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ARTICLE

On stability of rainbow matchings

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Abstract

We show that for any integer $k \geq 1$ there exists an integer $t_0(k)$ such that, for integers $t, k_1, \ldots, k_{t+1}, n$ with $t > t_0(k)$, $\max\{k_1, \ldots, k_{t+1}\} \leq k$, and $n > 2k(t+1)$, the following holds: If F_i is a k_i-uniform hypergraph with vertex set $[n]$ and more than $\binom{n}{k_i} - \binom{n-t}{k_i} - \binom{n-t-k}{k_i-1} + 1$ edges for all $i \in [t+1]$, then either $\{F_1, \ldots, F_{t+1}\}$ admits a rainbow matching of size $t+1$ or there exists $W \in \binom{[n]}{t}$ such that W intersects F_i for all $i \in [t+1]$. This may be viewed as a rainbow non-uniform extension of the classical Hilton-Milner theorem. We also show that the same holds for every t and $n > 2k^3t$, generalizing a recent stability result of Frankl and Kupavskii on matchings to rainbow matchings.

Keywords: Rainbow matching; hypergraph matching; vertex cover; stability

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1. Introduction

A classical problem in extremal set theory is to determine the maximum number of edges in a hypergraph with no matching of a given size. Let us write $\nu(H)$ for the size of the largest matching in a hypergraph H. Erdős [6] in 1965 made the following conjecture (known as the Erdős Matching Conjecture): For positive integers k, n, t with $n \geq kt$, every k-uniform hypergraph H (or k-graph for short) on n vertices with $\nu(H) < t$ satisfies $e(H) \leq \max\left\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\right\}$. This bound is tight for the following two extremal constructions: (a) the complete k-graph on $kt-1$ vertices, and (b) the k-graph on n vertices in which every edge intersects a fixed set of $t-1$ vertices. There has been recent progress on this conjecture, see [2, 3, 7, 9, 10, 15, 20]. In particular, Frankl [9] proved that if $n \geq (2t-1)k - (t-1)$ and $\nu(H) < t$ then $e(H) \leq \binom{n}{k} - \binom{n-t+1}{k}$, with further improvement by Frankl and Kupavskii [11].

The work in this paper was motivated by a recent result of Frankl and Kupavskii [12] on a stability version of the Erdős Matching Conjecture. We postpone the detailed discussion on this motivation to Section 6, since it uses technical notation that is not required for stating and proving our main result. For now, we simply mention that we are interested in extending a result of Frankl and Kupavskii [12] to rainbow matchings.

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Let $\mathcal{F} = \{F_1, \ldots, F_t\}$ be a family of hypergraphs. A set of pairwise disjoint edges, one from each F_i, is called a *rainbow matching* for $\mathcal{F}$, and we say that $\mathcal{F}$ or $\{F_1, \ldots, F_t\}$ *admits* a rainbow matching. Aharoni and Howard [1] (see also Huang, Loh and Sudakov [15]) conjectured the following rainbow version of the Erdős Matching Conjecture: If $E(F_i) \subseteq \binom{[n]}{k}$ and $e(F_i) > \max\left\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\right\}$ for $i \in [t]$ then $\{F_1, \ldots, F_t\}$ admits a rainbow matching. This conjecture was confirmed for $n > 3k^2t$ by Huang, Loh, and Sudakov [15]. The bound $3k^2t$ on n was improved to $12kt\log(e^2t)$ by Frankl and Kupavskii [13], and subsequently to $3ekt$ by Kupavskii [18]. Lu, Wang, and Yu [19] further improved this bound to $n > 2kt$ for t sufficiently large. Recently, Keevash, Lifshitz, Long, and Minzer [17] proved a more general result with $n \geq Ckt$ for some constant C using sharp threshold techniques developed in [16].

Frankl and Kupavskii [12] recently proved a result generalizing the the Hilton-Milner theorem by considering vertex covers of hypergraphs. A *vertex cover* of a hypergraph H is a set of vertices of H that meets all edges of H, and we use $\tau(H)$ to denote the minimum size of a vertex cover of H. Our main result below extends that result of Frankl and Kupavskii [12] to rainbow matchings. (By letting $k_i = k$ and $F_i = F$ for all $i \in [t+1]$, Theorem 1 gives that result of Frankl and Kupavskii.)

Theorem 1. *Let $k \geq 1$ be an integer. There exists $t_0 = t_0(k)$ such that, for positive integers $t, k_1, k_2, \ldots, k_{t+1}, n$ with $t > t_0$, $\max\{k_1, \ldots, k_{t+1}\} \leq k$, and $n > 2k(t+1)$, the following holds: If $E(F_i) \subseteq \binom{[n]}{k_i}$ and*

$$e(F_i) > \binom{n}{k_i} - \binom{n-t}{k_i} - \binom{n-t-k}{k_i - 1} + 1$$

for all $i \in [t+1]$, then either

(i) $\{F_1, \ldots, F_{t+1}\}$ admits a rainbow matching of size $t+1$; or

(ii) there exists $W \in \binom{[n]}{t}$ such that W is a vertex cover of F_i for all $i \in [t+1]$.

In Section 2, we first reduce the problem for finding rainbow matchings for non-uniform hypergraphs to one for uniform hypergraphs (through an operation called 'expansion'), and then reduce the problem to a matching problem for $\mathcal{F}^{t+1}(k, n)$, a special class of uniform hypergraphs. (We will see that the extremal hypergraphs of Theorem 1 naturally corresponds to a special class of $(k+1)$-graphs, denoted by $\mathcal{F}_{t+1}(n, k; t)$ and defined later.) This new matching problem for $\mathcal{F}^{t+1}(k, n)$ will be further reduced to a near perfect matching problem.

In Section 3, we prove Theorem 1 for small t by an inductive argument. We prove Theorem 1 in Section 4 when $\mathcal{F}^{t+1}(k, n)$ is close to $\mathcal{F}_{t+1}(k, n; t)$, in the sense that most edges of $\mathcal{F}_{t+1}(k, n; t)$ are also edges of $\mathcal{F}^{t+1}(k, n)$. In Section 5, we deal with the case when $\mathcal{F}^{t+1}(k, n)$ is not close to $\mathcal{F}_{t+1}(k, n; t)$. In Section 6, we discuss a conjecture of Frankl and Kupavskii [12] that motivates our work and conclude with a rainbow version of that conjecture.

2. Reductions

The goal of this section is to reduce the problem for finding rainbow matchings for non-uniform hypergraphs to a near perfect matching problem for uniform hypergraphs. We need to use shadows of hypergraphs. Let k, m be positive integers. The *k-cascade representation* of m is

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$$

where $a_k > a_{k-1} > \ldots > a_s \geq s \geq 1$ are integers. Given a family $\mathcal{F}$ of sets, the *shadow* $\partial\mathcal{F}$ of $\mathcal{F}$ is defined as

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$$\partial\mathcal{F} = \{E : E = F \setminus \{x\} \text{ for some } F \in \mathcal{F} \text{ and } x \in F\}.$$

For $i \geq 1$, we define $\partial^{i+1}\mathcal{F} = \partial(\partial^i\mathcal{F})$. The following result is known as the Kruskal-Katona theorem (see Theorem 1 in [8]).

