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Critical scaling profile for trees and connected subgraphs on the complete graph

Yucheng Liu and Gordon Slade

Abstract. We analyze generating functions for trees and for connected subgraphs on the complete graph, and identify a single scaling profile which applies for both generating functions in a critical window. Our motivation comes from the analysis of the finite-size scaling of lattice trees and lattice animals on a high-dimensional discrete torus, for which we conjecture that the identical profile applies in dimensions $d \geq 8$.

1 Main results

1.1 Results

The enumeration of trees and connected graphs has a long history. We are motivated by problems arising in the critical behavior of branched polymers in equilibrium statistical mechanics, to consider certain generating functions for the number of trees and connected subgraphs in the complete graph $\mathbb{K}_V$ on V labeled vertices. The vertices are labeled as $\mathbb{V} = \{0, \dots, V-1\}$ and the edge set is $\mathbb{E} = \{\{x, y\} : x, y \in \mathbb{V},\ x \neq y\}$. Our interest is in the asymptotic behaviour as $V \to \infty$.

We define *one-point functions*

$$G^t_{V,0}(p) = \sum_{T \ni 0} \left(\frac{p}{eV}\right)^{|T|}, \qquad G^a_{V,0}(p) = \sum_{A \ni 0} \left(\frac{p}{eV}\right)^{|A|}, \tag{1.1}$$

where the first sum is over all labeled trees T in $\mathbb{K}_V$ containing the vertex 0, the second sum is over all labeled connected subgraphs A containing 0, and $|T|$ and $|A|$ denote the number of edges in T and A. The division of p by eV is a normalisation to make $p = 1$ correspond to a critical value. We also study the *two-point functions*

$$G^t_{V,01}(p) = \sum_{T \ni 0,1} \left(\frac{p}{eV}\right)^{|T|}, \qquad G^a_{V,01}(p) = \sum_{A \ni 0,1} \left(\frac{p}{eV}\right)^{|A|}, \tag{1.2}$$

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where the sums now run over trees or connected subgraphs containing the distinct vertices $0, 1$. To avoid repetition, when a formula applies to both trees and connected subgraphs we often omit the superscripts t, a. With this convention, we define the *susceptibility*

$$\chi_V(p) = G_{V,0}(p) + (V-1)G_{V,01}(p). \tag{1.3}$$

We are particularly interested in values of p in a *critical window* $p = 1 + sV^{-1/2}$ around the critical point, with $s \in \mathbb{R}$.

We define the profile

$$I(s) = \frac{e}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}x^2 + sx} \frac{1}{\sqrt{x}} \mathrm{d}x \qquad (s \in \mathbb{R}). \tag{1.4}$$

The profile can be rewritten in terms of a *Faxén integral* [25, p. 332] as $I(s) = e\pi^{-1/2}2^{-5/4}\mathrm{Fi}(\frac{1}{2}, \frac{1}{4}; \sqrt{2}s)$, and its asymptotic behavior is given by [25, Example 7.3, p. 84] to be

$$I(s) \sim \begin{cases} e|2s|^{-1/2} & (s \to -\infty) \\ es^{-1/2}e^{s^2/2} & (s \to +\infty), \end{cases} \tag{1.5}$$

where $f \sim g$ means $\lim f/g = 1$. Our main result is the following theorem.

Theorem 1.1 *For both trees and connected subgraphs, and for all $s \in \mathbb{R}$, as $V \to \infty$ we have*

$$G_{V,01}(1 + sV^{-1/2}) \sim V^{-3/4} I(s), \tag{1.6}$$

$$\chi_V(1 + sV^{-1/2}) \sim V^{1/4} I(s). \tag{1.7}$$

The proof of Theorem 1.1 uses a uniform bound on the one-point function. The following theorem gives a statement that is more precise than a bound. It involves the principal branch W_0 of the Lambert function [5], which solves $W_0 e^{W_0} = z$ and has power series

$$W_0(z) = \sum_{n=1}^\infty \frac{(-n)^{n-1}}{n!} z^n. \tag{1.8}$$

The solution to $W_0 e^{W_0} = -1/e$ is achieved by the particular value $W_0(-1/e) = -1$.

Theorem 1.2 *For both trees and connected subgraphs, for all $s \geq 0$, and for all sequences p_V with $p_V \leq 1 + sV^{-1/2}$ and $\lim_{V\to\infty} p_V = p \in [0, 1]$,*

$$\lim_{V\to\infty} G_{V,0}(p_V) = \sum_{n=1}^\infty \frac{n^{n-1}}{n!} \Big(\frac{p}{e}\Big)^{n-1} = -\frac{e}{p} W_0\Big(-\frac{p}{e}\Big). \tag{1.9}$$

In particular, if $p = 1$ then $\lim_{V\to\infty} G_{V,0}(p_V) = e$.

Notation: We write $f \lesssim g$ if there is a $C > 0$ such that $f(x) \leq Cg(x)$ for all x of interest.

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1.2 Method of proof

To prove (1.6), it suffices to prove (1.7) and (1.9), since when $p = 1 + sV^{-1/2}$, by definition of χ_V we then have

$$G_{V,01} = \frac{\chi_V - G_{V,0}}{V-1} \sim \frac{\chi_V}{V}. \tag{1.10}$$

1.2.1 Trees

By Cayley's formula, the number of trees on n labeled vertices is n^{n-2}. By decomposing the sum defining $G^t_{V,0}(p)$ according to the number n of vertices in the tree, and by counting the number of ways to choose $n-1$ vertices other than 0, we have

$$G^t_{V,0}(p) = \sum_{n=1}^{V} \binom{V-1}{n-1} n^{n-2} \Big(\frac{p}{eV}\Big)^{n-1}. \tag{1.11}$$

Similarly, by counting the number of ways to choose $n-2$ vertices other than 0 and 1, we have

$$G^t_{V,01}(p) = \sum_{n=2}^{V} \binom{V-2}{n-2} n^{n-2} \Big(\frac{p}{eV}\Big)^{n-1}. \tag{1.12}$$

Since

$$\binom{V-1}{n-1} + (V-1)\binom{V-2}{n-2} = n\binom{V-1}{n-1}, \tag{1.13}$$

it follows from (1.3) that the susceptibility is given by

$$\chi^t_V(p) = \sum_{n=1}^{V} \binom{V-1}{n-1} n^{n-1} \Big(\frac{p}{eV}\Big)^{n-1}. \tag{1.14}$$

For trees, we prove Theorems 1.1 and 1.2 by directly analyzing the above series for χ^t_V and $G^t_{V,0}$. The profile $I(s)$ for $\chi^t_V(1 + sV^{-1/2})$ arises from a Riemann sum limit.

