Econometric Theory, **24**, 2008, 1607–1627. Printed in the United States of America.
doi:10.1017/S0266466608080638

ASYMPTOTIC PROPERTIES OF NONPARAMETRIC FRONTIER ESTIMATORS

LAJOS HORVÁTH
University of Utah

ZSUZSANNA HORVÁTH
University of Utah

WANG ZHOU
National University of Singapore

Aragon, Daouia, and Thomas-Agnan (2005, *Econometric Theory* 21, 358–389) introduced a new nonparametric frontier estimation. We prove the weak convergence of the empirical conditional quantile function. The distribution of the limit depends on the unknown conditional quantile density function. We provide a method to construct uniform confidence bands without estimating the conditional quantile density.

1. INTRODUCTION

Starting with the pioneering works of Koopmans (1951) and Debreu (1951) (see Shephard, 1970), the production set and the frontier function are used to describe production and productivity. A set of p inputs $\mathbf{X}$ is used to produce a univariate output Y. Following Cazals, Florens, and Simar (2002) and Aragon, Daouia, and Thomas-Agnan (2005), we define the production function $\varphi(\mathbf{x})$, $\mathbf{x} \in R^p$, as the upper end point of the conditional distribution of Y given that the input $\mathbf{X} \le \mathbf{x}$. For the vector $\mathbf{y} = (y_1, \ldots, y_p)$ and $\mathbf{x} = (x_1, \ldots, x_p)$ we write $\mathbf{y} \le \mathbf{x}$ if $y_i \le x_i$ for all $1 \le i \le p$. Hence

$$\varphi(\mathbf{x}) = \sup\{y : F_{\mathbf{x}}(y) < 1\},$$

where

$$F_{\mathbf{x}}(y) = P(Y \le y | \mathbf{X} \le \mathbf{x}) = \frac{F(y, \mathbf{x})}{F^*(\mathbf{x})}$$

Lajos Horváth is supported in part by NATO grant PST.EAP.CLG 980599 and by NSF grant INT-0223262. Zsuzsanna Horváth is supported in part by NSF and NSF-VIGRE grants. Wang Zhou is supported in part by the grant R-155-050-055-133/101 at the National University of Singapore, Singapore. Address correspondence to Lajos Horváth, Department of Mathematics, University of Utah, 155 South 1440 East, Salt Lake City, UT 84112-0090, USA; e-mail: zhorvath@math.utah.edu.

© 2008 Cambridge University Press 0266-4666/08 $15.00

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

denotes the conditional distribution function of Y given $\mathbf{X} \le \mathbf{x}$ and $F(y,\mathbf{x}) = P(Y \le y, \mathbf{X} \le \mathbf{x}), F^*(\mathbf{x}) = P(\mathbf{X} \le \mathbf{x})$ are the corresponding distribution functions.

Several methods have been developed to estimate $\varphi(\mathbf{x})$ from the random sample $(Y_i, \mathbf{X}_i)$, $1 \le i \le n$. We assume that $(Y_i, \mathbf{X}_i) \in R^{p+1}$ are independent and identically distributed random vectors with distribution function $F(y,\mathbf{x})$. For a survey of methods and comparisons we refer to Aragon et al. (2005), in which it is noted that some of the estimators used in the literature are not robust and subject to outliers.

Cazals et al. (2002) suggested a nonparametric envelope estimator using the concept of maximal output frontier of order m. A different nonparametric approach was developed by Aragon et al. (2005), which we follow in this paper. We use the conditional quantile

$$F_{\mathbf{x}}^{-1}(t) = \inf\{y : F_{\mathbf{x}}(y) \ge t\}, \qquad 0 < t < 1,$$

i.e., the left-continuous inverse function of $F_{\mathbf{x}}(y)$. We define $F_{\mathbf{x}}^{-1}(0) = \lim_{t \downarrow 0} F_{\mathbf{x}}^{-1}(t)$ and $F_{\mathbf{x}}^{-1}(1) = \lim_{t \uparrow} F_{\mathbf{x}}^{-1}(t)$. The natural estimator for $F_{\mathbf{x}}^{-1}(t)$ is the empirical quantile

$$F_{n,\mathbf{x}}^{-1}(t) = \inf\{y : F_{n,\mathbf{x}}(y) \ge t\},$$

where

$$F_{n,\mathbf{x}}(y) = \frac{F_n(y,\mathbf{x})}{F_n^*(\mathbf{x})}$$

and

$$F_n(y,\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} I\{Y_i \le y, \mathbf{X}_i \le \mathbf{x}\}, \qquad F_n^*(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} I\{\mathbf{X}_i \le \mathbf{x}\}$$

are the corresponding empirical functions. Let

$$R_{t,\mathbf{x}}(t) = n^{1/2}(F_{n,\mathbf{x}}^{-1}(t) - F_{\mathbf{x}}^{-1}(t)).$$

Aragon et al. (2005) showed that for any $\mathbf{x}$ such that $F^*(\mathbf{x}) > 0$ and $0 < t_0 < 1$, $F_{n,\mathbf{x}}^{-1}(t_0) \xrightarrow{P} F_{\mathbf{x}}^{-1}(t_0)$. The asymptotic normality of $R_{n,\mathbf{x}}(t)$ was also established by Aragon et al. (2005) assuming the existence of the conditional density and that the conditional quantile density function is positive at $F_{\mathbf{x}}^{-1}(t_0)$. Daouia (2005) considered the Bahadur–Kiefer representation and the weak convergence of $R_{n,\mathbf{x}}(t)$ for a fixed $\mathbf{x}$. We consider the weak convergence of the process $R_{n,\mathbf{x}}(t)$ uniformly in $\mathbf{x}$ and in t. We assume that the set C satisfies the following conditions:

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

$$\inf_{\mathbf{x}\in C} F^*(\mathbf{x}) > 0, \tag{1.1}$$

$$f_{\mathbf{x}}(y) = \partial F_{\mathbf{x}}(y)/\partial y \quad \text{exists and is continuous in } y, \tag{1.2}$$

and

$$\inf_{\mathbf{x}\in C} \inf_{\epsilon\le t\le 1-\epsilon} f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(t)) > 0 \quad \text{for all } 0 < \epsilon < \tfrac{1}{2}. \tag{1.3}$$

THEOREM 1.1. *If (1.1) and (1.3) hold and (1.2) is satisfied for all* $\mathbf{x} \in C$, *then for all* $0 < \epsilon < \frac{1}{2}$,

$$\sup_{\mathbf{x}\in C} \sup_{\epsilon\le t\le 1-\epsilon} |F_{n,\mathbf{x}}^{-1}(t) - F_{\mathbf{x}}^{-1}(t)| = O((\log\log n/n)^{1/2}) \quad a.s.$$

Theorem 1.1 extends Theorem 4.1 in Aragon et al. (2005). The pointwise convergence in probability is replaced by almost sure uniform convergence.

Next, we consider the asymptotic normality of $R_{n,\mathbf{x}}(t)$. Let $\mathbf{x} \wedge \mathbf{y} = (\min(x_1, y_1), \ldots, \min(x_p, y_p))$, $\mathbf{x} = (x_1, \ldots, x_p)$, $\mathbf{y} = (y_1, \ldots, y_p)$.

THEOREM 1.2. *If (1.1) and (1.3) hold and (1.2) is satisfied for all* $\mathbf{x} \in C$, *then we can define a sequence of processes* $K_{n,\mathbf{x}}(t)$ *such that*

$$\sup_{\mathbf{x}\in C} \sup_{\epsilon\le t\le 1-\epsilon} |R_{n,\mathbf{x}}(t) - K_{n,\mathbf{x}}(t)| = o_P(1)$$

for all $0 < \epsilon < \frac{1}{2}$. *Also,* $K_{n,\mathbf{x}}(t)$ *is Gaussian with* $\mathrm{E}K_{n,\mathbf{x}}(t) = 0$ *and*

$$\begin{aligned}\mathrm{E}K_{n,\mathbf{x}}(t)K_{n,\mathbf{z}}(s) = {} & \frac{1}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))} \frac{1}{f_{\mathbf{z}}(F_{\mathbf{z}}^{-1}(s))} \frac{F^*(\mathbf{x}\wedge\mathbf{z})}{F^*(\mathbf{x})F^*(\mathbf{z})} \\ & \times (F_{\mathbf{x}\wedge\mathbf{z}}(F_{\mathbf{x}}^{-1}(t) \wedge F_{\mathbf{z}}^{-1}(s)) - sF_{\mathbf{x}\wedge\mathbf{z}}(F_{\mathbf{x}}^{-1}(t)) \\ & - tF_{\mathbf{x}\wedge\mathbf{z}}(F_{\mathbf{z}}^{-1}(s)) + ts).\end{aligned}$$

We mention that for any $\mathbf{x} \in C$ and n

$$\{K_{n,\mathbf{x}}(t), 0 < t < 1\} \overset{D}{\equiv} \left\{ \frac{1}{f_{\mathbf{x}}(F_{\mathbf{x}}(t))} \frac{1}{(F^*(\mathbf{x}))^{1/2}} B(t), 0 < t < 1 \right\},$$

where $\{B(t), 0 \le t \le 1\}$ stands for a Brownian bridge.