Theorem 2 (Kruskal and Katona). *Let n, k be positive integers and $\mathcal{F} \subseteq \binom{[n]}{k}$. If $|\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$ where $a_k > a_{k-1} > \ldots > a_s \geq s \geq 1$ are integers. Then*

$$|\partial\mathcal{F}| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_s}{s-1}.$$

Applying Theorem 2 repeatedly, we obtain the following

Corollary 3. *Let n, k be positive integers and $\mathcal{F} \subseteq \binom{[n]}{k}$. If $|\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1}$ where $a_k > a_{k-1} > \ldots > a_1 \geq 1$ are integers. Let $1 \leq r \leq k-1$. Then*

$$|\partial^{k-r}\mathcal{F}| \geq \binom{a_k}{r} + \binom{a_{k-1}}{r-1} + \cdots + \binom{a_{k-r}}{0}.$$

The first step of our reduction needs the following connection between the number of edges in an r-graph and its k-uniform 'expansion'.

Lemma 4. *Let $n \geq k > r \geq 1$ be integers. Let $\mathcal{A} \subseteq \binom{[n]}{r}$ and let $\mathcal{B} = \{e \cup f : e \in \mathcal{A}, f \in \binom{[n]\setminus e}{k-r}\}$. If $|\mathcal{A}| > \binom{n}{r} - \binom{n-t}{r} - \binom{n-t-k}{r-1} + 1$, then $|\mathcal{B}| > \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1} + 1$.*

Proof. Suppose the conclusion is false. Then the number of elements in $\binom{[n]}{k} \setminus \mathcal{B}$, satisfies

$$\begin{aligned}\left|\binom{[n]}{k} \setminus \mathcal{B}\right| &\geq \binom{n-t}{k} + \binom{n-t-k}{k-1} - 1 \\ &= \binom{n-t}{k} + \sum_{i=1}^{k-1} \binom{n-t-k-i}{k-i},\end{aligned}$$

where in the second equality we repeatedly used the formula $\binom{x}{y} = \binom{x-1}{y} + \binom{x-1}{y-1}$ to $\binom{n-t-k}{k-1}$.

Thus, by applying Corollary 3, we have

$$\begin{aligned}\left|\partial^{k-r}\left(\binom{[n]}{k} \setminus \mathcal{B}\right)\right| &\geq \binom{n-t}{r} + \sum_{i=1}^{r} \binom{n-t-k-i}{r-i} \\ &= \binom{n-t}{r} + \binom{n-t-k}{r-1}\end{aligned}$$

Since $\partial^{k-r}(\binom{[n]}{k} \setminus \mathcal{B}) \cap \mathcal{A} = \emptyset$ (by definition), $|\mathcal{A}| \leq \binom{n}{r} - \binom{n-t}{r} - \binom{n-t-k}{r-1}$, a contradiction. □

By Lemma 4, Theorem 1 follows from the following result.

Theorem 5. *Let $k \geq 3$ be a positive integer. There exists $t_0 = t_0(k)$ such that, for integers t, n with $t > t_0$ and $n > 2k(t+1)$, if $E(F_i) \subseteq \binom{[n]}{k}$ and*

$$e(F_i) > \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1} + 1,$$

for $i \in [t+1]$, then

(i) $\{F_1, \ldots, F_{t+1}\}$ admits a rainbow matching; or

(ii) there exists $W \in \binom{[n]}{t}$ such that W is a vertex cover of F_i for all $i \in [t+1]$.

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To prove Theorem 5, we convert this rainbow matching problem on k-graphs to a matching problem for a special class of $(k+1)$-graphs. Let $F_1, \dots, F_t$ be a family of k-graphs and $X := \{x_1, \dots, x_t\}$ be a set of t vertices disjoint from $[n]$. We use $\mathcal{F}^t(k,n)$ to denote the $(k+1)$-graph with vertex set $X \cup [n]$ and edge set

$$E(\mathcal{F}^t(k,n)) = \bigcup_{i=1}^{t} \{\{x_i\} \cup e : e \in F_i\}.$$

Observation 6. *$\{F_1, \dots, F_t\}$ admits a rainbow matching if, and only if, $\mathcal{F}^t(k,n)$ has a matching of size t.*

Hence, to prove Theorem 5, we need to see when $\mathcal{F}^{t+1}(k,n)$ has a matching of size $t+1$. We further reduce this problem to a near perfect matching problem.

Write $n - kt = km + r$, where $0 \le r \le k-1$. Let $F_1, \dots, F_t \subseteq \binom{[n]}{k}$, and let $F_i = \binom{[n]}{k}$ for $i = t+1, \dots, t+m$. Let $Q = \{x_1, \dots, x_{m+t}\}$ disjoint from $[n]$ and let $\mathcal{H}^t(k,n)$ be the $(k+1)$-graph with vertex set $Q \cup [n]$ and edge set

$$E(\mathcal{H}^t(k,n)) = \left(\bigcup_{i=1}^{t} \{\{x_i\} \cup e : e \in F_i\}\right) \bigcup \left(\bigcup_{i=t+1}^{t+m} \left\{\{x_i\} \cup e : e \in \binom{[n]}{k}\right\}\right).$$

When $F_1 = \cdots = F_t = H_k(s,n)$, we denote $\mathcal{H}^t(k,n)$ by $\mathcal{H}_t(k,n;s)$. Note that $\nu(\mathcal{H}_t(k,n;t)) = t + m = (n-r)/k$, i.e., $\mathcal{H}_t(k,n;t)$ has a matching covering all but at most $r < k$ vertices (and such a matching is said to be *near perfect*).

The following lemma provides an equivalent condition on matchings in $\mathcal{F}^t(k,n)$ and matchings in $\mathcal{H}_t(k,n;t)$. See Lemma 2.1 in [19].

Lemma 7. *Let n, k, t be positive integers and let $F_1, \dots, F_t \subseteq \binom{[n]}{k}$. Then $\mathcal{F}^t(k,n)$ has a matching of size t if, and only if, $\mathcal{H}_t(k,n;t)$ has a matching of size $m+t$, where $m + t = \lfloor n/k \rfloor$.*

For the proof of Theorem 5, we need to tell how close a hypergraph is to another hypergraph. Given two k-hypergraphs H_1, H_2 with $V(H_1) = V(H_2)$, let $c(H_1, H_2)$ be the minimum of $|E(H_1) \backslash E(H')|$ taken over all isomorphic copies H' of H_2 with $V(H') = V(H_2)$. For a real number $\varepsilon > 0$, we say that H_2 is *ε-close* to H_1 if $V(H_1) = V(H_2)$ and $c(H_1, H_2) \le \varepsilon |V(H_1)|^k$. If $F_1 = \cdots = F_t = H_k(s,n)$, where $1 \le s < n$ and $H_k(s,n)$ denotes the k-graph with vertex set $[n]$ and edge set $\binom{[n]}{k} \setminus \binom{[n]\setminus[s]}{k}$, then we denote such $\mathcal{F}^t(k,n)$ by $\mathcal{F}_t(k,n;s)$. The following is straightforward to verify.