1.2.2 Connected subgraphs

For connected subgraphs, we will show that the contribution to $\chi^a_V, G^a_{V,01}$ from connected subgraphs with cycles is much smaller than the contribution from trees. Let $C(n, n-1+\ell)$ denote the number of connected graphs on n labeled vertices with exactly $n-1+\ell$ edges, i.e., with ℓ surplus edges. The surplus must be zero for $n = 1, 2$. For $n \geq 3$, we define the *surplus generating function*

$$S(n, z) = \sum_{\ell=1}^{\infty} C(n, n-1+\ell) z^{\ell}. \tag{1.15}$$

Note that terms in the above sum are zero unless $\ell \leq \binom{n}{2} - (n-1)$, and that the tree term ($\ell = 0$) is absent.

We decompose the sums defining $G^a_{V,0}$ and $G^a_{V,01}$ according to the number n of vertices in the connected subgraph, and we further distinguish whether or not the subgraph contains surplus edges. This leads to the decomposition

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$$G^a_{V,0}(p) = G^t_{V,0}(p) + \Delta_{V,0}(p), \tag{1.16}$$

$$\chi^a_V(p) = \chi^t_V(p) + \Delta_V(p), \tag{1.17}$$

with

$$\Delta_{V,0}(p) = \sum_{n=3}^{V} \binom{V-1}{n-1} S\Big(n, \frac{p}{eV}\Big)\Big(\frac{p}{eV}\Big)^{n-1}, \tag{1.18}$$

$$\Delta_V(p) = \sum_{n=3}^{V} \binom{V-1}{n-1} n S\Big(n, \frac{p}{eV}\Big)\Big(\frac{p}{eV}\Big)^{n-1}. \tag{1.19}$$

Given Theorems 1.1 and 1.2 for trees, we prove Theorems 1.1 and 1.2 for connected subgraphs by showing that, for all $s \in \mathbb{R}$,

$$\lim_{V\to\infty} \Delta_{V,0}(1 + sV^{-1/2}) = 0, \tag{1.20}$$

$$\lim_{V\to\infty} V^{-1/4}\Delta_V(1 + sV^{-1/2}) = 0. \tag{1.21}$$

The proof is more subtle than for trees and requires estimates on the surplus generating function. As we discuss later, a precise but cumbrous asymptotic formula for $C(n, n+k)$ is given in [3, Corollary 1]. We use that formula to prove the following useful explicit bound. By convention, $k^k = 1$ when $k = 0$.

Proposition 1.3 *Let $n \geq 3$ and $N = \binom{n}{2}$. For $0 \leq k \leq n$, we have*

$$C(n, n+k) \lesssim \binom{N}{n+k}\Big(\frac{2}{e}\Big)^n \Big(\frac{en}{k}\Big)^{k/2}. \tag{1.22}$$

Proposition 1.3 is most useful when the surplus $\ell = k + 1$ is small but of order n. This is a delicate region when controlling the surplus generating function, and the precise constant e in the last factor of (1.22) is important. For a larger surplus, we simply bound $C(n, n+k)$ by the total number of graphs (connected or not) on n vertices with $n + k$ edges, which is $\binom{N}{n+k}$. Together, these bounds provide enough control on $S(n, p/(eV))$ to prove (1.20) and (1.21).

1.3 Motivation

Theorem 1.1 is motivated by a broader emerging theory of finite-size scaling in statistical mechanical models above their upper critical dimensions. The theory involves a family of profiles expressed in terms of the functions

$$I_k(s) = \int_0^\infty x^k e^{-\frac{1}{4}x^4 - \frac{1}{2}sx^2}\,\mathrm{d}x \qquad (s \in \mathbb{R},\ k > -1). \tag{1.23}$$

A change of variables transforms the profile I of (1.4) into $I(s) = e2^{1/4}\pi^{-1/2}I_0(-\sqrt{2}s)$. The general theory is described in [21] with references to the extensive physics and mathematics literature.

Given an integer $d \geq 2$, infinite-volume models can be formulated on a transitive graph $\mathbb{G} = (\mathbb{Z}^d, \mathbb{E})$, whose edge set $\mathbb{E}$ has a finite number of edges containing the

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origin and is invariant under the symmetries of $\mathbb{Z}^d$. Above an *upper critical dimension* d_c, for many models it has been proven that the critical exponents that describe the critical behavior are the same as the corresponding exponents when the model is formulated on a regular tree or on the complete graph. The tree and complete graph settings are easy to analyze. Finite-volume models (with periodic boundary conditions) can instead be formulated on a discrete torus $\mathbb{G}_r = (\mathbb{T}_r^d, \mathbb{E}_r)$ of period r. At and above the upper critical dimension, the torus models are known or conjectured to have critical behavior analogous to that seen on the complete graph, with an interesting "plateau" phenomenon involving a universal profile which is often expressed in terms of I_k. The value of k depends on the model. Dimensions $d < d_c$ are conjectured to exhibit different scaling, with no plateau or profile.

Lattice trees and lattice animals: A *lattice animal* is a finite connected subgraph of $\mathbb{G}$, and a *lattice tree* is an acyclic lattice animal. The critical behavior of lattice trees and lattice animals is at least as difficult as is the case for the notoriously difficult self-avoiding walk. Despite significant interest from chemists and physicists for over half a century, due to applications to branched polymers [15], the critical behavior is understood mathematically only in dimensions $d > d_c = 8$. For $d > 8$, it has been proved using the lace expansion that for sufficiently large edge sets $\mathbb{E}$ (or for nearest-neighbor edges with d sufficiently large), lattice trees and lattice animals at the critical point both have the same behavior as a critical branching process [4, 6, 12, 13].