Theorem 1.2 provides the weak convergence of the conditional quantile process. In principle, it could be used to construct confidence bands for $F_{\mathbf{x}}^{-1}(t)$ uniformly in $\mathbf{x}$ and/or in t. However, the distribution function of the supremum of the limiting Gaussian process is unknown, and it is unlikely that it can be computed explicitly. Also, even simulations of the limiting Gaussian process would require the estimation of the quantile density function $f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))$. In general, density estimators converge very slowly, and they depend very heavily

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

on the choice of the smoothing parameter (see Silverman, 1986). Because of the appearance of the quantile density in the limit, it is not obvious how to use resampling, such as bootstrap. In the next section, we discuss a method to construct confidence intervals and bands without inference on the quantile density function. All the technical proofs are given in the last two sections.

2. CONFIDENCE INTERVALS AND BANDS FOR $F_{\mathbf{x}}^{-1}(t)$

In this section, we discuss a method of constructing confidence intervals and bands for the conditional quantile without the estimation or knowledge of the conditional quantile density function. We adapt the method of Csörgő and Révész (1984). For further discussion and applications of this method we refer to Csörgő and Horváth (1989) and Chung, Csörgő, and Horváth (1990). The method is based on the observation that for any distribution function G, we have

$$G(x) \geq t \quad \text{if and only if } G^{-1}(t) \leq x, \tag{2.1}$$

$$G(x) < t \quad \text{if and only if } G^{-1}(t) > x, \tag{2.2}$$

if $0 < t < 1$ (see Shorack and Wellner, 1986, pp. 5–8). We use the confidence interval

$$F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \leq F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2}). \tag{2.3}$$

THEOREM 2.1. *If* $\mathbf{x} \in C$, $0 < t < 1$, *and*

$$F_{\mathbf{x}}(y) \quad \textit{is continuous in } y, \tag{2.4}$$

then

$$\lim_{n\to\infty} P(F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \leq F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2})) = 2\Phi(c(t(1-t)/F^*(\mathbf{x}))^{-1/2}) - 1,$$

for all $c > 0$, *where* Φ *denotes the standard normal distribution function.*

We note that if $\mathbf{x} \in C$, then (2.4) holds if and only if Y has a continuous distribution function.

Because $F^*(\mathbf{x})$ is unknown, it must be estimated to use Theorem 2.1 to construct confidence intervals. Let

$$d_n = t(1-t)/F_n^*(\mathbf{x}).$$

COROLLARY 2.1. *If the conditions of Theorem 2.1 are satisfied, then*

$$\lim_{n\to\infty} P(F_{n,\mathbf{x}}^{-1}(t - cd_n^{1/2}n^{-1/2}) \leq F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cd_n^{1/2}n^{-1/2})) = 2\Phi(c) - 1,$$

where Φ *denotes the standard normal distribution function.*

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

Next, we show that the confidence interval in Corollary 2.1 can be used as a confidence band in t for any given $\mathbf{x}$, when c is computed from the distribution of the supremum of the absolute value of a Brownian bridge.

THEOREM 2.2. *If (1.1) and (2.4) hold,* $\mathbf{x} \in C$, *and* $0 < a < b < 1$, *then*

$$\lim_{n\to\infty} P(F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2}) \quad \text{for all } a \le t \le b)$$

$$= P\Big(\sup_{a\le t\le b} |B(t)| \le c(F^*(\mathbf{x}))^{1/2}\Big),$$

where $\{B(t), 0 \le t \le 1\}$ *denotes a Brownian bridge.*

Because $F_n^*(\mathbf{x})$ is a consistent estimator of $F^*(\mathbf{x})$, we have the following corollary.

COROLLARY 2.2. *If the conditions of Theorem 2.2 are satisfied, then*

$$\lim_{n\to\infty} P\left(F_{n,\mathbf{x}}^{-1}\left(t - \frac{c}{(nF_n^*(\mathbf{x}))^{1/2}}\right) \le F_{\mathbf{x}}^{-1}(t)\right.$$

$$\left. < F_{n,\mathbf{x}}^{-1}\left(t + \frac{c}{(nF_n^*(\mathbf{x}))^{1/2}}\right) \quad \text{for all } a \le t \le b\right)$$

$$= P\Big(\sup_{a\le t\le b} |B(t)| \le c\Big),$$

where $\{B(t), 0 \le t \le 1\}$ *denotes a Brownian bridge.*

Corollaries 2.1 and 2.2 provide distribution-free confidence intervals and bands for the conditional quantile $F_{\mathbf{x}}^{-1}(t)$ for any given $\mathbf{x}$. We note that a formula for the distribution function of $\sup_{a\le t\le b}|B(t)|$ can be found in Shorack and Wellner (1986, p. 38).

We can use Theorem 2.2 or Corollary 2.2 to construct confidence bands on $[a, b]$, $0 < a < b < 1$. The next result shows that, at least asymptotically, we can have confidence bands on $(0, 1)$. It would be nearly impossible to derive even asymptotic confidence bands on $(0, 1)$ from Theorem 1.2 because the limit of $n^{1/2}(F_{n,\mathbf{x}}^{-1}(t) - F_{\mathbf{x}}^{-1}(t))$ may not exist on (0,1).

THEOREM 2.3. *If (1.1) and (2.4) hold,* $\mathbf{x} \in C$,

$$\epsilon_n \to 0 \quad \text{and} \quad n^{1/2}\epsilon_n \to 0, \tag{2.5}$$

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

then

$$\lim_{n\to\infty} P(F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2}), \quad \text{for all } \epsilon_n \le t \le 1 - \epsilon_n)$$

$$= P\Big(\sup_{0\le t\le 1} |B(t)| \le c(F^*(\mathbf{x}))^{1/2}\Big),$$

where $\{B(t), 0 \le t \le 1\}$ *denotes a Brownian bridge.*

Similarly to Corollary 2.2, we have the following result.

COROLLARY 2.3. *If the conditions of Theorem 2.3 are satisfied, then*

$$\lim_{n\to\infty} P\left(F_{n,\mathbf{x}}^{-1}\left(t - \frac{c}{(nF_n^*(\mathbf{x}))^{1/2}}\right) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}\left(t + \frac{c}{(nF_n^*(\mathbf{x}))^{1/2}}\right) \quad \text{for all } \epsilon_n \le t \le 1 - \epsilon_n\right)$$

$$= P\Big(\sup_{0\le t\le 1} |B(t)| \le c\Big),$$

where $\{B(t), 0 \le t \le 1\}$ *denotes a Brownian bridge.*

We are not aware of any closed formula for the distribution function of $\sup_{a\le t\le b}|B(t)|$ if $a \ne 0$ or $b \ne 1$. However, it is easy to get it via simulations. The distribution function of $\sup_{0\le t\le 1}|B(t)|$ is well known, and tables are widely available (see Shorack and Wellner, 1986).

Our last result discusses confidence sets for $F_{\mathbf{x}}^{-1}(t)$ uniformly in $\mathbf{x}$ and t.