Observation 8. *(i) If $\mathcal{F}^t(k,n)$ is ε-close to $\mathcal{F}_t(k,n;s)$ then $\mathcal{H}^t(k,n)$ is ε-close to $\mathcal{H}_t(k,n;s)$; (ii) For $n \le 2k^3 \min\{t,s\}$, if $\mathcal{H}^t(k,n)$ is ε-close to $\mathcal{H}_t(k,n;s)$ then $\mathcal{F}^t(k,n)$ is 6ε-close to $\mathcal{F}_t(k,n;s)$.*

Proof. We first prove (i). By definition, we have

$$|E(\mathcal{F}_t(k,n;s)) \setminus E(\mathcal{F}^t(k,n))| \le \varepsilon |V(\mathcal{F}^t(k,n))|^{k+1}. \tag{1}$$

Thus, we can infer that

$$\begin{aligned} |E(\mathcal{H}_t(k,n;s)) \setminus E(\mathcal{H}^t(k,n))| &= |E(\mathcal{F}_t(k,n;s)) \setminus E(\mathcal{F}^t(k,n))| \\ &\le \varepsilon |V(\mathcal{F}^t(k,n))|^{k+1} \quad \text{(by (1))} \\ &\le \varepsilon |V(\mathcal{H}^t(k,n))|^{k+1}. \end{aligned}$$

It follows that $\mathcal{H}^t(k,n)$ is ε-close to $\mathcal{H}_t(k,n;s)$.

Next, we prove (ii). By definition, we have

$$|E(\mathcal{H}_t(k,n;s)) \setminus E(\mathcal{H}^t(k,n))| \le \varepsilon |V(\mathcal{H}^t(k,n))|^{k+1}. \tag{2}$$

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One can that

$$|V(\mathcal{H}^t(k,n))| \le n + n/k. \tag{3}$$

Thus we can get

$$\begin{aligned} |E(\mathcal{F}_t(k,n;s)) \setminus E(\mathcal{F}^t(k,n))| &= |E(\mathcal{H}_t(k,n;s)) \setminus E(\mathcal{H}^t(k,n))| \\ &\le \varepsilon(n+n/k)^{k+1} \quad \text{(by (2) and (3))} \\ &\le 6\varepsilon n^{k+1}. \end{aligned}$$

So it follows that $\mathcal{F}^t(k,n)$ is 6ε-close to $\mathcal{F}_t(k,n;s)$. This completes the proof of Observation 8. □

Our proof of Theorem 5 will be divided into two parts, according to whether or not $\mathcal{F}^{t+1}(k,n)$ is close to $\mathcal{F}_{t+1}(n,k;t)$. If $\mathcal{F}^{t+1}(k,n)$ is close to $\mathcal{F}_{t+1}(n,k;t)$, we will apply greedy argument to construct a matching of size t. If $\mathcal{F}^{t+1}(k,n)$ is not close to $\mathcal{F}_{t+1}(n,k;t)$, then by Observation 8, $\mathcal{H}^{t+1}(k,n)$ is not close to $\mathcal{H}_{t+1}(n,k;t)$ (see proof of Theorem 5), and we will show that $\mathcal{H}^{t+1}(k,n;t)$ has a small matching M_1 with nice absorbing properties and $\mathcal{H}^{t+1}(k,n;t) - V(M_1)$ has a spanning subgraph in which we can find a large matching M_2 that can be extended to a near perfect matching by using M_1.

3. Stability results on small matchings

We begin with the result of Huang, Loh, and Sudakov [15] mentioned previously, and state a corollary which generalizes it to non-uniform hypergraphs.

Theorem 9 (Huang, Loh, and Sudakov). *Let n, k, t be three positive integers such that $n > 3k^2t$. Let $\mathcal{F} = \{F_1, \dots, F_t\}$ be a family of k-graphs. If*

$$e(F_i) > \binom{n}{k} - \binom{n-t+1}{k}$$

for all $1 \le i \le t$, then $\mathcal{F}$ admits a rainbow matching.

Corollary 10. *Let $t, n, k_1, \dots, k_t, k$ be positive integers such that $k = \max\{k_1, \dots, k_t\}$ and $n > 3k^2t$. Let $\mathcal{F} = \{F_1, \dots, F_t\}$ be a family where $E(F_i) \subseteq \binom{[n]}{k_i}$ for $i \in [t]$. If*

$$e(F_i) > \binom{n}{k_i} - \binom{n-t+1}{k_i}$$

for all $1 \le i \le t$, then $\mathcal{F}$ admits a rainbow matching.

Proof. For $i \in [t]$, we define the expansion of $E(F_i)$ as

$$E(G_i) = \left\{ e \cup f : e \in F_i, f \in \binom{[n] \setminus e}{k - k_i} \right\}.$$

Note that each G_i is a k-graph.

We claim that $|\binom{[n]}{k} \setminus E(G_i)| < \binom{n-t+1}{k}$ for all $i \in [t]$. For, otherwise, there exists some $i \in [t]$ such that $|\binom{[n]}{k} \setminus E(G_i)| \ge \binom{n-t+1}{k}$. Then by Theorem 2, $|\partial^{k-k_i}\left(\binom{[n]}{k} \setminus E(G_i)\right)| \ge \binom{n-t+1}{k_i}$. Since $\partial^{k-k_i}\left(\binom{[n]}{k} \setminus E(G_i)\right) \cap E(F_i) = \emptyset$ (by definition), $e(F_i) \le \binom{n}{k_i} - \binom{n-t+1}{k_i}$, a contradiction.

Therefore, $e(G_i) > \binom{n}{k} - \binom{n-t+1}{k}$. Hence, by Theorem 9, $\{G_1, \dots, G_t\}$ admits a rainbow matching, which implies that $\mathcal{F}$ admits a rainbow matching. □

We also need a classical theorem of Mörs, which is Theorem 6 in [21].

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Theorem 11 (Mörs, [21]). *Let n, k, l be three positive integers such that $n \geq k + l$. Let $\mathcal{A} \in \binom{[n]}{k}$ and $\mathcal{B} \in \binom{[n]}{l}$. Suppose $\bigcap_{C \in \mathcal{A} \cup \mathcal{B}} C = \emptyset$ and $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then either*

$$|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-1-l}{k-1} + 1$$

or

$$|\mathcal{B}| \leq \binom{n-1}{l-1} - \binom{n-1-k}{l-1} + 1.$$

To prove Theorem 1, we need to establish the following result, which may be viewed as a stability version (as well as a generalization) of the above mentioned result of Huang, Loh, and Sudakov [15] as well as an earlier result of Bollobás, Daykin, and Erdős [5]. Note the following theorem holds for every positive integer t.

Theorem 12. *Let $n, t, k_1, k_2, \ldots, k_{t+1}$ be positive integers such that $k_1 \geq k_2 \geq \ldots \geq k_{t+1} \geq 2$ and $n \geq 2k_1^2 k_2 t$. Let $E(F_i) \subseteq \binom{[n]}{k_i}$, $i \in [t+1]$, such that, for $i \in [2]$,*

$$e(F_i) > \binom{n}{k_i} - \binom{n-t}{k_i} - \binom{n-t-k_{3-i}}{k_i - 1} + 1,$$

and for $i \in [t+1] \setminus [2]$,

$$e(F_i) > \binom{n}{k_i} - \binom{n-t}{k_i} - \binom{n-t-k_3}{k_i - 1} + 1.$$

Then one of the following holds:

(i) $\{F_1, \ldots, F_{t+1}\}$ admits a rainbow matching of size $t+1$; or

(ii) there exists $W \in \binom{[n]}{t}$ such that W is a vertex cover of F_i for all $i \in [t+1]$.

