{N}$. Moreover, the initial values $\sigma_0^2, \sigma_{-1}^2, \ldots, \sigma_{1-p}^2$ and the parameters $\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p$ are nonnegative, and $\alpha_0 > 0$. Here $X = (X_n)_{n\in\mathbb{N}}$ is referred to as a GARCH(p,q) process, and σ^2 is its latent conditional variance process.

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When one moves from a scalar X to a d-dimensional X, the variance process σ^2 becomes a $d \times d$ covariance matrix process Σ, and one uses the vec (or alternatively vech) transformation to specify the model. The vec transformation maps the $d \times d$ matrices bijectively to $\mathbb{R}^{d^2}$ by stacking the columns of a matrix below one another. This leads to the vec model (Engle and Kroner, 1995), which is given by

$$X_n = \Sigma_n^{1/2} \epsilon_n, \tag{3}$$

$$\operatorname{vec}(\Sigma_n) = \operatorname{vec}(C) + \sum_{i=1}^{q} \tilde{A}_i \operatorname{vec}(X_{n-i} X_{n-i}^T) + \sum_{j=1}^{p} \tilde{B}_j \operatorname{vec}(\Sigma_{n-j}) \tag{4}$$

for $n \in \mathbb{N}$ where $(\epsilon_n)_{n \in \mathbb{N}}$ is now an $\mathbb{R}^d$-valued i.i.d. sequence and $\Sigma_n^{1/2}$ denotes the unique positive semidefinite matrix whose square is Σ_n, i.e., $\Sigma_n^{1/2} \in \mathbb{S}_d^+$ and $\Sigma_n^{1/2} \Sigma_n^{1/2} = \Sigma_n$. To ensure the positive semidefiniteness of the process Σ the initial values and C have to be positive semidefinite, and $\tilde{A}_1, \ldots, \tilde{A}_q, \tilde{B}_1, \ldots, \tilde{B}_p$ need to be $d^2 \times d^2$ matrices mapping the vectorized positive semidefinite matrices into themselves.

For notational convenience we shall use not only the vec model in the following discussion but also an obviously equivalent specification defined directly on the symmetric matrices. This model, referred to as the "general d-dimensional GARCH(p,q) model" in what follows, is given by

$$X_n = \Sigma_n^{1/2} \epsilon_n, \tag{5}$$

$$\Sigma_n = C + \sum_{i=1}^{q} A_i X_{n-i} X_{n-i}^T + \sum_{j=1}^{p} B_j \Sigma_{n-j}. \tag{6}$$

The only difference from the vec model is that $A_1, \ldots, A_q$ and $B_1, \ldots, B_p$ are now linear operators from $\mathbb{S}_d$ to $\mathbb{S}_d$ that map the positive semidefinite $d \times d$ matrices into themselves, i.e., $A_i(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$ and $B_j(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$ for $i = 1, \ldots, q$ and $j = 1, \ldots, p$.

The restrictions on the linear operators A_i and B_j (or $\tilde{A}_i$ and $\tilde{B}_j$ in the vec model) necessary to ensure positive semidefiniteness led to the introduction of the so-called BEKK model (see again Engle and Kroner, 1995), which automatically ensures positive semidefiniteness:

$$X_n = \Sigma_n^{1/2} \epsilon_n, \tag{7}$$

$$\Sigma_n = C + \sum_{i=1}^{q} \sum_{k=1}^{l_i} \bar{A}_{i,k} X_{n-i} X_{n-i}^T \bar{A}_{i,k}^T + \sum_{j=1}^{p} \sum_{r=1}^{s_j} \bar{B}_{j,r} \Sigma_{n-j} \bar{B}_{j,r}^T, \tag{8}$$

where $\bar{A}_{i,k}, \bar{B}_{j,r}$ are now arbitrary elements of $M_d(\mathbb{R})$. It is standard that the BEKK model is equivalent to the vec model with $\tilde{A}_i = \sum_{k=1}^{l_i} \bar{A}_{i,k} \otimes \bar{A}_{i,k}$ and $\tilde{B}_j = \sum_{r=1}^{s_j} \bar{B}_{j,r} \otimes \bar{B}_{j,r}$ with $\otimes$ denoting the tensor (Kronecker) product.

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3. THE RELATIONSHIP BETWEEN THE VEC MODEL AND THE BEKK MODEL

From the definitions of the models it is clear that studying the relationships between the vec (or general) multivariate GARCH model and the BEKK model further is intrinsically related to characterizing the linear operators on $\mathbb{S}_d$ that map the positive semidefinite matrices into themselves. The latter has been studied for a long time in linear algebra under the general topic "Linear Preserver Problems" (see, e.g., the overview papers by Pierce, Lim, Loewy, Li, Tsing, McDonald, and Beasley, 1992; Li and Pierce, 2001). From the results obtained there we need the following ones.