For $x \in \mathbb{Z}^d$, let $c_m(x)$ denote the number of lattice trees or lattice animals containing $0, x$ and having exactly m bonds. The one-point functions, two-point functions, and susceptibilities are defined by

$$g(z) = \sum_{m=0}^{\infty} c_m(0) z^m, \quad G_z(x) = \sum_{m=0}^{\infty} c_m(x) z^m, \quad \chi(z) = \sum_{x \in \mathbb{Z}^d} G_z(x). \tag{1.24}$$

The radius of convergence z_c (the *critical point*) of these series is finite and positive, and is strictly smaller for animals than for trees [7]. High-dimensional versions and extensions of Theorem 1.2 for $g(z_c)$ are proved in [19, 23]. The analogous quantities for trees and animals on the torus $\mathbb{T}_r^d$ are denoted $g_r(z)$, $G_{r,z}(x)$, $\chi_r(z)$. These are polynomials in z, so they define entire functions of z. Nevertheless, for large r the infinite-volume critical point z_c plays a role in the scaling. We denote the volume of the torus by $V = r^d$.

Our computation of the profile I for the two-point function and susceptibility in Theorem 1.1 supports the following conjecture from [22] that the profile I_0 (just a rescaled I) occurs for both lattice trees and lattice animals on the torus, above the upper critical dimension.

Conjecture 1.4 *For lattice trees and lattice animals on $\mathbb{T}_r^d$ with $d > 8$, there are constants $a_d < 0$ and $b_d > 0$ (different constants for trees and animals) such that, as $V = r^d \to \infty$,*

$$\begin{aligned} G_{r,z_c+sV^{-1/2}}(x) - G_{z_c}(x) &\sim b_d V^{-3/4} I_0(a_d s), \\ \chi_r(z_c + sV^{-1/2}) &\sim b_d V^{1/4} I_0(a_d s). \end{aligned} \tag{1.25}$$

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In (1.25), the torus point x is identified with its representative in $\mathbb{Z}^d \cap (-\frac{r}{2}, \frac{r}{2}]^d$ in the evaluation of $G_{z_c}(x)$. For $d > 8$, $G_{z_c}(x)$ decays as $|x|^{-(d-2)}$ [10, 11], and the constant term of order $V^{-3/4} = r^{-3d/4}$ dominates the Gaussian decay over most of the torus. This is the "plateau" phenomenon. On the complete graph, the decaying term $|x|^{-(d-2)}$ is absent, and only the constant term occurs for $G_{V,01}$, as in (1.6). For $d = d_c = 8$, the conjecture is modified to include logarithmic corrections to the window scale $V^{-1/2}$, the plateau scale $V^{-3/4}$, and the susceptibility scale $V^{1/4}$, but with the identical profile I_0.

Self-avoiding walk: Self-avoiding walk on the complete graph $\mathbb{K}_V$ is exactly solvable [27]. For $1 \le n \le V-1$, let $c_{V,n}(0,1) = \prod_{j=2}^{n}(V-j)$ denote the number of n-step self-avoiding walks from 0 to 1 on $\mathbb{K}_V$. Let $S_{V,01}(p) = \sum_{n=1}^{V-1} c_{V,n}(0,1)(p/V)^n$ and let $\chi_V^{\mathrm{SAW}}(p) = 1 + (V-1)S_{V,01}(p)$. It is proved in [27] (see also [24, Appendix B]) that, as $V \to \infty$,

$$\begin{aligned} S_{V,01}(1 + sV^{-1/2}) &\sim (2V)^{-1/2} I_1(-\sqrt{2}s), \\ \chi_V^{\mathrm{SAW}}(1 + sV^{-1/2}) &\sim 2^{-1/2} V^{1/2} I_1(-\sqrt{2}s). \end{aligned} \tag{1.26}$$

In [24, 26], the same profile I_1 is conjectured to apply to the self-avoiding walk on $\mathbb{T}_r^d$ for $d \ge 4$, in the sense that the two-point function and susceptibility obey the analog of (1.25) with the right-hand sides replaced respectively by $b_d V^{-1/2} I_1(a_d s)$ and $b_d V^{1/2} I_1(a_d s)$. The conjectured log corrections for $d = 4$ are indicated in [24, Section 1.6.3].

Spin systems: The plateau for spin systems in dimensions $d \ge d_c = 4$ is discussed in [20, 21, 26], including rigorous results for a hierarchical $|\varphi|^4$ model and conjectures for spin systems on the torus. The relevant profile for n-component spin systems is

$$f_n(s) = \frac{\int_{\mathbb{R}^n} |x|^2 e^{-\frac{1}{4}|x|^4 - \frac{s}{2}|x|^2} dx}{n \int_{\mathbb{R}^n} e^{-\frac{1}{4}|x|^4 - \frac{s}{2}|x|^2} dx} = \frac{I_{n+1}(s)}{n I_{n-1}(s)}. \tag{1.27}$$

The profile f_1 has been proven to occur for the Ising model on the complete graph (Curie–Weiss model); a recent reference is [2]. As $n \to 0$, the profile $f_n(s)$ converges to $I_1(s)$, which is consistent with the conventional wisdom that the spin model with $n = 0$ corresponds to the self-avoiding walk.

Percolation: Percolation has been extensively studied both on infinite lattices [9] and on the complete graph (the Erdős–Rényi random graph) [17]. This is a probabilistic model in which the cluster containing 0 is a connected subgraph $A \ni 0$ with weight $p^{|A|}(1-p)^{|\partial A|}$, where $|A|$ denotes the number of edges in A, and ∂A denotes the set of edges which are not in A but are incident to one or two vertices in A. On the complete graph, we divide p by V (not by eV as in (1.2)) to make the critical value $p = 1$. Thus we define the *two-point function*

$$\tau_{V,01}(p) = \mathbb{P}_{p/V}(0 \leftrightarrow 1) = \sum_{A \ni 0,1} \left(\frac{p}{V}\right)^{|A|} \left(1 - \frac{p}{V}\right)^{|\partial A|} \tag{1.28}$$

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and the *susceptibility* (expected cluster size) $\chi_V^{\mathrm{perc}}(p) = 1 + (V-1)\tau_{V,01}(p)$. Our conjecture for an analog of Theorem 1.1 for percolation on the complete graph is as follows. It involves the Brownian excursion W^* of length 1, and the moment generating function $\Psi(x) = \mathbb{E}\exp[x\int_0^1 W^*(t)\mathrm{d}t]$ for the Brownian excursion area.