Proof. We apply induction on t. For the base case, suppose $t = 1$. Then for $i \in [2]$,

$$e(F_i) > \binom{n}{k_i} - \binom{n-1}{k_i} - \binom{n-1-k_{3-i}}{k_i - 1} + 1 = \binom{n-1}{k_i - 1} - \binom{n-1-k_{3-i}}{k_i - 1} + 1;$$

so (i) of Theorem 12 follows from Theorem 11.

Now suppose $t \geq 2$ and the conclusion holds with t hypergraphs. Moreover, we may assume (i) does not hold, i.e., $\{F_1, \ldots, F_{t+1}\}$ does not admit a rainbow matching.

Since $k_1 \geq k_i$ for $i \in [t+1]$, we have from the assumption of Theorem 12 that

$$e(F_i) \geq \binom{n}{k_i} - \binom{n-t}{k_i} - \binom{n-t-k_1}{k_i - 1} + 1 \quad \text{for } i \in [t+1]. \tag{4}$$

Hence, since $\binom{n-t}{k_i-1} \geq \binom{n-t-k_j}{k_i-1}$ and $\binom{n-(t-1)}{k_i} = \binom{n-t}{k_i} + \binom{n-t}{k_i-1}$,

$$e(F_i) > \binom{n}{k_i} - \binom{n-(t-1)}{k_i} \quad \text{for } i \in [t].$$

Thus by Corollary 10, $\{F_1, \ldots, F_t\}$ admits a rainbow matching, say M_1. Therefore, since $\{F_1, \ldots, F_{t+1}\}$ does not admit a rainbow matching, every edge in F_{t+1} must intersect $V(M_1)$. Thus the maximum degree $\Delta(F_{t+1}) \geq e(F_{t+1})/|V(M_1)|$. Let $v \in [n]$ such that $d_{F_{t+1}}(v) = \Delta(F_{t+1})$. Then, since

$$\binom{n}{k_i} = \binom{n-1}{k_i - 1} + \binom{n-1}{k_i} = \sum_{i=1}^{t-1} \binom{n-i}{k_i - 1} + \binom{n-t}{k_i} \geq (t-1)\binom{n-t}{k_i - 1} + \binom{n-t+1}{k_i},$$

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then by (4), we have

$$e(F_i) \geq (t-1)\binom{n-t}{k_i-1} \quad \text{for } i \in [t+1].$$

Thus if follow that

$$d_{F_{t+1}}(v) \geq \frac{e(F_{t+1})}{|V(M_1)|} > \frac{(t-1)\binom{n-t}{k_{t+1}-1}}{\sum_{i=1}^{t} k_i}.$$

Hence, since $n \geq 2k_1^2k_2t$ and $k_1 \geq k_2 \geq \max\{k_i \mid 3 \leq i \leq t+1\}$,

$$\begin{aligned} d_{F_{t+1}}(v) &> \frac{(t-1)\binom{n-t}{k_{t+1}-1}}{\sum_{i=1}^{t} k_i} \\ &\geq \frac{n(t-1)(1-1/2k_1^2k_2)^{k_{t+1}-1}\binom{n}{k_{t+1}-2}}{(k_{t+1}-1)\sum_{i=1}^{t} k_i} \\ &> (1-\frac{k_{t+1}-1}{2k_1^2k_2})\frac{n(t-1)\binom{n}{k_{t+1}-2}}{(k_{t+1}-1)\sum_{i=1}^{t} k_i} \\ &> \frac{n(t-1)\binom{n}{k_{t+1}-2}}{k_t\sum_{i=1}^{t} k_i} \\ &> \left(\sum_{i=1}^{t} k_i\right)\binom{n-2}{k_{t+1}-2}. \end{aligned}$$

If $\{F_1 - v, \ldots, F_t - v\}$ admits a rainbow matching, say M_2, then $d_{F_{t+1}}(v) > |V(M_2)|\binom{n-2}{k_{t+1}-2}$ as $|V(M_2)| = \sum_{i=1}^{t} k_i$. So there exists an edge $e \in F_{t+1}$ such that $v \in e$ and $e \cap V(M_2) = \emptyset$. Now $M_2 \cup \{e\}$ is a desired rainbow matching for $\{F_1, \ldots, F_t, F_{t+1}\}$, a contradiction.

So $\{F_1 - v, \ldots, F_t - v\}$ does not admit any rainbow matching. Note that, for $i \in [2]$,

$$e(F_i - v) \geq e(F_i) - \binom{n-1}{k_i-1} > \binom{n-1}{k_i} - \binom{(n-1)-(t-1)}{k_i} - \binom{(n-1)-(t-1)-k_{3-i}}{k_i-1} + 1,$$

and, for $i \in [t] \setminus [2]$,

$$e(F_i - v) \geq e(F_i) - \binom{n-1}{k_i-1} > \binom{n-1}{k_i} - \binom{(n-1)-(t-1)}{k_i} - \binom{(n-1)-(t-1)-k_3}{k_i-1} + 1.$$

Thus, by inductive hypothesis (applied to $\{F_1 - v, \ldots, F_t - v\}$), there exists $W' \in \binom{[n]\setminus\{v\}}{t-1}$ such that W' is a vertex cover of $F_i - v$ for all $i \in [t]$.

Write $W = W' \cup \{v\}$. We may assume that there exists $f \in F_{t+1}$ such that $f \cap W = \emptyset$; for otherwise (ii) holds. Note that the number of edges in F_i (for each $i \in [t]$) intersecting both f and W is

$$e_i(f, W) \leq \binom{n}{k_i} - \binom{n-k_{t+1}}{k_i} - \binom{n-t}{k_i} + \binom{n-t-k_{t+1}}{k_i}.$$

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Hence, for each $i \in [t]$, since every edge of F_i intersects W, the number of edges in F_i disjoint from f is

$$\begin{aligned} e(F_i - f) &= e(F_i) - e_i(f, W) \\ &> \left(\binom{n}{k_i} - \binom{n-t}{k_i} - \binom{n-t-k_3}{k_i-1} + 1\right) - e(f, W) \\ &\geq \binom{n-k_{t+1}}{k_i} - \binom{n-t-k_3}{k_i-1} - \binom{n-t-k_{t+1}}{k_i} + 1 \\ &> \binom{n-k_{t+1}}{k_i} - \binom{(n-k_{t+1})-(t-1)}{k_i} \end{aligned}$$

Hence by Corollary 10, $\{F_1 - V(f), F_2 - V(f), \ldots, F_t - V(f)\}$, i.e. deleting all vertices in f, admits a rainbow matching M_3 of size t. Therefore, $M_3 \cup \{f\}$ is a rainbow matching of size $t+1$ which satisfies (i). □

As a consequence of Theorem 12, we have the following conclusion.