PROPOSITION 3.1. *Let* $\mathbf{A}:\mathbb{S}_d \to \mathbb{S}_d$ *be a linear operator. Then*

(i) $\mathbf{A}(\mathbb{S}_d^+) = \mathbb{S}_d^+$, *if and only if there exists a matrix* $A \in GL_d(\mathbb{R})$ *such that* $\mathbf{A}$ *can be represented as* $X \mapsto AXA^T$;
(ii) for $d = 2$, $\mathbf{A}(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$, *if and only if there is an* $r \in \mathbb{N}$ *and* $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_r \in M_d(\mathbb{R})$ *such that* $\mathbf{A}$ *can be represented as*

$$X \mapsto \sum_{i=1}^{r} \bar{A}_i X \bar{A}_i^T.$$

Proof. Part (i) was initially proved in Schneider (1965); alternatively a more general proof in a Hilbert space context may be found in Li, Rodman, and Semrl (2003). Part (ii) was established in Størmer (1963) (cf. also Loewy, 1992). ■

From this we can immediately infer the relations between the general (or equivalently vec) multivariate GARCH model and the BEKK model:

THEOREM 3.2.

(i) For $d \leq 2$ *the general (or vec) multivariate GARCH model and the BEKK model are equivalent.*
(ii) Every general multivariate GARCH(p,q) model satisfying $A_i(\mathbb{S}_d^+) = \mathbb{S}_d^+$ *and* $B_j(\mathbb{S}_d^+) = \mathbb{S}_d^+$ *for* $i = 1,2,\ldots,q$ *and* $j = 1,2,\ldots,p$ *can be represented as a BEKK GARCH(p,q) model with* $l_i = s_j = 1 \ \forall \ i,j$ *and* $\bar{A}_{i,1}, \bar{B}_{j,1} \in GL_d(\mathbb{R})$.

For the vec model $A_i(\mathbb{S}_d^+) = \mathbb{S}_d^+$ and $B_j(\mathbb{S}_d^+) = \mathbb{S}_d^+$ for $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,q$ translates into demanding that $\tilde{A}_i$ and $\tilde{B}_j$ map the vectorized positive semidefinite matrices onto themselves.

The preceding result means that when a general (or vec) multivariate GARCH model does not have a BEKK representation with $l_i = s_j = 1 \ \forall \ i,j$ and invertible $\bar{A}_{i,1}, \bar{B}_{j,1}$ it has to be the case that A_i or B_j map the positive semidefinite

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matrices into a strict subset of themselves for some $i = 1, 2, \ldots, q$ or $j = 1, 2, \ldots, p$. This is a somehow degenerated situation, because it may imply that the distribution of $\Sigma_n - C$ for all $n \in \mathbb{N}$ (and thus any limiting or stationary distribution) is concentrated on a subset of the positive semidefinite matrices.

Let us now turn to providing an example for a vec model that cannot be represented in the BEKK form. Consider $d = 3$ and the linear operator $D : \mathbb{S}_d \to \mathbb{S}_d$ given by

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} + 2x_{22} & -x_{12} & -x_{13} \\ -x_{12} & x_{22} + 2x_{33} & -x_{23} \\ -x_{13} & -x_{23} & x_{33} + 2x_{11} \end{pmatrix}. \tag{9}$$

It has been shown by Choi (1975) that $D(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$ and that there exist no $r \in \mathbb{N}$ and $E_1, E_2, \ldots, E_r \in M_3(\mathbb{R})$ such that

$$DX = \sum_{i=1}^{r} E_i X E_i^T \quad \text{for all } X \in \mathbb{S}_d.$$

Hence, using D as some A_i or B_j gives a general three-dimensional multivariate GARCH model having no BEKK representation. Clearly this means that the corresponding vec models have no BEKK representation. Because the operator D was defined on $\mathbb{S}_d$ only and not on $M_d(\mathbb{R})$, the corresponding 9×9 matrix in the vec model is not unique, but the corresponding vec models are unique. One of the possible 9×9 matrices the operator D corresponds to is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It should be noted that D is an invertible linear operator, as can easily be seen, and that Proposition 3.1 implies $D(\mathbb{S}_d^+) \subset \mathbb{S}_d^+$. An example for a positive semidefinite matrix not being the image of another positive semidefinite matrix under D is

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = D \begin{pmatrix} 1/9 & 0 & 0 \\ 0 & 4/9 & 0 \\ 0 & 0 & -2/9 \end{pmatrix}.$$

So we have given an example showing the following proposition.

PROPOSITION 3.3. *For $d \geq 3$ there exist general (or vec) multivariate GARCH models that cannot be represented in the BEKK form.*

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