Conjecture 1.5 *For $s \in \mathbb{R}$, let*

$$f_{\mathrm{perc}}(s) = \int_0^\infty x^2 \mathrm{d}\sigma_s, \qquad \mathrm{d}\sigma_s = \frac{1}{\sqrt{2\pi}} x^{-5/2} \Psi(x^{3/2}) e^{-\frac{1}{6}x^3 + \frac{s}{2}x^2 - \frac{s^2}{2}x} \mathrm{d}x. \tag{1.29}$$

Then, for some $a, b > 0$, as $V \to \infty$ we have

$$\begin{aligned} \tau_{V,01}(1 + sV^{-1/3}) &\sim bV^{-2/3} f_{\mathrm{perc}}(as), \\ \chi_V^{\mathrm{perc}}(1 + sV^{-1/3}) &\sim bV^{1/3} f_{\mathrm{perc}}(as). \end{aligned} \tag{1.30}$$

Note the different powers of V in (1.30) compared to (1.25) and (1.6)–(1.7). The powers of V in (1.30) are well-known, but to our knowledge the occurrence of the profile has not been proved. On the torus $\mathbb{T}_r^d$ with $d > 6$, the powers $V^{-1/3}$, $V^{-2/3}$, $V^{1/3}$ are proved in [14], and the role of f_{perc} was first conjectured in [20, Appendix C].

The origin of the conjecture is as follows. The properly rescaled cluster size (without expectation) is known to converge in distribution to a random variable described by the Brownian excursion [1], and the limiting random variable is characterized by a point process [18]. The measure σ_s is the intensity of the point process and is found in [18, Theorem 4.1]. The point process describes cluster sizes, in the sense that

$$n^{-2k/3} \sum_i |C_i|^k \Rightarrow \int_0^\infty x^k \mathrm{d}\sigma_s \qquad (k \geq 2) \tag{1.31}$$

in distribution [1]. The $k = 2$ case corresponds to χ_V^{perc} and identifies $f_{\mathrm{perc}}(s)$.

2 Proof for trees

We begin with an elementary lemma.

Lemma 2.1 *Let $\gamma \geq 0$, $\kappa > 0$, and $\lambda \in \mathbb{R}$. There is a constant $C_{\kappa,\lambda} > 0$ such that*

$$\sum_{n=\lceil b\sqrt{V}\rceil}^{V} \frac{1}{n^\gamma} e^{-\kappa n^2/V} e^{\lambda n/\sqrt{V}} \leq C_{\kappa,\lambda} b^{-\gamma} V^{(1-\gamma)/2} \tag{2.1}$$

for all V and for all b sufficiently large (depending on κ, λ).

Proof Since $n \geq b\sqrt{V}$ and $\gamma \geq 0$, the left-hand side of (2.1) is bounded by

$$\frac{1}{b^\gamma V^{\gamma/2}} \sum_{n=\lceil b\sqrt{V}\rceil}^{\infty} e^{-\kappa n^2/V} e^{\lambda n/\sqrt{V}}. \tag{2.2}$$

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For b sufficiently large (depending on κ, λ), the summand above is monotone decreasing in n, so we can bound the sum by the integral

$$\int_{b\sqrt{V}-1}^{\infty} e^{-\kappa(y/\sqrt{V})^2} e^{\lambda(y/\sqrt{V})} \mathrm{d}y \le C_{\kappa,\lambda}\sqrt{V}, \tag{2.3}$$

and the desired result follows. ∎

Proof of Theorem 1.1 for trees We use (1.14) and drop the superscript t. Fix $s \in \mathbb{R}$. For $p = 1 + sV^{-1/2}$, by combining $V^{-(n-1)}$ with the binomial coefficient, we have

$$\chi_V(1 + sV^{-1/2}) = \sum_{n=1}^{V} \Big(\prod_{j=1}^{n-1} (1 - \frac{j}{V}) \Big) \frac{1}{(n-1)!} \frac{n^{n-1}}{e^{n-1}} \Big(1 + \frac{s}{\sqrt{V}}\Big)^{n-1}. \tag{2.4}$$

Let $0 < a < 1 < b < \infty$. We divide the sum over n into three parts $\chi_V^{(1)}, \chi_V^{(2)}, \chi_V^{(3)}$, which respectively sum over n in the intervals $[1, a\sqrt{V})$, $[a\sqrt{V}, b\sqrt{V}]$, $(b\sqrt{V}, V]$. We will prove that

$$\chi_V^{(1)} \lesssim e^{a|s|}(1 + a^{1/2}V^{1/4}), \qquad \chi_V^{(3)} \le C_{|s|} b^{-1/2} V^{1/4} \tag{2.5}$$

for all $a > 0$ and all b sufficiently large, and that

$$\lim_{V\to\infty} V^{-1/4}\chi_V^{(2)} = \int_a^b f(x)\mathrm{d}x, \qquad f(x) = \frac{e}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{x}} e^{sx} \tag{2.6}$$

for all a, b. These claims imply that

$$\int_a^b f(x)\mathrm{d}x \le \liminf_{V\to\infty} \frac{\chi_V}{V^{1/4}} \le \limsup_{V\to\infty} \frac{\chi_V}{V^{1/4}} \le Ce^{a|s|}a^{1/2} + \int_a^b f(x)\mathrm{d}x + C_{|s|}b^{-1/2} \tag{2.7}$$

for all $a > 0$ and all b sufficiently large. Since χ_V does not depend on a or b, by taking the limits $a \to 0$, $b \to \infty$, we obtain $\lim_{V\to\infty} V^{-1/4}\chi_V = \int_0^\infty f$, which is the desired result (1.7).

It remains to prove the claims (2.5) and (2.6). Let

$$b_n = \frac{n^{n-1}}{(n-1)!e^{n-1}}, \tag{2.8}$$

which obeys $b_n \lesssim 1/\sqrt{n}$, by Stirling's formula. Using this in the sum for $\chi_V^{(1)}$, and using $1 + s/\sqrt{V} \le e^{|s|/\sqrt{V}}$, we get

$$\chi_V^{(1)} \lesssim \sum_{n=1}^{\lfloor a\sqrt{V}\rfloor} 1 \frac{1}{\sqrt{n}} e^{|s|n/\sqrt{V}} \le e^{a|s|} \sum_{n=1}^{\lfloor a\sqrt{V}\rfloor} \frac{1}{\sqrt{n}} \lesssim e^{a|s|}(1 + a^{1/2}V^{1/4}), \tag{2.9}$$

as claimed. For $\chi_V^{(3)}$, we also need a bound on the product over j. Using $1 - x \le e^{-x}$, we have

$$\prod_{j=1}^{n-1} \big(1 - \frac{j}{V}\big) \le \exp\Big\{ -\frac{1}{V} \sum_{j=1}^{n-1} j \Big\} = \exp\Big\{ -\frac{n(n-1)}{2V} \Big\}. \tag{2.10}$$