Corollary 13. *Let $t, k_1, k_2, \ldots, k_{t+1}, n$ be positive integers, let $k = \max\{k_1, \ldots, k_{t+1}\}$, and let $E(F_i) \subseteq \binom{[n]}{k_i}$ for $i \in [t+1]$. Suppose $n \geq 2k^3t$ and $e(F_i) > \binom{n}{k_i} - \binom{n-t}{k_i} - \binom{n-t-k}{k_i-1} + 1$ for all $i \in [t+1]$. Then*

(i) $\{F_1, F_2, \cdots, F_{t+1}\}$ admits a rainbow matching of size $t+1$; or

(ii) there exists $W \in \binom{[n]}{t}$ such that W is a vertex cover of F_i for all $i \in [t+1]$.

By taking $k_1 = k_2 = \cdots = k_{t+1} = k$ in Corollary 13, we have the following

Corollary 14. *Let t, k, n be positive integers and let $E(F_i) \subseteq \binom{[n]}{k}$ for $i \in [t+1]$. Suppose $n \geq 2k^3t$ and $e(F_i) > \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1} + 1$ for all $i \in [t+1]$. Then either*

(i) $\{F_1, F_2, \cdots, F_{t+1}\}$ admits a rainbow matching of size $t+1$; or

(ii) there exists $W \in \binom{[n]}{t}$ such that W is a vertex cover of F_i for all $i \in [t+1]$.

4. Extremal case

Recall $\mathcal{F}^{t+1}(k, n)$ is the $(k+1)$-graph with vertex set $X \cup [n]$ and edge set $E(\mathcal{F}^{t+1}(k, n)) = \bigcup_{i=1}^{t+1}\{\{x_i\} \cup e : e \in F_i\}$. We also recall that $\mathcal{F}_{t+1}(k, n; s)$ is the $(k+1)$-graph with vertex set $X \cup [n]$ and edge set $E(\mathcal{F}_{t+1}(k, n; s)) = \bigcup_{i=1}^{t+1}\{\{x_i\} \cup e : e \in H_k(s, n)\}$ where $H_k(s, n)$ denotes the k-graph with vertex set $[n]$ and edge set $\binom{[n]}{k} \setminus \binom{[n]\setminus[s]}{k}$.

In this section, we prove Theorem 1 for the case when $\mathcal{F}^{t+1}(k, n)$ is ε-close to the extremal configuration $\mathcal{F}_{t+1}(k, n; t)$ for some $\varepsilon \ll 1/k$. We write $a \ll b$ to mean that there exists an increasing function f such that our result holds whenever $a \leq f(b) < b$.

Let H be a $(k+1)$-graph and $v \in V(H)$. We define the neighborhood $N_H(v)$ of v in H to be the set $\{S \in \binom{V(H)}{k} : S \cup \{v\} \in E(H)\}$. Let H be a $(k+1)$-graph with the same vertex set as $\mathcal{F}_{t+1}(k, n; t)$. Given real number α with $0 < \alpha < 1$, a vertex v in H is called *α-good* with respect to $\mathcal{F}_{t+1}(k, n; t)$ if

$$\left|N_{\mathcal{F}_{t+1}(k,n;t)}(v) \setminus N_H(v)\right| \leq \alpha n^k.$$

Clearly, if H is ε-close to $\mathcal{F}_{t+1}(k, n; t)$, then at most $(k+1)(1+1/k)^{k+1}\varepsilon n/\alpha$ vertices of H are not α-good with respect to $\mathcal{F}_{t+1}(k, n; t)$.

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First we deal with the case when all vertices of $\mathcal{F}^{t+1}(k,n)$ are α-good with respect to $\mathcal{F}_{t+1}(k,n;t)$. Let Q, V be two disjoint sets. A $(k+1)$-graph H with vertex set $Q \cup V$ is called $(1,k)$-*partite* with *partition classes* Q, V if, for each edge $e \in E(H)$, $|e \cap Q| = 1$ and $|e \cap V| = k$. Clearly, $\mathcal{F}^{t+1}(k,n)$ and $\mathcal{F}_{t+1}(k,n;t)$ are $(1,k)$-partite graphs with partition classes $X, [n]$.

Lemma 15. *Let ζ, α be real numbers and n, k, t be positive integers such that $0 < \alpha \ll \zeta < 1$, $\alpha \ll 1/k$, $n \geq 8k^4$, $t \geq n/(2k^3)$ and $t+1 < (1-\zeta)n/k$. Let H be a $(1,k)$-partite $(k+1)$-graph with the same partition classes as $\mathcal{F}_{t+1}(k,n;t)$. If every vertex of H is α-good with respect to $\mathcal{F}_{t+1}(k,n;t)$, then $\nu(H) \geq t$ with equality only if H is a subgraph of $\mathcal{F}_{t+1}(k,n;t)$.*

Proof. Let $X := \{x_1, x_2, \ldots, x_{t+1}\}$, $W := [t]$, and $U := [n] \setminus [t]$, such that $X, [n]$ are the partition classes of H and $\mathcal{F}_{t+1}(k,n;t)$.

If H is a subgraph of $\mathcal{F}_{t+1}(k,n;t)$, then we define $e_0 = \emptyset$. Otherwise, there exists an edge e_0 in H such that $|e_0 \cap X| = 1$ and $e_0 \cap W = e_0 \cap [t] = \emptyset$. Without loss of generality, we may assume $e_0 \cap X = \{x_{t+1}\}$.

Write $H' = H - V(e_0)$, i.e. deleting all vertices in e_0. Let M be a maximum matching in H' such that $|e \cap X| = |e \cap W| = 1$ for all $e \in M$. Thus $|M| \leq |W| = t$. Let $X' = X \setminus (V(M) \cup V(e_0))$, $W' = W \setminus (V(M) \cup V(e_0))$, and $U' = U \setminus (V(M) \cup V(e_0))$. Thus $|W'| = |W| - |M|$ and $|U'| \geq |U| - |M|(k-1) - k$.

We claim that $|M| \geq n/(4k^3)$. For, suppose $|M| < n/(4k^3)$. Consider any vertex $x \in X'$. Since x is α-good with respect to $\mathcal{F}_{t+1}(k,n;t)$, we have

$$\left| \left(W \times \binom{U}{k-1} \right) \setminus N_H(x) \right| \leq \alpha n^k.$$

Since $t+1 < (1-\zeta)n/k$, $|U'| \geq |U| - |M|(k-1) - k \geq (n-t) - t(k-1) - k > \zeta n$. Hence, since $t \geq n/2k^3$,

$$\left| W' \times \binom{U'}{k-1} \right| \geq (|W| - |M|) \binom{|U| - |M|(k-1) - k}{k-1} > \frac{n}{4k^3} \binom{\zeta n}{k-1} > \frac{n}{4k^3} \frac{(\zeta n/2)^{k-1}}{(k-1)!},$$

Therefore, $\left| W' \times \binom{U'}{k-1} \right| > \alpha n^k$, as $\alpha < \zeta^{k-1}(k^3 2^{k+1}(k-1)!)^{-1}$. Hence, there exists $f \in N_H(x) \cap \left(W' \times \binom{U'}{k-1} \right)$. Let $f' = \{x\} \cup f$, Then $f' \in E(H)$, $|f' \cap X| = |f' \cap W| = 1$, and $f' \cap V(M) = \emptyset$. Now $M' = M \cup \{f'\}$ is a matching of size $|M|+1$ in H, and $|e \cap X| = |e \cap W| = 1$ for all $e \in M'$. Thus, M' contradicts the choice of M, completing the proof of the claim.