THEOREM 2.4. *If (1.1) holds,* $A \subseteq C$, $0 < a \le t \le b < 1$, *and*

$$F_{\mathbf{x}}(y) \quad \text{is continuous on } A \times R, \tag{2.6}$$

then

$$\lim_{n\to\infty} P(F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2}) \quad \text{for all } \mathbf{x} \in A, a \le t \le b)$$

$$= P\Big(\sup_{\mathbf{x}\in A}\sup_{a\le t\le b} |L_{\mathbf{x}}(t)| \le c\Big),$$

where $L_{\mathbf{x}}(t)$ *is a Gaussian process with* $\mathrm{E}L_{\mathbf{x}}(t) = 0$ *and*

$$\mathrm{E}L_{\mathbf{x}}(t)L_{\mathbf{z}}(s) = \frac{F^*(\mathbf{x}\wedge\mathbf{z})}{F^*(\mathbf{x})F^*(\mathbf{z})}\,(F_{\mathbf{x}\wedge\mathbf{z}}(F_{\mathbf{x}}^{-1}(t)\wedge F_{\mathbf{z}}^{-1}(s)) - sF_{\mathbf{x}\wedge\mathbf{z}}(F_{\mathbf{x}}^{-1}(t)) - tF_{\mathbf{x}\wedge\mathbf{z}}(F_{\mathbf{z}}^{-1}(s)) + st).$$

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

The Gaussian process $L_{\mathbf{x}}(t)$ has a very complicated covariance structure that depends on the unknown conditional distribution and quantile functions, and so it is impossible to simulate the sample paths of $L_{\mathbf{x}}(t)$. The bootstrap method provides an easy way to find the distribution function of functionals of $L_{\mathbf{x}}(t)$.

Given the original sample $(Y_i, \mathbf{X}_i)$, $1 \le i \le n$, we create the bootstrap sample $(Y_i^*, \mathbf{X}_i^*)$, $1 \le i \le m$, by sampling with equal probabilities from the original sample with replacement. So, conditionally on $(Y_i, \mathbf{X}_i)$, $1 \le i \le n$, the bootstrap variables are independent and identically distributed with distribution function $F_n(y, \mathbf{x})$. Using the bootstrap sample, we define the bootstrap versions of all empirical distribution functions introduced so far. Let

$$H_m(y, \mathbf{x}) = \frac{1}{m} \sum_{i=1}^{m} I(Y_i^* \le y, \mathbf{X}_i^* \le \mathbf{x}), \qquad H_m^*(\mathbf{x}) = \frac{1}{m} \sum_{i=1}^{m} I(\mathbf{X}_i \le \mathbf{x}),$$

and

$$H_{m,\mathbf{x}}(y) = \frac{H_m(y, \mathbf{x})}{H_m^*(\mathbf{x})}.$$

The bootstrap conditional quantile function is $H_{m,\mathbf{x}}^{-1}(t)$, the generalized right-continuous inverse of $H_{m,\mathbf{x}}(y)$.

The next result can be used to get the value of c in Theorem 2.4 via simulations.

THEOREM 2.5. *If the conditions of Theorem 2.4 are satisfied and* $m = m(n)$ *and* $0 < \liminf m/n \le \limsup n/m < \infty$, *then for all* $0 < c < \infty$,

$$\lim_{n\to\infty} P\left(\sup_{\mathbf{x}\in C} \sup_{a\le t\le b} m^{1/2} |H_{m,\mathbf{x}}(F_{n,\mathbf{x}}^{-1}(t)) - F_{n,\mathbf{x}}(F_{n,\mathbf{x}}^{-1}(t))| \le c \,|\, (Y_i, \mathbf{X}_i),\ 1 \le i \le n \right)$$

$$= P\left(\sup_{\mathbf{x}\in A} \sup_{a\le t\le b} |L_{\mathbf{x}}(t)| \le c \right) \quad a.s.,$$

where $L_{\mathbf{x}}(t)$ *is the Gaussian process in Theorem 2.4.*

Remark 2.1. We note that condition (2.6) is used only to prove that $\sup_{\mathbf{x}\in A} \sup_{a\le t\le b} |L_{\mathbf{x}}(t)|$ has a continuous distribution function. Because for any $\mathbf{x}$, $\{L_{\mathbf{x}}(t), 0 < t < 1\}$ is a Brownian bridge, we conclude that $\sup_{\mathbf{x}\in A} \sup_{a\le t\le b} |L_{\mathbf{x}}(t)|$ will have a continuous distribution function without assuming (2.6), if A is a countable set. Hence in the case of a countable A, Theorems 2.4 and 2.5 will be true assuming only (2.4) for all $\mathbf{x} \in A$.

In this section we described a method for obtaining asymptotic confidence intervals and bands for the conditional quantile function $F_{\mathbf{x}}^{-1}(t)$. The same arguments can be used to construct asymptotic lower and upper confidence limits and boundaries for $F_{\mathbf{x}}^{-1}(t)$.

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

3. SIMULATIONS

Let $0 < \alpha < 1$. According to Corollary 2.1, $I_{n,\mathbf{x}}(t) = [F_{n,\mathbf{x}}^{-1}(t - c_1 d_n^{1/2} n^{-1/2}), F_{n,\mathbf{x}}^{-1}(t + c_1 d_n^{1/2} n^{-1/2})]$ is an asymptotic $1 - \alpha$ confidence interval for $F_{\mathbf{x}}^{-1}(t)$, where c_1 satisfies the equation $\Phi(c_1) = 1 - \alpha/2$. It is easy to prove for the length of $I_{n,\mathbf{x}}(t)$ that

$$n^{1/2}(F_{n,\mathbf{x}}^{-1}(t + c_1 d_n^{1/2} n^{-1/2}) - F_{n,\mathbf{x}}^{-1}(t - c_1 d_n^{1/2} n^{-1/2})) \xrightarrow{P} \frac{2c_1}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))} \left(\frac{t(1-t)}{F^*(\mathbf{x})} \right)^{1/2}. \tag{3.1}$$

The asymptotic length of $I_{n,\mathbf{x}}(t)$ is the same as the length of the confidence interval obtained by Aragon et al. (2005) using the asymptotic normality of $F_{n,\mathbf{x}}^{-1}(t)$. Let c_2 be defined by $P(\sup_{0 \le t \le 1} |B(t)| \le c_2) = 1 - \alpha$. By Corollary 2.3, $J_{n,\mathbf{x}}(\epsilon_n, 1 - \epsilon_n) = [F_{n,\mathbf{x}}^{-1}(t - c_2(nF_n^*(\mathbf{x}))^{-1/2}), F_{n,\mathbf{x}}^{-1}(t + c_2(nF_n^*(\mathbf{x}))^{-1/2})$ $\epsilon_n \le t \le 1 - \epsilon_n]$ is an asymptotic $1 - \alpha$ confidence band in t for $F_{\mathbf{x}}^{-1}(t)$ for any fixed $\mathbf{x}$. Similarly to (3.1) we have

$$n^{1/2}(F_{n,\mathbf{x}}^{-1}(t + c_2(nF_n^*(\mathbf{x}))^{-1/2}) - F_{n,\mathbf{x}}^{-1}(t - c_2(nF_n^*(\mathbf{x}))^{-1/2})) \xrightarrow{P} \frac{2c_2}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))} \left(\frac{1}{F^*(\mathbf{x})} \right)^{1/2}. \tag{3.2}$$

This means that the width of $J_{n,\mathbf{x}}(\epsilon_n, 1 - \epsilon_n)$ depends on t. We point out that if Theorem 1.2 were used to derive confidence bands in t for $F_{\mathbf{x}}^{-1}(t)$ we would have the same asymptotic width as in (3.2). Similarly, Theorem 2.4 yields that $S_{n,C}(t) = [F_{n,\mathbf{x}}^{-1}(t - c_3 n^{-1/2}), F_{n,\mathbf{x}}^{-1}(t + c_3 n^{-1/2})$ $\mathbf{x} \in C]$ can be used as a confidence band, uniform in $\mathbf{x}$, where c_3 is defined by $P(\sup_{\mathbf{x} \in C} |L_{\mathbf{x}}(t)| \le c_3) = 1 - \alpha$. For the asymptotic width of $S_{n,C}(t)$ we have (3.2) with c_2 replaced by c_3.