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By Lemma 2.1 with $\gamma = \kappa = \frac{1}{2}$ and $\lambda = |s|$, this implies that, for all b sufficiently large,

$$\chi_V^{(3)} \lesssim \sum_{n=\lceil b\sqrt{V}\rceil}^{V} e^{-n^2/2V} e^{n/2V} \frac{1}{\sqrt{n}} e^{|s|n/\sqrt{V}} \lesssim e^{1/2} b^{-1/2} V^{1/4}. \tag{2.11}$$

Finally, for $\chi_V^{(2)}$ we fix a, b and use the asymptotic formulas

$$\Big(1+\frac{s}{\sqrt{V}}\Big)^{n-1} = \exp\Big\{(n-1)\log(1+\frac{s}{\sqrt{V}})\Big\} = e^{sn/\sqrt{V}}\Big[1+O\Big(\frac{1}{\sqrt{V}}\Big)+O\Big(\frac{n}{V}\Big)\Big], \tag{2.12}$$

$$\prod_{j=1}^{n-1}\Big(1-\frac{j}{V}\Big) = \exp\Big\{\sum_{j=1}^{n-1}\log(1-\frac{j}{V})\Big\} = e^{-n^2/2V}\Big[1+O\Big(\frac{n}{V}\Big)+O\Big(\frac{n^3}{V^2}\Big)\Big], \tag{2.13}$$

which follow from Taylor expansion of the logarithm (the constants here depend on s). Since $n \in [a\sqrt{V}, b\sqrt{V}]$, the above, together with the fact that $b_n = \frac{e}{\sqrt{2\pi n}}[1+O(1/n)]$ by Stirling's formula, give

$$\chi_V^{(2)} = \sum_{n=\lceil a\sqrt{V}\rceil}^{\lfloor b\sqrt{V}\rfloor} e^{-n^2/2V} \frac{e}{\sqrt{2\pi n}} e^{sn/\sqrt{V}}\Big[1+O\Big(\frac{1}{\sqrt{V}}\Big)\Big]. \tag{2.14}$$

The desired limit then follows from the observations that the leading term of $V^{-1/4}\chi_V^{(2)}$ is a Riemann sum for the integral $\int_a^b f$ with mesh size $V^{-1/2}$. ∎

Proof of Theorem 1.2 for trees We use (1.11) and again drop the superscript t. Fix $s \geq 0$. Let p_V be a sequence with $p_V \leq 1 + sV^{-1/2}$ and $p_V \to p \in [0,1]$. Similarly to (2.4) and with an additional factor of n in the denominator,

$$G_{V,0}(p_V) = \sum_{n=1}^{V}\Big(\prod_{j=1}^{n-1}(1-\frac{j}{V})\Big)\frac{1}{n!}\frac{n^{n-1}}{e^{n-1}}p_V^{n-1}. \tag{2.15}$$

Let $N, b \geq 1$. We divide the sum over n into three parts $G_V^{(1)}, G_V^{(2)}, G_V^{(3)}$, which respectively sum over n in the intervals $[1, N]$, $(N, b\sqrt{V}]$, $(b\sqrt{V}, V]$. For a fixed N, we immediately get

$$\lim_{V\to\infty} G_V^{(1)}(p_V) = \sum_{n=1}^{N}\frac{1}{n!}\frac{n^{n-1}}{e^{n-1}}p^{n-1}, \tag{2.16}$$

which dominates the sum. Indeed, using monotonicity of the generating function, for $G_V^{(2)}$ we can proceed as in (2.9) to bound

$$G_V^{(2)}(p_V) \leq G_V^{(2)}(1+sV^{-1/2}) \leq e^{bs}\sum_{n=N}^{\lfloor b\sqrt{V}\rfloor}\frac{1}{n^{3/2}} \lesssim \frac{e^{bs}}{\sqrt{N}}. \tag{2.17}$$

For $G_V^{(3)}$, we can argue as in (2.11) but with an additional factor n in the denominator, and use Lemma 2.1 with $\gamma = \frac{3}{2}$ to get $G_V^{(3)}(p_V) \lesssim b^{-3/2}V^{-1/4}$ for b sufficiently large. Together, we obtain

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$$\sum_{n=1}^{N} \frac{1}{n!} \frac{n^{n-1}}{e^{n-1}} p^{n-1} \le \liminf_{V\to\infty} G_{V,0} \le \limsup_{V\to\infty} G_{V,0} \le \sum_{n=1}^{N} \frac{1}{n!} \frac{n^{n-1}}{e^{n-1}} p^{n-1} + \frac{Ce^{bs}}{\sqrt{N}} \tag{2.18}$$

for all $N \ge 1$ and all b sufficiently large. Since $G_{V,0}$ does not depend on N, we can take the limit $N \to \infty$ to conclude the desired result (1.9). ∎

3 Proof for connected subgraphs

3.1 Bound on $C(n, n+k)$

We use the asymptotic formula for $C(n, n+k)$ proved in [3]. We follow the notation in [3] and write

$$x = 1 + \frac{k}{n}. \tag{3.1}$$

For $x > 1$, we define the function $y = y(x) \in (0,1)$ implicitly by

$$x = \frac{1}{2y} \log\left(\frac{1+y}{1-y}\right) = \frac{1}{y} \operatorname{arctanh} y = \sum_{m=0}^{\infty} \frac{y^{2m}}{2m+1}, \tag{3.2}$$

and we define the functions $\varphi(x)$, $a(x)$ by

$$e^{\varphi(x)} = \frac{2e^{-x} y^{1-x}}{\sqrt{1-y^2}}, \tag{3.3}$$

$$a(x) = x(x+1)(1-y) + \log(1-x+xy) - \frac{1}{2}\log(1-x+xy^2). \tag{3.4}$$

Both φ and a extend continuously to $x = 1$ by defining $y^{1-x} = 1$ at $x = 1$ and defining $a(1) = 2 + \frac{1}{2}\log\frac{3}{2}$.

Let $N = \binom{n}{2}$. It is proved in [3, Corollary 1] that there are constants $w_k = 1 + O(1/k)$ for which

$$C(n, n+k) = w_k \binom{N}{n+k} e^{n\varphi(x)} e^{a(x)} \left[1 + O\left(\frac{(k+1)^{1/16}}{n^{9/50}}\right)\right] \tag{3.5}$$

uniformly in $0 \le k \le N - n$. The constants w_k are related to Wright's constants for the asymptotics of $C(n, n+k)$ with k fixed [29], and they are related to the Brownian excursion area [28]. We will simply bound w_k by a constant. The next lemma gives estimates for $\varphi(x)$ and $a(x)$.