Let $\{u_1, \ldots, u_{k+1}\} \subseteq V(H) \setminus V(M)$, where $u_1 \in X'$, $u_{k+1} \in W'$ and $u_i \in U'$ for $i \in [k] \setminus \{1\}$. Since $|M| \geq n/(4k^3) \geq 2k$, let $\{e_1, \ldots, e_k\}$ be an arbitrary k-subset of M, and let $e_i := \{v_{i,1}, v_{i,2}, \ldots, v_{i,k+1}\}$ with $v_{i,1} \in X$, $v_{i,k+1} \in W$, and $v_{i,j} \in U$ for $i \in [k]$ and $j \in [k] \setminus \{1\}$. For $j \in [k+1]$, let $f_j := \{u_j, v_{1,j+1}, v_{2,j+2}, \ldots, v_{k,j+k}\}$ with addition in the subscripts modulo $k+1$ (except we write $k+1$ instead of 0). Note that $f_1, \ldots, f_{k+1}$ are pairwise disjoint.

If $f_j \in E(H)$ for all $j \in [k+1]$ then $M' := (M \cup \{f_1, \ldots, f_{k+1}\}) \setminus \{e_1, \ldots, e_k\}$ is a matching in H such that $|M'| = |M| + 1 > |M|$ and $|f \cap X| = |f \cap W| = 1$ for all $f \in M'$, contradicting the choice of M. Hence, $f_j \notin E(H)$ for some $j \in [k+1]$.

Note that there are $\binom{|M|}{k} k!$ choices of ordered k-tuples $(e_1, \ldots, e_k) \in M^k$ and that any two different such choices correspond to different f_j. Hence,

$$\begin{aligned}
&|\{f \in E(\mathcal{F}_{t+1}(k,n;t)) \setminus E(H) : |f \cap \{u_i : i \in [k+1]\}| = 1\}| \\
&\geq |M|(|M|-1)\cdots(|M|-k+1) \\
&> \left(n/(4k^3) - k\right)^k \\
&> \left(n/(8k^3)\right)^k \quad (\text{since } n \geq 8k^4) \\
&> (k+1)\alpha n^k \quad (\text{since } \alpha < ((k+1)8^k k^{3k}))^{-1}.
\end{aligned}$$

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This implies that there exists $i \in [k+1]$ such that $|N_{\mathcal{F}_{t+1}(k,n)}(u_i) \setminus N_H(u_i)| > \alpha n^k$, contradicting the fact that all u_i's are α-good with respect to $\mathcal{F}_{t+1}(k, n; t)$.

Therefore, H has a matching M of size t. Moreover, $M \cup \{e_0\}$ is a matching of size $t+1$ unless $e_0 = \emptyset$ in which case H is a subgraph of $\mathcal{F}_{t+1}(k, n; t)$. □

We can now prove Theorem 1 when $\mathcal{F}^{t+1}(k, n)$ is ε-close to $\mathcal{F}_{t+1}(k, n; t)$.

Lemma 16. *Let $k \geq 3$, $t \geq 1$ and n be integers, and let ε, ζ be real numbers such that $0 < 1/n \ll \varepsilon \ll \zeta \ll 1/k < 1$, $t+1 < (1-\zeta)(1-4k^2\sqrt{\varepsilon})n/k$. Let $\mathcal{F} = \{F_1, \ldots, F_{t+1}\}$ be a family of k-graphs such that $e(F_i) > \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1} + 1$ for $i \in [t+1]$. Suppose $\mathcal{F}^{t+1}(k, n)$ is ε-close to $\mathcal{F}_{t+1}(k, n; t)$. Then $\mathcal{F}^{t+1}(k, n)$ has a matching of size $t+1$ or $\mathcal{F}^{t+1}(k, n)$ is a subgraph of $\mathcal{F}_{t+1}(k, n; t)$.*

Proof. We may assume $n \leq 2k^3 t$ as otherwise the assertion follows from Corollary 13. Let $X, [n]$ be the partition classes of $\mathcal{F}_{t+1}(k, n; t)$, and let $X := \{x_1, x_2, \ldots, x_{t+1}\}$. Note that each edge of $\mathcal{F}_{t+1}(k, n; t)$ intersects $[t]$.

Let B denote the set of vertices in $\mathcal{F}^{t+1}(k, n)$ that are not $\sqrt{\varepsilon}$-good with respect to $\mathcal{F}_{t+1}(k, n; t)$. Since $\mathcal{F}^{t+1}(k, n)$ is ε-close to $\mathcal{F}_{t+1}(k, n; t)$,

$$|B \cap X| \leq (1+1/k)^{k+1}\sqrt{\varepsilon}n \leq 4\sqrt{\varepsilon}n,$$

and

$$|B \cap [n]| \leq k(1+1/k)^{k+1}\sqrt{\varepsilon}n \leq 4k\sqrt{\varepsilon}n.$$

Let $b := \max\{|B \cap X|, |B \cap [t]|\}$; so $b \leq 4k\sqrt{\varepsilon}n$. We choose $X_1 \subseteq X$ and $W_1 \subseteq [t]$ such that $B \cap X \subseteq X_1$, $B \cap [t] \subseteq W_1$, and $|X_1| = |W_1| = b$.

For each $x_i \in X \setminus X_1$, x_i is $\sqrt{\varepsilon}$-good with respect to $\mathcal{F}_{t+1}(k, n; t)$ and we let $\mathcal{F}_i = \mathcal{F}^{t+1}(k, n)[X_1 \cup W_1 \cup U \cup \{x_i\}]$. For every $x \in X_1 \cup \{x_i\}$, there exists $j \in [t+1]$ such that $x = x_j$, and we have

$$\begin{aligned}
|N_{\mathcal{F}_i}(x)| &\geq |N_{\mathcal{F}^{t+1}(k,n)}(x_j)| - \left(\binom{n}{k} - \binom{n - |[t] \setminus W_1|}{k}\right) \\
&= e(F_j) - \left(\binom{n}{k} - \binom{n-(t-b)}{k}\right) \\
&> \binom{n-(t-b)}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1} + 1 \\
&= \binom{n-(t-b)}{k} - \binom{(n-(t-b))-b}{k} - \binom{(n-(t-b))-b-k}{k-1} + 1.
\end{aligned}$$

Since $n - (t-b) > n/2 \geq 2k^3(k+1)\sqrt{\varepsilon}n > 2k^3 b$, it follows from Corollary 13 that the family $\{N_{\mathcal{F}_i}(x) : x \in X_1 \cup \{x_i\}\}$ admits a rainbow matching of size $b+1$, or there exists $W^i \in \binom{[n]\setminus([t]\setminus W_1)}{b}$ such that W^i is a vertex cover of $N_{\mathcal{F}_i}(x)$ for all $x \in X_1 \cup \{x_i\}$.