We use both examples in Aragon et al. (2005) to illustrate our method. We simulate a sample of size $n = 500$ according to the Cobb–Douglas log-linear function model given by $Y = X^{1/2} \exp(-U)$, where X is uniform on (0,1) and U is an exponential random variable with $EU = \frac{1}{3}$. Elementary calculations yield

$$F_x(y) = \frac{1}{x} \left(3y^2 - \frac{2y^3}{\sqrt{x}} \right), \quad \text{if } 0 \le y \le \sqrt{x},$$

and so $F_x^{-1}(t)$ can be computed numerically. The graph of $F_x^{-1}(t)$ is displayed in Figures 1–3 with the corresponding confidence intervals and bands. Figure 1 contains the graphs of $I_{500,0.9}(t)$, if $t = 0.1$, 0.25, 0.5, 0.75, and 0.9, and Figure 2

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

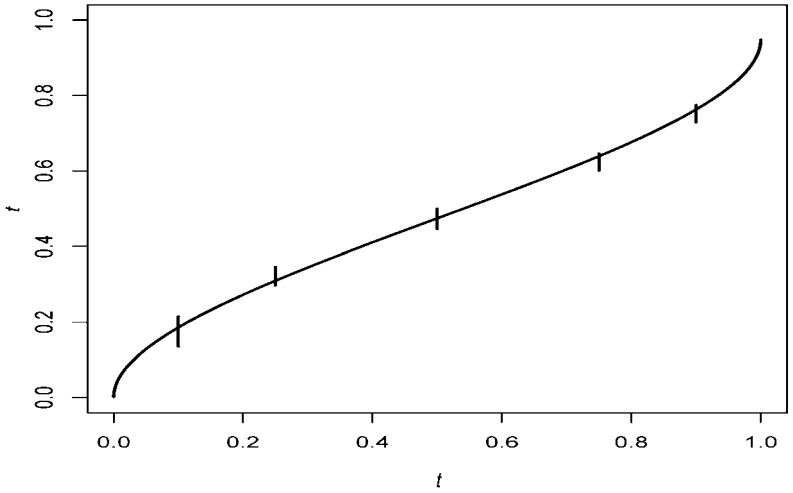


FIGURE 1. 90% confidence intervals $I_{500,0.9}(t)$ with $x = 0.9$, $t = 0.1$, 0.25, 0.5, 0.75, and 0.9 based on the Cobb-Douglas model.

displays $J_{500,0.9}(0.1, 0.9)$. The computation of $S_{500,[0,0.9]}(0.9)$ needs the value of c_3. We apply the bootstrap method of Theorem 2.5 using 2,000 repetitions. The graph of $S_{500,[0,0.9]}(0.9)$ is in Figure 3.

Next we consider the nonconvex production set given by the model

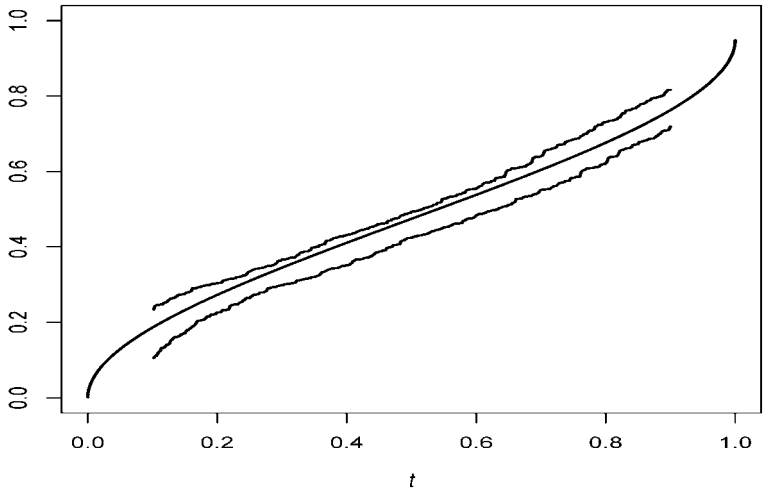


FIGURE 2. 90% band (uniform in t on [0.1, 0.9] and $x = 0.9$) $J_{500,0.9}(0.1, 0.9)$ based on the Cobb–Douglas model.

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

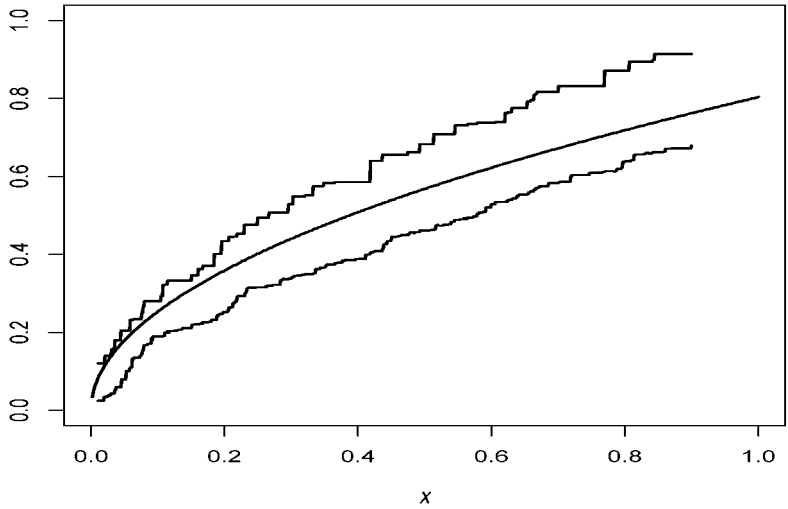


FIGURE 3. 90% band (uniform in x on $[0, 0.9]$ and $t = 0.9$) $S_{500,[0,0.9]}(0.9)$ based on the Cobb–Douglas model.

$$Y = \frac{\exp(-5 + 10X)}{1 + \exp(-5 + 10X)} \exp(-U), \tag{3.3}$$

where, as before, X is uniform on (0,1) and U is exponential with $EU = \frac{1}{3}$. In this case

$$\begin{aligned} F_x(y) = \frac{1}{x}\Bigg\{ & \frac{1}{2} - \frac{1}{10}\log\left(\frac{1}{y} - 1\right) \\ & + y^3\left(x - \frac{3}{10}\,e^{5-10x} - \frac{3}{20}\,e^{10-20x} - \frac{1}{30}\,e^{15-30x} - \frac{1}{2}\right. \\ & \quad + \frac{1}{10}\log\left(\frac{1}{y} - 1\right) + \frac{3}{10}\left(\frac{1}{y} - 1\right) + \frac{3}{20}\left(\frac{1}{y} - 1\right)^2 \\ & \quad \left. + \frac{1}{30}\left(\frac{1}{y} - 1\right)^3\right)\Bigg\}, \end{aligned} \tag{3.4}$$

if $0 \le y \le \exp(-5 + 10x)/(1 + \exp(-5 + 10x))$. The inverse $F_x^{-1}(t)$ was computed numerically, and it is in Figures 4–7 together with the corresponding intervals and bands. Figures 4–6 have the graphs of $I_{500,0.9}(t)$, $t = 0.1$, 0.25, 0.5, 0.75, 0.9, $J_{500,0.9}(0.1, 0.9)$, and $S_{500,[0,0.9]}(0.9)$. We note that the confidence intervals in Figure 4 are somewhat narrower than than the band in Figure 5. So

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

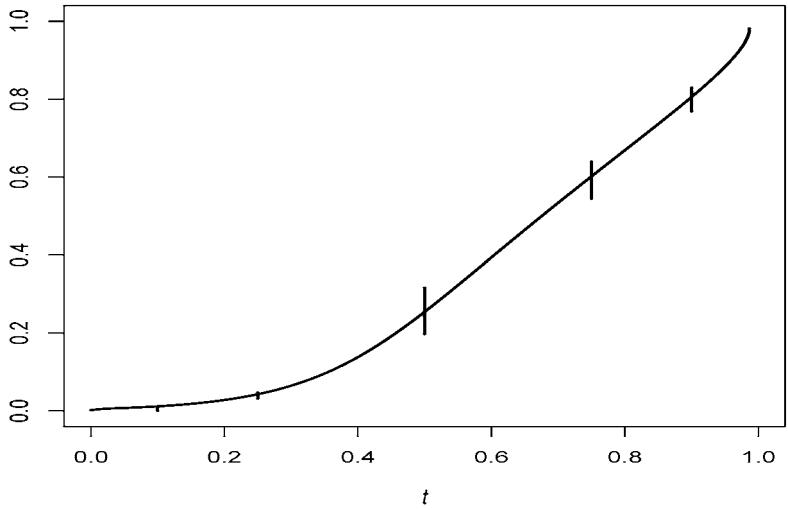


FIGURE 4. 90% confidence intervals $I_{500,0.9}(t)$, with $x = 0.9$, $t = 0.1$, 0.25, 0.5, 0.75, and 0.9 based on model (3.3).

even if we are interested in confidence intervals for $F_{\mathbf{x}}(t)$ at few selected t's, the application of the Bonferroni inequality will not provide a much better result.