Lemma 3.1 *Let $x \ge 1$.*

(i) *The function $a(x)$ is bounded.*

(ii) *Let $t = \sqrt{3e}$ and $y = y(x)$. Then*

$$e^{\varphi(x)} \le \frac{2}{e} \exp\left\{-\frac{1}{3} y^2 \log\frac{y}{t}\right\}, \tag{3.6}$$

and the right-hand side is monotonically increasing for $0 < y \le t/\sqrt{e}$.

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By considering the limit $x \to \infty$ ($y \to 1$), we expect that the inequality (3.6) becomes optimal with $t = (e/2)^3 \approx 2.51$, but we do not pursue this. The weaker version with $t = \sqrt{3e}$ is sufficient for our purposes, but to show the role of t we keep it in our formulas.

Proof (i) The function $a(x)$ is continuous on $[1, \infty)$ by definition, and it satisfies $|a(x)| \lesssim x^2(1-y) \sim 2x^2 e^{-2x}$ as $x \to \infty$ by [3, Lemma 3.2], so it is bounded.

(ii) By the definitions of $\varphi(x)$ and x, and by the Taylor series for $\log(1-y^2)$,

$$\begin{aligned}\varphi(x) - \log\frac{2}{e} &= (1-x)(1+\log y) - \frac{1}{2}\log(1-y^2) \\ &= -\sum_{m=1}^{\infty}\frac{y^{2m}}{2m+1}(1+\log y) + \sum_{m=1}^{\infty}\frac{y^{2m}}{2m} \\ &= -\frac{1}{3}y^2\log y + \frac{1}{6}y^2 + \sum_{m=2}^{\infty}\frac{y^{2m}}{2m+1}\Big(-\log y + \frac{1}{2m}\Big). \end{aligned} \tag{3.7}$$

We bound the series in the last line by a quadratic function, term by term. For any $m \geq 2$, by calculus,

$$\max_{0\leq y\leq 1}\big[y^{2m-2}(-2m\log y + 1)\big] = \frac{2m}{2m-2}e^{-1/m}. \tag{3.8}$$

Then, with $K = \max_{m\geq 2}\{\frac{2m}{2m-2}e^{-1/m}\} = 2e^{-1/2}$, by [8, 0.234.8] we have

$$\sum_{m=2}^{\infty}\frac{y^{2m}}{2m+1}\Big(-\log y + \frac{1}{2m}\Big) \leq \sum_{m=2}^{\infty}\frac{Ky^2}{(2m+1)(2m)} = (1-\log 2 - \frac{1}{6})Ky^2. \tag{3.9}$$

Therefore,

$$\varphi(x) - \log\frac{2}{e} \leq -\frac{1}{3}y^2\log y + \Big[\frac{1}{6} + (1-\log 2 - \frac{1}{6})K\Big]y^2. \tag{3.10}$$

This implies (3.6) with any t that obeys $\frac{1}{3}\log t \geq \frac{1}{6} + (1-\log 2 - \frac{1}{6})K \approx 0.3367$. In particular, we can take any $t \geq 2.75$, including $t = \sqrt{3e} \approx 2.85$. Monotonicity of the upper bound in $0 < y \leq t/\sqrt{e}$ is another calculus exercise. ∎

We now restate and prove Proposition 1.3.

Proposition 3.2 *Let $n \geq 3$, $N = \binom{n}{2}$, and $t = \sqrt{3e}$. For $0 \leq \frac{k}{n} \leq \frac{t^2}{3e}$, we have*

$$C(n, n+k) \lesssim \binom{N}{n+k}\Big(\frac{2}{e}\Big)^n\Big(\frac{t^2 n}{3k}\Big)^{k/2}. \tag{3.11}$$

Proof We use the asymptotic formula (3.5), and use that $w_k = 1 + O(1/k)$ is bounded. The error term in (3.5) is bounded by a constant since k is at most linear in n. The factor $e^{a(x)}$ is also bounded by a constant, by Lemma 3.1(i).

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We therefore only need to estimate $e^{n\varphi(x)}$. Since $0 \le \frac{k}{n} \le \frac{t^2}{3e}$ and $x = 1 + \frac{k}{n}$, by (3.2) we have $y(x) \le \sqrt{3(x-1)} = \sqrt{3k/n} \le t/\sqrt{e}$, so Lemma 3.1(ii) gives

(3.12)
$$e^{\varphi(x)} \le \frac{2}{e} \exp\Big\{ -\frac{1}{3} y^2 \log \frac{y}{t} \Big\} \le \frac{2}{e} \exp\Big\{ -\frac{x-1}{2} \log \frac{3(x-1)}{t^2} \Big\} = \frac{2}{e} \Big(\frac{t^2 n}{3k} \Big)^{k/2n}.$$

The desired result then follows by inserting the above into (3.5). ∎

For larger $\frac{k}{n}$ we simply use the fact that $C(n, n+k)$ is less than the total number of graphs (connected or not) on n vertices with $n+k$ edges, which is $\binom{N}{n+k}$. For all $n \ge 2$ and $k \ge -1$, we have

$$C(n, n+k) \le \binom{N}{n+k} \le \frac{N^{n+k}}{(n+k)!}. \tag{3.13}$$

3.2 Bound on the surplus generating function

We now prove useful bounds on the surplus generating function defined in (1.15):

$$S(n,z) = \sum_{\ell=1}^{\infty} C(n, n-1+\ell) z^{\ell} = \sum_{k=0}^{\infty} C(n, n+k) z^{k+1}. \tag{3.14}$$

The terms in the series are zero unless $k \le \binom{n}{2} - n$. The goal is to prove that $S(n,z)$ is small relative to the number of trees $C(n, n-1) = n^{n-2}$. We do this by decomposing the series into two parts corresponding to sparse and dense graphs. We define

(3.15)
$$A(n,z) = \frac{1}{n^{n-2}} \sum_{k=0}^{n} C(n, n+k) z^{k+1}, \quad B(n,z) = \frac{1}{n^{n-2}} \sum_{k=\lfloor \frac{1}{2} n \rfloor}^{\infty} C(n, n+k) z^{k+1},$$

so that

$$S(n,z) \le n^{n-2} \big(A(n,z) + B(n,z) \big). \tag{3.16}$$

Lemma 3.3 (Sparse connected graphs) *Let $n \ge 3$, $z \ge 0$, and $t = \sqrt{3e}$.*

(i) *If $n^{3/2} z \le b$, then $A(n,z) \le C_b n^{3/2} z$ for some $C_b > 0$.*

(ii) *If $\varepsilon > 0$, then*

$$A(n,z) \le C_\varepsilon \exp\Big\{ \Big(\frac{1}{24} + \varepsilon \Big) e t^2 z^2 n^3 \Big\} \tag{3.17}$$

for some $C_\varepsilon > 0$.