Suppose there exists $x_i \in X \setminus X_1$ such that $\{N_{\mathcal{F}_i}(x) : x \in X_1 \cup \{x_i\}\}$ admits a rainbow matching M_i of size $b+1$. Write $H := \mathcal{F}^{t+1}(k, n)[X \cup [n] \setminus (V(M_i) \cup B)]$. Let $n' = |V(H) \cap [n]|$. We relabel vertices so that $[t] \setminus W_1$ corresponds to $[t-b]$. Since $b \leq 4k\sqrt{\varepsilon}n$ and $\varepsilon \ll 1/k$, for every $y \in V(H)$, we have

$$\begin{aligned}
|N_{\mathcal{F}_{t-b}(k,n';t-b)}(y) \setminus N_H(y)| &\leq |N_{\mathcal{F}_{t+1}(k,n;t)}(y) \setminus N_{\mathcal{F}^{t+1}(n,b)}(y)| \\
&\leq \sqrt{\varepsilon}(n+t+1)^k \\
&< \varepsilon^{1/3}(n/2)^k \\
&< \varepsilon^{1/3}(n+t+1-(k+1)(b+1))^k \\
&= \varepsilon^{1/3}|V(\mathcal{F}_{t-b}(k, n'; t-b))|^k.
\end{aligned}$$

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Hence every vertex in H is $\varepsilon^{1/3}$-good with respect to $\mathcal{F}_{t-b}(k, n'; t-b)$. Note that

$$n' = n - kb \geq n - 4k^2\sqrt{\varepsilon}n > (t-b)k/(1-\zeta).$$

By Lemma 15, H has a matching M' of size $t-b$. Therefore, $M_1 \cup M'$ is a matching in $\mathcal{F}^{t+1}(k, n)$ of size $t+1$.

Hence, we may assume that for each $x_i \in X \setminus X_1$, there exists $W^i \in \binom{[n]\setminus([t]\setminus W_1)}{b}$ such that W^i is a vertex cover of $\mathcal{F}_i$. If $W := W^i = W^j$ for all $x_i, x_j \in X \setminus X_1$ then $([t] \setminus W_1) \cup W$ is a vertex cover of F_i for all $i \in [t+1]$; so $\mathcal{F}^{t+1}(k, n)$ is a subgraph of $\mathcal{F}_{t+1}(k, n; t)$. Hence, we may assume that there exist $i \neq j$ such that $W^i \setminus W^j \neq \emptyset$.

Now fix $x \in X_1$. Note that each edge in $N_{\mathcal{F}^{t+1}(k,n)}(x)$ either intersects $([t] \setminus W_1) \cup (W^i \cap W^j)$ or intersects both $W^i \setminus W^j$ and $W^j \setminus W^i$. Recall that $|W^i| = |W^j| = b$. Let $r := |W^i \setminus W^j| = |W^j \setminus W^i|$; then $|([t] \setminus W_1) \cup (W^i \cap W^j)| = (t-b) + (b-r) = t - r$. Hence,

$$|N_{\mathcal{F}^{t+1}(k,n)}(x)| \leq \left(\binom{n}{k} - \binom{n-(t-r)}{k}\right) + \binom{n-t-r}{k-2}r^2. \tag{5}$$

Note that

$$\begin{aligned}\binom{n-(t-r)}{k} - \binom{n-t}{k} &\geq r\binom{n-t}{k-1} \\ &= \frac{(r-1)(n-t)}{k-1}\binom{n-t-1}{k-2} + \binom{n-t}{k-1},\end{aligned}$$

i.e.,

$$\binom{n-(t-r)}{k} \geq \frac{(r-1)(n-t)}{k-1}\binom{n-t-1}{k-2} + \binom{n-t}{k-1} + \binom{n-t}{k}. \tag{6}$$

If $r = 1$, then by (5), we have

$$\begin{aligned}|N_{\mathcal{F}^{t+1}(k,n)}(x)| &\leq \binom{n}{k} - \binom{n-(t-1)}{k} + \binom{n-t-1}{k-2} \\ &= \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-1}{k-1} \\ &< \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1},\end{aligned}$$

This leads to a contradiction as $e(F_i) > \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1} + 1$ for $i \in [t+1]$.

Now we may assume that $r \geq 2$. Combining (5) and (6), it follows that

$$\begin{aligned}|N_{\mathcal{F}^{t+1}(k,n)}(x)| &\leq \binom{n}{k} - \binom{n-t}{k} - \binom{n-t}{k-1} - \frac{(r-1)(n-t)}{k-1}\binom{n-t-1}{k-2} + \binom{n-t-1}{k-2}r^2 \\ &< \binom{n}{k} - \binom{n-t}{k} - \binom{n-t}{k-1} - \binom{n-t-1}{k-2} \quad (\text{as } n > 10kb \geq 10kr \text{ and } t \leq n/k) \\ &< \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-1}{k-1} \\ &< \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1},\end{aligned}$$

contradicting that $e(F_i) > \binom{n}{k} - \binom{n-t}{k} - \binom{n-t-k}{k-1} + 1$ for $i \in [t+1]$. This completes the proof. □

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5. Non-extremal case

To deal with the case when $\mathcal{F}^{t+1}(k,n)$ is not ε-close to $\mathcal{F}_{t+1}(k,n;t)$, we need two lemmas proved by the present authors in [19], both with a slight variation but same proof.

The first is Lemma 4.2 in [19]. The only difference between that lemma and the statement below is that we replace the assumption '$d_H(x_i) > \binom{n}{k} - \binom{n-t+1}{k}$ for $i \in [t]$' in Lemma 4.2 of [19] by '$d_H(x_i) > \binom{n}{k} - \binom{n-t+1}{k} - \rho n^k$ for $i \in [t]$'. We omit the proof as it is almost identical to the proof of Lemma 4.2 in [19].

Lemma 17. *Let n, k, t be integers and ζ, ρ be real numbers, such that $1/n \ll \rho \ll \zeta \ll 1/k$ and $n/(2k^3) \le t \le (1-\zeta)n/k$. Let H be a $(1,k)$-partite $(k+1)$-graph with partition classes $\{x_1, \dots, x_{\lfloor n/k \rfloor}\}, [n]$ such that $d_H(x_i) > \binom{n}{k} - \binom{n-t+1}{k} - \rho n^k$ for $i \in [t]$ and $d_H(x_i) = \binom{n}{k}$ for $i = t+1, \dots, \lfloor n/k \rfloor$. Then for any c with $0 < c \ll \zeta$, there exists a matching M in H such that $|M| \le 2kcn$ and, for any balanced subset $S \subseteq V(H)$ with $|S| \le (k+1)c^{1.5}n/2$, $H[V(M) \cup S]$ has a perfect matching.*

The second result we need is Lemma 6.4 in [19], with the assumption $n/(3k^2) \le t$ replaced by $n/(2k^3) \le t$ in the statement below. Again, the same proof in [19] works. Recall the definition of $\mathcal{H}_t(k,n;t)$.

Lemma 18. *Let $k \ge 3$ be an integer, $0 < c \ll \rho \ll \varepsilon \ll 1$ be real numbers, $n \in k\mathbb{Z}$ sufficiently large, and let t be an integer with $n/(2k^3) \le t \le (\frac{1}{2} + c)n/k$. Let H be a $(1,k)$-partite $(k+1)$-graph with partition classes $X, [n]$ such that $k|X| = n$. Let A_1 and A_2 be a partition of A such that $|A_1| = t$ and $|A_2| = n/k - t$. Suppose that $d_H(x) > \binom{n}{k} - \binom{n-t+1}{k} - \rho n^k$ for all $x \in A_1$ and $d_H(x) = \binom{n}{k}$ for all $x \in A_2$. If H is not ε-close to $\mathcal{H}_t(k,n;t)$, then there exists a spanning subgraph H' of H such that the following conditions hold:*

(1) For all $x \in V(H')$, with at most $n^{0.99}$ exceptions, $d_{H'}(x) = (1 \pm n^{-0.01})n^{0.2}$.