We choose $x = 0.9$ and note that the width is increasing as t increases. The same can be observed in the graphs of $I_{500,x}$ and $J_{500,x}$ using different x's. The

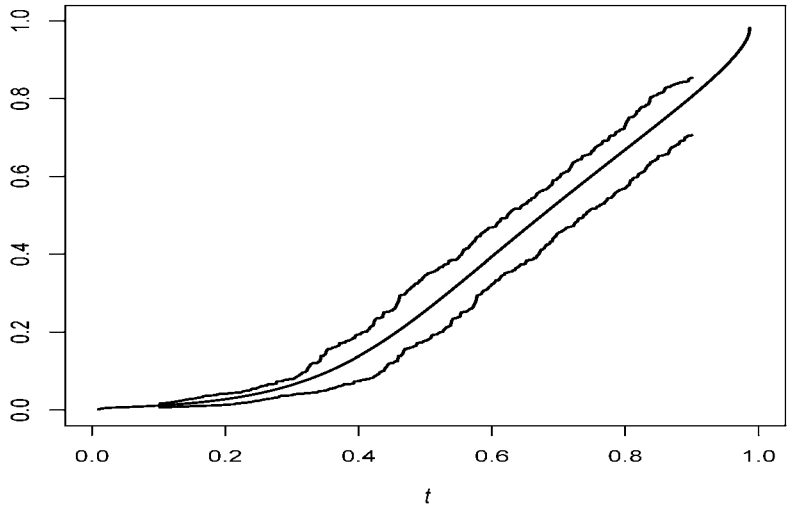


FIGURE 5. 90% band (uniform in t on [0.1, 0.9] and $x = 0.9$) $J_{500,0.9}(0.1, 0.9)$ based on model (3.3).

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

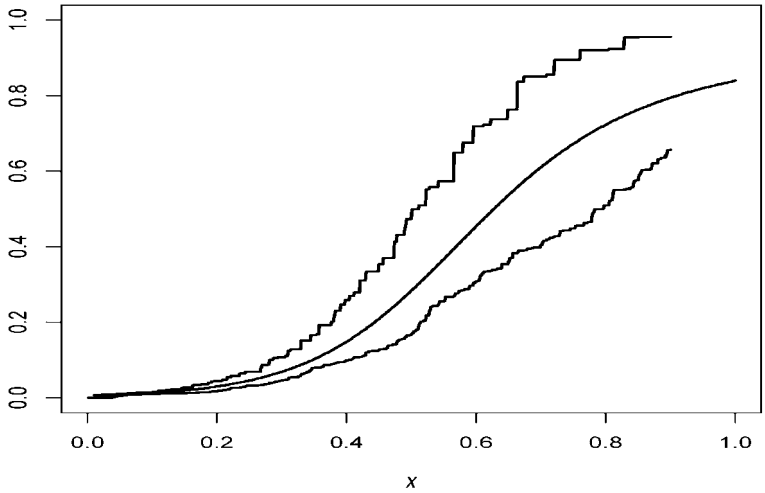


FIGURE 6. 90% band (uniform in x on [0, 0.9] and $t = 0.9$) $S_{500,[0,0.9]}(0.9)$ based on model (3.3).

confidence band on Figure 6 is very wide at the end. Large values of x yield more y's, and some of them are very large. The 90th percentile is determined by large observations. For comparison, $S_{500,[0,0.9]}(0.75)$ is also included in Figure 7. The shape of the confidence band is the same as in Figure 6 but not as wide on the right tail.

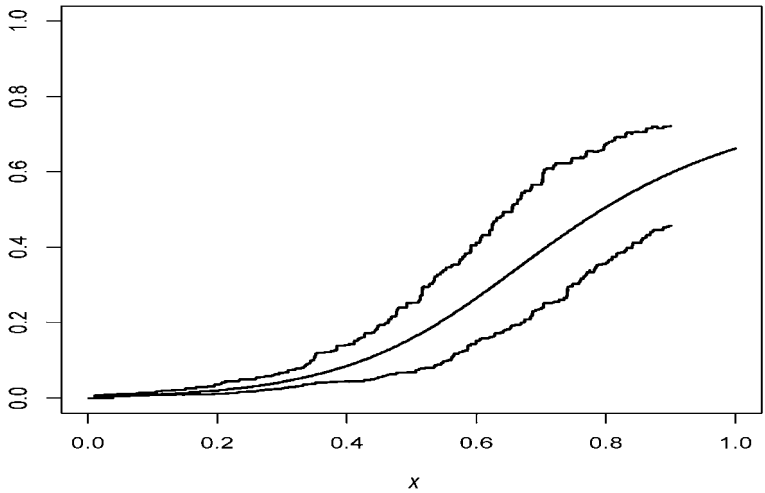


FIGURE 7. 90% band (uniform in x on [0, 0.9] and $t = 0.75$) $S_{500,[0,0.9]}(0.75)$ based on model (3.3).

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

4. PROOF OF THEOREMS 1.1 AND 1.2

We use the general method of getting limit results for quantiles. For a survey, we refer to Chapter 6 in Csörgő and Horváth (1993). Let

$$U_{n,\mathbf{x}}(t) = F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t)).$$

Then

$$U_{n,\mathbf{x}}^{-1}(t) = F_{\mathbf{x}}(F_{n,\mathbf{x}}^{-1}(t)).$$

Because $F_{n,\mathbf{x}}(y)$ is close to $F_{\mathbf{x}}(y)$ (uniformly in $\mathbf{x}$ and y), the process $U_{n,\mathbf{x}}(t)$ should be close to t. As a result of the geometric interpretation of the inverse of nondecreasing functions, the distance between $U_{n,\mathbf{x}}^{-1}(t)$ and t can be bounded by the distance between $U_{n,\mathbf{x}}(t)$ and t. This argument is utilized in Horváth (1984) (see also Csörgő and Horváth, 1993).

LEMMA 4.1. *If (1.1) holds and*

Y has a continuous distribution function, **(4.1)**

then we have

$$\sup_{\mathbf{x}\in C}\sup_{0\le t\le 1}|U_{n,\mathbf{x}}^{-1}(t) - t| = O((\log\log n/n)^{1/2}) \quad a.s.$$

Proof. By Lemma 1.1 of Csörgő and Horváth (1993, p. 24), we have

$$\begin{aligned}\sup_{\mathbf{x}\in C}\sup_{0\le t\le 1}|U_{n,\mathbf{x}}^{-1}(t) - t| &\le \sup_{\mathbf{x}\in C}\sup_{0\le t\le U_{n,\mathbf{x}}^{-1}(1)}|U_{n,\mathbf{x}}(t) - t| \\ &\le \sup_{\mathbf{x}\in C}\sup_{0\le t\le 1}|U_{n,\mathbf{x}}(t) - t| \\ &\le \sup_{\mathbf{x}\in C}\sup_{-\infty<y<\infty}|F_{n,\mathbf{x}}(y) - F_{\mathbf{x}}(y)|.\end{aligned}$$

(We note that Lemma 1.1 in Csörgő and Horváth, 1993, p. 24, is for the right-continuous inverse. However, the same argument also holds in our case.)