Proof Since $\frac{t^2}{3e} = 1$, we can apply Proposition 3.2 to estimate $C(n, n+k)$. For the binomial coefficient in (3.11), we use Stirling's formula, $n + k \ge n$, and $N = \binom{n}{2} = \frac{1}{2} n(n-1)$ to see that

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$$\binom{N}{n+k} \le \frac{N^{n+k}}{(n+k)!} \lesssim \frac{1}{\sqrt{n+k}} \left(\frac{eN}{n+k}\right)^{n+k} \le \frac{1}{\sqrt{n}} \left(\frac{e(n-1)}{2}\right)^{n+k}. \tag{3.18}$$

Then, by extending the sum to run over all $k \ge 0$, we obtain

$$\begin{aligned} \frac{1}{z} A(n,z) &= \frac{1}{n^{n-2}} \sum_{k=0}^{n} C(n,n+k) z^k \\ &\lesssim \frac{1}{n^{n-2}\sqrt{n}} \sum_{k=0}^{n} \left(\frac{en}{2}\right)^{n+k} \left(\frac{2}{e}\right)^{n} \left(\frac{t^2 n}{3k}\right)^{k/2} z^k \le n^{3/2} \sum_{k=0}^{\infty} \frac{1}{k^{k/2}} \left(\frac{etn^{3/2}z}{2\sqrt{3}}\right)^k, \end{aligned} \tag{3.19}$$

which converges for all $z > 0$.

(i) If $n^{3/2}z \le b$ then the series on the right-hand side is bounded by a constant C_b, as required.

(ii) We set $x = \frac{etn^{3/2}z}{2\sqrt{3}}$ and use the asymptotic formula [16, Lemma 4.1(i)]

$$\sum_{k=0}^{\infty} \frac{1}{k^{k/2}} x^k \sim (4\pi e^{-1})^{1/2} x e^{\frac{1}{2e}x^2} \qquad \text{as } x \to \infty \tag{3.20}$$

to get a bound for large x. For smaller $x \ge 0$, we simply bound by a constant. The desired result then follows by absorbing the prefactor of (3.20) and another factor of $n^{3/2}z = \text{const}\, x$ into the exponential. This completes the proof. ∎

Lemma 3.4 (Dense connected graphs) *Let $n \ge 3$ and $z \le \frac{3}{en}$. Then $B(n,z) \lesssim z^2$.*

Proof Let $v = \lfloor n/2 \rfloor \ge 1$. The crude bound (3.13) gives

$$\begin{aligned} B(n,z) \le \frac{1}{n^{n-2}} \sum_{k=v}^{\infty} \frac{N^{n+k}}{(n+k)!} z^{k+1} &= \frac{z^{1+v}}{n^{n-2}} \frac{N^{n+v}}{(n+v)!} \sum_{k=v}^{\infty} (Nz)^{k-v} \frac{(n+v)!}{(n+k)!} \\ &\le \frac{z^{1+v}}{n^{n-2}} \frac{N^{n+v}}{(n+v)!} \sum_{m=0}^{\infty} \left(\frac{Nz}{n+v}\right)^m, \end{aligned} \tag{3.21}$$

since $(n+k)! \ge (n+v)!(n+v)^{k-v}$. Note that by our hypothesis

$$\frac{Nz}{n+v} \le \frac{\frac{1}{2}n(n-1)z}{n+(\frac{1}{2}n-\frac{1}{2})} < \frac{n(n-1)}{3n-3} z = \frac{1}{3}nz \le \frac{1}{e}, \tag{3.22}$$

so the geometric series in (3.21) is bounded by a constant. For the prefactor in (3.21), since $1+v \ge 2$ and $z \le \frac{3}{en}$ by hypothesis, we have

$$z^{1+v} = z^2 z^{v-1} \le \frac{e}{3} z^2 \left(\frac{3}{e}\right)^v n^{1-v}. \tag{3.23}$$

Also, using $N = \frac{1}{2}n(n-1)$ and Stirling's formula,

$$\frac{N^{n+v}}{(n+v)!} \lesssim \frac{[\frac{1}{2}n(n-1)]^{n+v}}{\sqrt{n}(\frac{n+v}{e})^{n+v}} = \frac{n^{n+v}}{\sqrt{n}} \left(\frac{e}{2}\right)^{n+v} \left(\frac{n-1}{n+v}\right)^{n+v}. \tag{3.24}$$

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Since $\frac{n-1}{n+v} \le \frac{2}{3}$, together we obtain

$$\frac{z^{1+v}}{n^{n-2}} \frac{N^{n+v}}{(n+v)!} \lesssim z^2 \Big(\frac{3}{e}\Big)^v n^{5/2} \Big(\frac{e}{3}\Big)^{n+v} = z^2 n^{5/2} \Big(\frac{e}{3}\Big)^n. \tag{3.25}$$

It follows that $B(n,z) \lesssim z^2 \sup_{n\ge 3}\{n^{5/2}(e/3)^n\}$, and the proof is complete since $e < 3$. ∎

3.3 Proof for connected subgraphs

Proof of Theorem 1.1 for connected subgraphs Fix $s \in \mathbb{R}$ and let $p = 1 + sV^{-1/2}$. We assume V is large enough so that $p \le 3$. As discussed around (1.21), it suffices to prove

$$\lim_{V\to\infty} V^{-1/4} \Delta_V (1 + sV^{-1/2}) = 0. \tag{3.26}$$

By the definition of Δ_V in (1.19) and by (3.16),

$$\Delta_V(p) \le \sum_{n=3}^{V} \binom{V-1}{n-1} \Big(\frac{p}{eV}\Big)^{n-1} n^{n-1} \Big(A\Big(n, \frac{p}{eV}\Big) + B\Big(n, \frac{p}{eV}\Big)\Big). \tag{3.27}$$

We write the part of the upper bound (3.27) that contains A, B as $\Delta_V^{(A)}, \Delta_V^{(B)}$ respectively.