By the law of the iterated logarithm for empirical processes we have

$$\sup_{y,\mathbf{x}}|F_n(y,\mathbf{x}) - F(y,\mathbf{x})| = O((\log\log n/n)^{1/2}) \quad \text{a.s.} \qquad \textbf{(4.2)}$$

Observing that

$$\begin{aligned}|F_{n,\mathbf{x}}(y) - F_{\mathbf{x}}(y)| &= \left|\frac{F_n(y,\mathbf{x})}{F_n^*(\mathbf{x})} - \frac{F(y,\mathbf{x})}{F^*(\mathbf{x})}\right| \\ &\le \frac{|F_n(y,\mathbf{x}) - F(y,\mathbf{x})|}{F_n^*(\mathbf{x})} + \frac{|F_n^*(\mathbf{x}) - F^*(\mathbf{x})|}{F_n^*(\mathbf{x})F^*(\mathbf{x})},\end{aligned}$$

Lemma 4.1 follows from (4.2) and (1.1). ■

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

Proof of Theorem 1.1. We write

$$F_{n,\mathbf{x}}^{-1}(t) - F_{\mathbf{x}}^{-1}(t) = F_{\mathbf{x}}^{-1}(F_{\mathbf{x}}(F_{n,\mathbf{x}}^{-1}(t))) - F_{\mathbf{x}}^{-1}(t)$$

$$= \frac{1}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(\xi_{n,\mathbf{x}}(t)))}\,(U_{n,\mathbf{x}}^{-1}(t) - t),$$

where $\xi_{n,\mathbf{x}}(t)$ is between $U_{n,\mathbf{x}}^{-1}(t)$ and t. Condition (1.3) and Lemma 4.1 yield

$$\sup_{\mathbf{x}\in C}\sup_{\epsilon\le t\le 1-\epsilon}\frac{1}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(\xi_{n,\mathbf{x}}(t)))} = O(1) \quad \text{a.s.},$$

and therefore Theorem 1.1 follows from Lemma 4.1. ■

Next we obtain approximations for

$$\alpha_{n,\mathbf{x}}(y) = n^{1/2}(F_{n,\mathbf{x}}(y) - F_{\mathbf{x}}(y)).$$

LEMMA 4.2. *We can define a sequence of Gaussian processes* $\tilde{\Gamma}_{n,\mathbf{x}}(y)$ *such that*

$$\sup_{\mathbf{x}\in C}\sup_{y}|\alpha_{n,\mathbf{x}}(y) - \tilde{\Gamma}_{n,\mathbf{x}}(y)| = o_P(1) \tag{4.3}$$

for all C satisfying (1.1). Also, $\mathrm{E}\tilde{\Gamma}_{n,\mathbf{x}}(y) = 0$ *and*

$$\mathrm{E}\tilde{\Gamma}_{n,\mathbf{x}}(y)\tilde{\Gamma}_{n,\mathbf{x}'}(y')$$

$$= \frac{1}{F^*(\mathbf{x})F^*(\mathbf{x}'}))(F(y\wedge y',\mathbf{x}\wedge\mathbf{x}') - F_{\mathbf{x}'}(y')F(y,\mathbf{x}\wedge\mathbf{x}')$$

$$- F_{\mathbf{x}}(y)F(y',\mathbf{x}\wedge\mathbf{x}') + F_{\mathbf{x}}(y)F_{\mathbf{x}'}(y')F^*(\mathbf{x}\wedge\mathbf{x}'))$$

$$= \frac{F^*(\mathbf{x}\wedge\mathbf{x}')}{F^*(\mathbf{x})F^*(\mathbf{x}'}))(F_{\mathbf{x}\wedge\mathbf{x}'}(y\wedge y') - F_{\mathbf{x}'}(y')F_{\mathbf{x}\wedge\mathbf{x}'}(y)$$

$$- F_{\mathbf{x}}(y)F_{\mathbf{x}\wedge\mathbf{x}'}(y') + F_{\mathbf{x}}(y)F_{\mathbf{x}'}(y')). \tag{4.4}$$

Proof. Easy calculations and the multivariate Glivenko–Cantelli theorem give

$$\sup_{\mathbf{x}\in C}\sup_{-\infty<y<\infty}\left|\alpha_{n,\mathbf{x}}(y) - \left(\frac{n^{1/2}(F_n(y,\mathbf{x}) - F(y,\mathbf{x}))}{F^*(\mathbf{x})} - \frac{n^{1/2}(F_n^*(\mathbf{x}) - F^*(\mathbf{x}))}{F^*(\mathbf{x})}\,\frac{F(y,\mathbf{x})}{F^*(\mathbf{x})}\right)\right|$$

$$= o_P(1). \tag{4.5}$$

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

Philipp and Pinzur (1980) constructed a sequence of Gaussian processes $\Gamma_n(y,\mathbf{x})$ such that

$$\sup_{y,\mathbf{x}} |n^{1/2}(F_n(y,\mathbf{x}) - F(y,\mathbf{x})) - \Gamma_n(y,\mathbf{x})| = o_P(1). \tag{4.6}$$

Introducing $\Gamma_n^*(\mathbf{x}) = \lim_{y\to\infty}\Gamma_n(y,\mathbf{x})$, by (4.6) we have

$$\sup_{\mathbf{x}} |n^{1/2}(F_n^*(\mathbf{x}) - F^*(\mathbf{x})) - \Gamma_n^*(\mathbf{x})| = o_P(1). \tag{4.7}$$

Putting together (4.5)–(4.7), we get

$$\sup_{\mathbf{x}\in C}\sup_{y}\left|\alpha_{n,\mathbf{x}}(y) - \frac{1}{F^*(\mathbf{x})}(\Gamma_n(y,\mathbf{x}) - F_{\mathbf{x}}(y)\Gamma_n^*(\mathbf{x}))\right| = o_P(1). \tag{4.8}$$

We also note that $\Gamma_n(y,\mathbf{x})$ is a Gaussian with

$$\mathrm{E}\Gamma_n(y,\mathbf{x}) = 0, \qquad \mathrm{E}\Gamma_n(y,\mathbf{x})\Gamma_n(y',\mathbf{x}') = F(y\wedge y',\mathbf{x}\wedge\mathbf{x}') - F(y,\mathbf{x})F(y',\mathbf{x}').$$

Hence the covariance structure of

$$\tilde{\Gamma}_{n,\mathbf{x}}(y) = \frac{1}{F^*(\mathbf{x})}(\Gamma_n(y,\mathbf{x}) - F_{\mathbf{x}}(y)\Gamma_n^*(\mathbf{x}))$$

can be easily computed, and thus we get (4.4). ■

LEMMA 4.3. *If (1.1) and (4.1) hold, then*

$$\sup_{\mathbf{x}\in C}\sup_{y}\sup_{|s|\le h}|\tilde{\Gamma}_{\mathbf{x}}(y) - \tilde{\Gamma}_{\mathbf{x}}(y+s)| = o_P(1) \quad \text{as } h\to 0,$$

where $\{\tilde{\Gamma}_{\mathbf{x}}(y), \mathbf{x}\in C, y\in R\} \overset{\mathcal{D}}{=} \{\tilde{\Gamma}_{n,\mathbf{x}}(y), \mathbf{x}\in C, y\in R\}$.

Proof. There is V, a $(p+1)$-variate distribution function with uniform marginals (see Wichura, 1973) such that

$$F(y,\mathbf{x}) = V(F_{(0)}(y), F_{(1)}(x_1), \ldots, F_{(p)}(x_p)),$$

where $F_{(0)}, F_{(1)}, \ldots, F_{(p)}$ are the marginal distribution functions of $(Y,\mathbf{X})$. Let $W_V(y,\mathbf{x})$ be a Gaussian process with $\mathrm{E}W_V(y,\mathbf{x}) = 0$ and $\mathrm{E}W_V(y,\mathbf{x})W_V(y',\mathbf{x}') = V(y\wedge y',\mathbf{x}\wedge\mathbf{x}')$. Computing the covariance functions, one can easily verify that

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

$$\{\tilde{\Gamma}_{\mathbf{x}}(y), \mathbf{x} \in C, y \in R\}$$

$$\stackrel{\mathcal{D}}{=} \left\{ \frac{1}{F^*(\mathbf{x})} \{W_V(F_{(0)}(y), F_{(1)}(x_1), \ldots, F_{(p)}(x_p)) - F(y, \mathbf{x}) W_V(1, 1, \ldots, 1) \right.$$

$$- F_{\mathbf{x}}(y)(W_V(1, F_{(1)}(x_1), \ldots, F_{(p)}(x_p))$$

$$\left. - V(1, F_{(1)}(x_1), \ldots, F_{(p)}(x_p)) W_V(1, 1, \ldots, 1))\}, \mathbf{x} \in C, y \in R \right\}.$$

Because $\sup_{\mathbf{x} \in C} \sup_y \sup_{|s| \le h} |F_{\mathbf{x}}(y + s) - F_{\mathbf{x}}(y)| \to 0$ $(h \to 0)$ we get that

$$\sup_{\mathbf{x} \in C} \sup_y \sup_{|s| \le h} |F_{\mathbf{x}}(y + s) - F_{\mathbf{x}}(y)| |W_V(1, F_{(1)}(x_1), \ldots, F_{(p)}(x_p))$$

$$- V(1, F_{(1)}(x_1), \ldots, F_{(p)}(x_p)) W_V(1, 1, \ldots, 1)|$$

$$= o_P(1),$$

as $h \to 0$ and

$$\sup_{\mathbf{x} \in C} \sup_y \sup_{|s| \le h} |F_{\mathbf{x}}(y + s) - F_{\mathbf{x}}(y)| \, |W_v(1, 1, \ldots, 1)| = o_P(1) \qquad (h \to 0).$$

Also,

$$\sup_{\mathbf{x} \in C} \sup_y \sup_{|s| \le h} |W_V(F_{(0)}(y + s), F_{(1)}(x_1), \ldots, F_{(p)}(x_p))$$

$$- W_V(F_{(0)}(y), F_{(1)}(x_1), \ldots, F_{(p)}(x_p))|$$

$$\le \sup_{0 \le s_1, \ldots, s_p \le 1} \sup_y \sup_{|s| \le h}$$

$$\times |W_V(F_{(0)}(y + s), s_1, \ldots, s_p) - W_V(F_{(0)}(y), s_1, \ldots, s_p)|.$$

Because $F_{(0)}(y)$ is uniformly continuous and $W_V(s_0, s_1, \ldots, s_p)$ has continuous sample paths with probability one, we conclude that

$$\sup_{0 \le s_1, \ldots, s_p \le 1} \sup_y \sup_{|s| \le h} |W_V(F_{(0)}(y + s), s_1, \ldots, s_p) - W_V(F_{(0)}(y), s_1, \ldots, s_p)|$$

$$= o_P(1) \qquad (h \to 0),$$

which also completes the proof of Lemma 4.3. ■

Proof of Theorem 1.2. Following the argument in Lemma 1.2 of Csörgő and Horváth (1993, p. 25) (see also Horváth, 1984) we have

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

$$\sup_{\mathbf{x}\in C}\sup_{0<t<1}|t-U_{n,\mathbf{x}}^{-1}(t)-U_{n,\mathbf{x}}(U_{n,\mathbf{x}}^{-1}(t))-U_{n,\mathbf{x}}^{-1}(t)|$$

$$\le \sup_{\mathbf{x}\in C}\sup_{0\le t<U_{n,\mathbf{x}}^{-1}(1)}|U_{n,\mathbf{x}}(t)-U_{n,\mathbf{x}}(t-)|$$

$$\le \sup_{\mathbf{x}\in C}\sup_{0\le t\le 1}|F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))-F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t)-)|, \tag{4.9}$$

where $g(t-)$ denotes the limit of the function g at t from the left. Because $F_{n,\mathbf{x}}(y)\neq F_{n,\mathbf{x}}(y-)$ if y is one of the Y_i's we get

$$\sup_{\mathbf{x}\in C}\sup_{0\le t\le 1}|F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))-F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t)-)|$$

$$\le \frac{1}{F_n^*(\mathbf{x})}\sup_y\frac{1}{n}\sum_{i=1}^n I(Y_i=y)$$

$$= O\left(\frac{1}{n}\right)\quad \text{a.s.}, \tag{4.10}$$

noting that by (4.1) ties can occur with probability zero.

Using (4.8) we obtain for any $0<\epsilon<\frac{1}{2}$,

$$\sup_{\mathbf{x}\in C}\sup_{\epsilon\le t\le 1-\epsilon}|n^{1/2}(U_{n,\mathbf{x}}(t)-t)-\tilde{\Gamma}_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))|=o_P(1). \tag{4.11}$$

So by (4.9) and (4.10) we have that

$$\sup_{\mathbf{x}\in C}\sup_{\epsilon\le t\le 1-\epsilon}|n^{1/2}(t-U_{n,\mathbf{x}}^{-1}(t))-\tilde{\Gamma}_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(U_{n,\mathbf{x}}^{-1}(t)))|=o_P(1).$$

Using Lemma 4.1 we conclude that

$$\sup_{\mathbf{x}\in C}\sup_{\epsilon\le t\le 1-\epsilon}|F_{\mathbf{x}}^{-1}(U_{n,\mathbf{x}}^{-1}(t))-F_{\mathbf{x}}^{-1}(t)|=o_P(1),$$

and so by Lemma 4.3 we conclude that

$$\sup_{\mathbf{x}\in C}\sup_{\epsilon\le t\le 1-\epsilon}|\tilde{\Gamma}_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(U_{n,\mathbf{x}}^{-1}(t)))-\tilde{\Gamma}_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))|=o_P(1). \tag{4.12}$$

Hence by (4.9)–(4.12), we have

$$\sup_{\mathbf{x}\in C}\sup_{\epsilon\le t\le 1-\epsilon}|n^{1/2}(t-U_{n,\mathbf{x}}^{-1}(t))-\hat{\Gamma}_{n,\mathbf{x}}(t)|=o_P(1), \tag{4.13}$$

where $\hat{\Gamma}_{n,\mathbf{x}}(t)=-\tilde{\Gamma}_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))$.

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

The mean value theorem gives

$$R_{n,\mathbf{x}}(t) = n^{1/2}(F_{n,\mathbf{x}}^{-1}(t) - F_{\mathbf{x}}^{-1}(t))$$
$$= n^{1/2}(F_{\mathbf{x}}^{-1}(U_{n,\mathbf{x}}^{-1}(t)) - F_{\mathbf{x}}^{-1}(t))$$
$$= \frac{1}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(\xi_{n,\mathbf{x}}(t)))}\, n^{1/2}(U_{n,\mathbf{x}}^{-1}(t) - t),$$

where $\xi_{n,\mathbf{x}}(t)$ is between $U_{n,\mathbf{x}}^{-1}(t)$ and t. Conditions (1.2), (1.3), and (4.13) give

$$\sup_{\mathbf{x}\in C}\sup_{\epsilon\le t\le 1-\epsilon}\left|\frac{1}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(\xi_{n,\mathbf{x}}(t)))} - \frac{1}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))}\right| = o_P(1),$$

and therefore Theorem 1.2 follows from (4.13) with

$$K_{n,\mathbf{x}}(t) = \frac{-1}{f_{\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))}\,\hat{\Gamma}_{n,\mathbf{x}}(t). \qquad \blacksquare$$

5. PROOFS OF THEOREMS 2.1–2.5

If

$$0 < t - cn^{-1/2} \le t + cn^{-1/2} < 1, \tag{5.1}$$

then according to (2.1) and (2.2),

$$F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2}) \tag{5.2}$$

holds if and only if

$$t - cn^{-1/2} \le F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t)) < t + cn^{-1/2}. \tag{5.3}$$

Proof of Theorem 2.1. We can assume that n is so large that (5.1) holds. It follows from (4.8) that

$$\lim_{n\to\infty} P(F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2}))$$
$$= \lim_{n\to\infty} P(n^{1/2}|F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t)) - t| \le c)$$
$$= P(|\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t))| \le c),$$

where $\tilde{\Gamma}(y)$ is a Gaussian process that equals in distribution to $\tilde{\Gamma}_{n,\mathbf{x}}(y)$ in Section 4. Clearly, $\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t))$ is normal with $\mathrm{E}\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)) = 0$ and $\mathrm{E}(\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)))^2 = t(1-t)/F^*(\mathbf{x})$. Therefore the proof is complete. $\blacksquare$

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

Proof of Corollary 2.1. By the Glivenko–Cantelli theorem for the multivariate empirical distribution function we have

$$F_n^*(\mathbf{x}) \to F^*(\mathbf{x}) \quad \text{a.s.}, \tag{5.4}$$

and so the result follows from Theorem 2.1. ■

Proof of Theorem 2.2. Following the proof of Theorem 2.1 we get

$$\begin{aligned} &P(F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2}) \quad \text{for all } a \le t \le b) \\ &\quad = P(-c \le n^{1/2} F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t) - t) < c \quad \text{for all } a \le t \le b), \end{aligned} \tag{5.5}$$

and therefore,

$$\begin{aligned} &P\left(\inf_{a \le t \le b} \tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)) > -c \quad \text{and} \quad \sup_{a \le t \le b} \tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)) < -c \right) \\ &\quad \le \liminf_{n \to \infty} P(-c \le n^{1/2} F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t) - t) < c \quad \text{for all } a \le t \le b) \\ &\quad \le \limsup_{n \to \infty} P(-c \le n^{1/2} F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t) - t) < c \quad \text{for all } a \le t \le b) \\ &\quad \le P\left(\inf_{a \le t \le b} \tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)) \ge -c \quad \text{and} \quad \sup_{a \le t \le b} \tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)) \le c \right). \end{aligned} \tag{5.6}$$

For any fixed $\mathbf{x}$ we have $\mathrm{E}\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)) = 0$ and

$$\mathrm{E}\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t))\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(s)) = \frac{1}{F^*(\mathbf{x})}(t \wedge s - ts).$$

Hence for any $\mathbf{x}$,

$$\{\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)),\ 0 < t < 1\} \overset{D}{=} \{(1/F^*(\mathbf{x}))^{1/2} B(t),\ 0 < t < 1\}, \tag{5.7}$$

where $\{B(t),\ 0 \le t \le 1\}$ is a Brownian bridge.