We start with $\Delta_V^{(B)}$ and use Lemma 3.4 to bound B. Let $z = p/(eV)$. Since $n \le V$ and $p \le 3$ (for large V), we have $nz \le p/e \le 3/e$, so Lemma 3.4 applies and gives $B(n,z) \lesssim V^{-2}$. Then, by comparing to χ_V^t in (1.14), we find that

$$\Delta_V^{(B)}(p) \lesssim V^{-2} \chi_V^t(p). \tag{3.28}$$

For $\Delta_V^{(A)}$, we claim that if both $b \ge 1$ and V are sufficiently large, then

$$\Delta_V^{(A)}(p) \le C_b V^{-1/4} \chi_V^t(p) + C_s b^{-1/2} V^{1/4}. \tag{3.29}$$

Since we already know that $V^{-1/4}\chi_V^t$ converges, (3.29) implies that

$$0 \le \limsup_{V\to\infty} \frac{\Delta_V}{V^{1/4}} \le \limsup_{V\to\infty} \frac{\Delta_V^{(A)} + \Delta_V^{(B)}}{V^{1/4}} \le C_s b^{-1/2} \tag{3.30}$$

for all b sufficiently large. But Δ_V does not depend on b, so by taking the limit $b \to \infty$, we obtain (3.26), as desired.

It remains to prove (3.29). We divide the sum defining $\Delta_V^{(A)}$ into two parts $\Delta_V^{(1)}, \Delta_V^{(2)}$, which sum over n in the intervals $[3, b\sqrt{V}]$, $(b\sqrt{V}, V]$ respectively.

For $\Delta_V^{(1)}$, we have $n \le bV^{1/2}$ so $n^{3/2}z = n^{3/2}p/(eV) \le c_b V^{-1/4}$, so we can apply Lemma 3.3(i) to obtain $A(n,z) \le C_b' n^{3/2} z \le C_b V^{-1/4}$. With the formula for χ_V^t in (1.14), this gives

$$\Delta_V^{(1)} \le C_b V^{-1/4} \sum_{n=3}^{\lfloor b\sqrt{V}\rfloor} \binom{V-1}{n-1} \Big(\frac{p}{eV}\Big)^{n-1} n^{n-1} \le C_b V^{-1/4} \chi_V^t(p). \tag{3.31}$$

This provides the first term on the right-hand side of (3.29).

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For $\Delta_V^{(2)}$, we use $z = p/(eV)$ and Lemma 3.3(ii) to see that

$$A(n,z) \le C_\varepsilon \exp\Big\{\Big(\frac{1}{24}+\varepsilon\Big)e^{-1}t^2p^2n^3/V^2\Big\}. \tag{3.32}$$

Since $t = \sqrt{3e}$, we have $\frac{1}{24}e^{-1}t^2 = \frac{1}{8}$. By choosing ε small, and by using $p = 1 + sV^{-1/2} \to 1$ as $V \to \infty$, for V sufficiently large (depending only on ε, s) we have

$$A(n,z) \le C_\varepsilon \exp\Big\{\frac{1}{5}\frac{n^3}{V^2}\Big\}. \tag{3.33}$$

For these values of V, we thus have

$$\Delta_V^{(2)}(1+sV^{-1/2}) \lesssim \sum_{n=\lceil b\sqrt{V}\rceil}^{V} \binom{V-1}{n-1}\Big(\frac{1+sV^{-1/2}}{eV}\Big)^{n-1} n^{n-1}\exp\Big\{\frac{1}{5}\frac{n^3}{V^2}\Big\}. \tag{3.34}$$

We now follow the argument used for $\chi_V^{(3)}$ of trees in the paragraph containing (2.11). Using $\frac{n^3}{V^2} \le \frac{n^2}{V}$ and Lemma 2.1 with $\gamma = \frac{1}{2}$ and $\kappa = \frac{1}{2} - \frac{1}{5} > 0$, we find that if b is sufficiently large then

$$\Delta_V^{(2)}(1+sV^{-1/2}) \lesssim \sum_{n=\lceil b\sqrt{V}\rceil}^{V} e^{-(\frac{1}{2}-\frac{1}{5})n^2/V}\frac{1}{\sqrt{n}}e^{|s|n/\sqrt{V}} \le C_{|s|}b^{-1/2}V^{1/4}. \tag{3.35}$$

This gives the second term on the right-hand side of (3.29) and concludes the proof. ∎

Proof of Theorem 1.2 for connected subgraphs As noted at (1.20), it suffices to prove that $\Delta_{V,0}(1+sV^{-1/2}) \to 0$ for all $s \ge 0$. We write $p = 1 + sV^{-1/2}$ and follow the proof of Theorem 1.1. Compared to Δ_V for the susceptibility, there is one less factor n in $\Delta_{V,0}$, so instead of (3.27) we now have

$$\Delta_{V,0}(p) \le \sum_{n=3}^{V}\binom{V-1}{n-1}\Big(\frac{p}{eV}\Big)^{n-1} n^{n-2}\Big(A\Big(n,\frac{p}{eV}\Big) + B\Big(n,\frac{p}{eV}\Big)\Big). \tag{3.36}$$

As in (3.28), the contribution from B obeys

$$\Delta_{V,0}^{(B)}(p) \lesssim V^{-2}G_{V,0}^t(p) \lesssim V^{-2}, \tag{3.37}$$

so it vanishes in the limit. For $\Delta_{V,0}^{(1)}(p)$, the same bound on A that was used in (3.31) now gives $\Delta_{V,0}^{(1)}(p) \le C_b V^{-1/4}G_{V,0}(p)$. For $\Delta_{V,0}^{(2)}(p)$, in (3.35) we now have an extra factor n in the denominator, so Lemma 2.1 with $\gamma = \frac{3}{2}$ gives

$$\Delta_{V,0}^{(2)}(1+sV^{-1/2}) \lesssim \sum_{n=\lceil b\sqrt{V}\rceil}^{V} e^{-(\frac{1}{2}-\frac{1}{5})n^2/V}\frac{1}{n^{3/2}}e^{|s|n/\sqrt{V}} \le C_{|s|}b^{-3/2}V^{-1/4}. \tag{3.38}$$

Altogether, we have $\Delta_{V,0}(p) \lesssim V^{-1/4} \to 0$, and the proof is complete. ∎

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