The distribution functions of $\inf_{a \le t \le b} B(t)$ and $\sup_{a \le t \le b} B(t)$ are continuous (see Shorack and Wellner, 1986, p. 38), and therefore,

$$P\left(\inf_{a \le t \le b} \tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)) = -c \right) = 0 \quad \text{and} \quad P\left(\sup_{a \le t \le b} \tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t)) = c \right) = 0$$

for all c. Hence by (5.5) and (5.6) we have

$$\begin{aligned} &\lim_{n \to \infty} P(F_{n,\mathbf{x}}^{-1}(t - cn^{-1/2}) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t + cn^{-1/2}) \quad \text{for all} \quad a \le t \le b) \\ &\quad = P\left(\sup_{a \le t \le b} |B(t)| \le c(F^*(\mathbf{x}))^{1/2} \right), \end{aligned}$$

completing the proof of Theorem 2.2. ■

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

Proof of Corollary 2.2. It is an immediate consequence of (5.4) and Theorem 2.2. ■

Proof of Theorem 2.3. By (2.5) we can assume that n is so large that (5.1) holds for all $t \in [\epsilon_n, 1-\epsilon_n]$. Using (5.2), (5.3), and (4.8) we get that

$$P(F_{n,\mathbf{x}}^{-1}(t-cn^{-1/2}) \le F_{\mathbf{x}}^{-1}(t) < F_{n,\mathbf{x}}^{-1}(t+cn^{-1/2}) \quad \text{for all } \epsilon_n \le t \le 1-\epsilon_n)$$

$$= P\left(\sup_{\epsilon_n \le t \le 1-\epsilon_n} n^{1/2}|F_{n,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t)) - t| \le c\right)$$

$$= P\left(\sup_{\epsilon_n \le t \le 1-\epsilon_n} |\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t))| \le c\right) + o(1) \qquad (n \to \infty).$$

In light of (5.7), it is enough to show that

$$\sup_{0 \le t \le \epsilon_n} |B(t)| \xrightarrow{P} 0 \tag{5.8}$$

and

$$\sup_{1-\epsilon_n \le t \le 1} |B(t)| \xrightarrow{P} 0. \tag{5.9}$$

As a result of symmetry, (5.8) and (5.9) are equivalent. Because $B(0) = 0$ a.s. and almost all sample paths of $B(t)$ are continuous, we have (5.8). Thus we have

$$\lim_{n\to\infty} P\left(\sup_{\epsilon_n \le t \le 1-\epsilon_n} |\tilde{\Gamma}(F_{\mathbf{x}}^{-1}(t))| \le c\right) = P\left(\left(\frac{1}{F^*(\mathbf{x})}\right)^{1/2} \sup_{0 \le t \le 1} |B(t)| \le c\right),$$

which completes the proof. ■

Proof of Theorem 2.4. We note that by Tsirelson (1975), $\sup_{\mathbf{x}\in A} \sup_{a \le t \le b} L_{\mathbf{x}}(t)$ has a continuous distribution. Hence Theorem 2.4 is an immediate consequence of (5.2), (5.3), and the approximation in (4.8). ■

Proof of Theorem 2.5. Using Bickel and Freedman (1981), we can define a Gaussian process $T_{m,\mathbf{x}}(y)$ such that for all $\epsilon > 0$,

$$\lim_{n,m\to\infty} P\left(\sup_{\mathbf{x}\in C} \sup_{y} |m^{1/2}(H_{m,\mathbf{x}}(y) - F_{m,\mathbf{x}}(y)) - T_{m,\mathbf{x}}(y)| > \epsilon \,|\, (Y_i, \mathbf{X}_i),\ 1 \le i \le n\right)$$

$$= 0, \quad \text{a.s.}$$

and

$$\{T_{m,\mathbf{x}}(y), \mathbf{x} \in C, y \in R\} \overset{D}{=} \{\tilde{\Gamma}_{n,\mathbf{x}}(y), \mathbf{x} \in C, y \in R\}.$$

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press

Also, $T_{m,\mathbf{x}}(y)$ has continuous sample paths with probability one. Using Theorem 1.1 and the almost sure continuity of $T_{m,\mathbf{x}}(y)$ we conclude that

$$\lim_{n\to\infty} P\left(\sup_{\mathbf{x}\in C}\sup_{a\le t\le b}|T_{m,\mathbf{x}}(F_{m,\mathbf{x}}^{-1}(t)) - T_{m,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))| > \epsilon \,|\, (Y_i,\mathbf{X}_i),\, 1\le i\le n\right)$$
$$= 0 \quad \text{a.s.}$$

for all $\epsilon > 0$ (note that the distribution of $T_{m,\mathbf{x}}(y)$ does not depend on m). Observing that $T_{m,\mathbf{x}}(F_{\mathbf{x}}^{-1}(t))$ and $L_{\mathbf{x}}(t)$ are equal in distribution for each m, we have completed the proof. ■

REFERENCES

Aragon, Y., A. Daouia, & C. Thomas-Agnan (2005) Nonparametric frontier estimation: A conditional quantile-based approach. *Econometric Theory* 21, 358–389.

Bickel, P.J. & D.A. Freedman (1981) Some asymptotic theory for the bootstrap. *Annals of Statistics* 9, 1196–1217.

Cazals, C., J.P. Florens, & L. Simar (2002) Nonparametric frontier estimation: A robust approach. *Journal of Econometrics* 1, 1–25.

Chung, C.-J., M. Csörgő, & L. Horváth (1990) Confidence bands for quantile function under random censorship. *Annals of the Institute of Statistical Mathematics* 42, 21–36.

Csörgő, M. & L. Horváth (1989) On confidence bands for the quantile function of a continuous distribution function. In I. Berkes, E. Czáki, & P. Révész (eds.), *Colloquia Mathematica Societatis János Bolyai*, vol. 57, *Limit Theorems of Probability Theory and Statistics*, pp. 95–106. North-Holland.

Csörgő, M. & L. Horváth (1993) *Weighted Approximations in Probability and Statistics*. Wiley.

Csörgő, M. & P. Révész (1984) Two approaches to constructing simultaneous confidence bounds for quantiles. *Probability and Mathematical Statistics* 4, 221–236.

Daouia, A. (2005) Asymptotic representation theory for nonstandard conditional quantiles. *Nonparametric Statistics* 17, 253–268.

Debreu, G. (1951) The coefficient of resource utilization. *Econometrica* 19, 273–292.

Horváth, L. (1984) Strong approximation of renewal processes. *Stochastic Processes and Their Applications* 18, 127–138.

Koopmans, T.C. (1951) An analysis of production as an efficient combination of activities. In T.C. Koopmans (ed.), *Activity Analysis of Production and Allocation*, Cowles Commission for Research in Economics, Monograph 13. Wiley.

Philipp, W. & L. Pinzur (1980) Almost sure approximation theorems for the multivariate empirical process. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* 54, 1–13.

Shephard, R.W. (1970) *Theory of Costs and Production Function.* Princeton University Press.

Shorack, G.R. & J.A. Wellner (1986) *Empirical Processes with Applications to Statistics.* Wiley.

Silverman, B.W. (1986) *Density Estimation for Statistics and Data Analysis.* Chapman and Hall.

Tsirelson, B.S. (1975) Density of the distribution of the maximum of a Gaussian process (in Russian). *Teorija Verojatnosty i Primenen* 20, 865–873.

Wichura, M.J. (1973) Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments. *Annals of Probability* 1, 272–296.

https://doi.org/10.1017/S0266466608080638 Published online by Cambridge University Press