(2) For all $x \in V(H')$, $d_{H'}(x) < 2n^{0.2}$.

(3) For any two distinct $x, y \in V(H')$, $d_{H'}(\{x,y\}) < n^{0.19}$.

Lemma 18 allows us to apply the following result attributed to Pippenger [22] (see Theorem 4.7.1 in [4]). An *edge cover* in a hypergraph H is a set of edges whose union is $V(H)$.

Theorem 19 (Pippenger). *For every integer $k \ge 2$ and reals $r \ge 1$ and $a > 0$, there are $\gamma = \gamma(k,r,a) > 0$ and $d_0 = d_0(k,r,a)$ such that for every n and $D \ge d_0$ the following holds: Every k-uniform hypergraph $H = (V,E)$ on a set V of n vertices in which all vertices have positive degrees and which satisfies the following conditions:*

(1) for all vertices $x \in V$ but at most γn of them, $d_H(x) = (1 \pm \gamma)D$;

(2) for all $x \in V$, $d_H(x) < rD$;

(3) for any two distinct $x, y \in V$, $d_H(\{x,y\}) < \gamma D$;

contains an edge cover of at most $(1+a)(n/k)$ edges.

Proof of Theorem 5. By Corollary 13, we may assume that $2k(t+1) < n < 2k^3(t+1)$. Let $0 < \varepsilon \ll 1/k < 1$ be sufficiently small and let $t > t_0(k) = n/(2k)^3$ be sufficiently large. By Observation 6, it suffices to show $\mathcal{F}^{t+1}(k,n)$ has a matching of size $t+1$. Applying Lemma 16 to $\mathcal{F}^{t+1}(k,n)$ (by choosing $\zeta < 1/3$ for instance), we may assume that $\mathcal{F}^{t+1}(k,n)$ is not ε-close to $\mathcal{F}_{t+1}(k,n;t)$. That is, $\mathcal{H}^{t+1}(k,n)$ is not $\varepsilon/6$-close to $\mathcal{H}_{t+1}(k,n;t+1)$ by Observation 8.

Now we apply Lemma 17 to $\mathcal{H}^{t+1}(k,n)$ with $t \le n/(2k)$ and sufficiently small ζ. Thus there exists some constant $0 < c \ll \varepsilon$ such that $n - 2k^2cn \ge (2k - 2k^4c)(t+1)$, and $\mathcal{H}^{t+1}(k,n)$ contains

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an absorbing matching M_1 with $m_1 := |M_1| \le 2kcn$ and for any balanced subset S of vertices with $|S| \le (k+1)c^{1.5}n$, $\mathcal{H}^{t+1}(k,n)[V(M_1) \cup S]$ has a perfect matching. Let $H := \mathcal{H}^{t+1}(k,n) - V(M_1)$ and $n_1 := n - km_1$.

Next, we see that H is not $(\varepsilon/12)$-close to $\mathcal{H}_{t+1}(k, n-km_1; t+1)$. For, suppose otherwise. Then

$$\begin{aligned}
&|E(\mathcal{H}_{t+1}(k,n;t+1)) \setminus E(\mathcal{H}^{t+1}(k,n))| \\
&\le |E(\mathcal{H}_{t+1}(k,n-km_1;t+1)) \setminus E(H)| + |e \in E(\mathcal{H}_{t+1}(k,n;t+1)) : e \cap V(M_1) \neq \emptyset| \\
&\le (\varepsilon/12)(n+n/k)^{k+1} + 2k(k+1)cn \cdot n^k \\
&\le \frac{1}{6}\varepsilon(n+n/k)^{k+1}.
\end{aligned}$$

This is a contradiction as $\mathcal{H}^{t+1}(k,n)$ is not ε-close to $\mathcal{H}_{t+1}(k,n;t+1)$.

Let $Q, [n]$ denote the partition classes of $\mathcal{H}^{t+1}(k,n)$, and let X consist of all vertices of Q contained in $\mathcal{F}^{t+1}(k,n)$. Let $A_1 = X \cap V(H)$ and $A_2 = (Q \setminus X) \cap V(H)$. Then we have $d_H(x) > \binom{n}{k} - \binom{n-t+1}{k} - \rho n^k$ for all $x \in A_1$ and $d_H(x) = \binom{n}{k}$ for all $x \in A_2$. Since $n_1 = n - km_1 \ge n - 2k^2cn \ge (2k - 2k^4c)(t+1)$, by Lemma 18 H has a spanning subgraph H_1 such that

(1) for all but at most $n_1^{0.99}$ vertices $x \in V(H_1)$, $d_{H_1}(x) = (1 \pm n_1^{-0.01})n_1^{0.2}$;
(2) for all $x \in V(H_1)$, $d_{H_1}(x) < 2n_1^{0.2}$;
(3) for any two distinct $x, y \in V(H_1)$, $d_{H_1}(\{x,y\}) < n_1^{0.19}$.

Hence by applying Lemma 19 to H_1 by choosing a with $0 < a \ll c^{1.5}$, H_1 contains an edge cover of size at most $(1+a)((n_1/k + n_1)/(k+1))$. Thus, at most $a(n_1/k + n_1)$ vertices are each covered by more than one edge in the cover. Hence, after removing at most $a(n_1/k + n_1)$ edges from the edge cover, we obtain a matching M_2 covering all but at most $(k+1)a(n_1/k + n_1) \le 3kan_1 \le 3kan$ vertices.

Now we may choose a subset S of $V(H) \setminus V(M_2)$ such that $|V(H) \setminus (V(M_2) \cup S)| < k$ and if $S \neq \emptyset$ then $|S \cap [n]| = k|S \cap Q|$. Since $|S| \le 3kan \le (k+1)c^{1.5}n$, $\mathcal{H}^{t+1}(k,n)[V(M_1) \cup S]$ has a perfect matching, say M_3. Thus, $M_2 \cup M_3$ is matching of $\mathcal{H}^{t+1}(k,n)$ covering all but at most k vertices, and, hence, has size $\lfloor n/k \rfloor$. Therefore, by Lemma 7, $\mathcal{F}^{t+1}(k,n)$ has a matching of size $t+1$. □

6. Conclusion

We proved stability results for rainbow matchings for families of non-uniform hypergraphs. In the uniform case, our result generalizes the result of Frankl and Kupavskii [12] on a conjecture which we now describe.

Let n, k, s be positive integers. For $i \in [k]$, define

$$\mathcal{A}_i^{(k)}(n,s) := \left\{ A \in \binom{[n]}{k} : |A \cap [(s+1)i - 1]| \ge i \right\}.$$

Note that $|\mathcal{A}_i^{(k)}(n,s)| \le \binom{n}{k} - \sum_{j=0}^{i-1} \binom{(s+1)i-1}{j}$. It is worth pointing out that $\mathcal{A}_k^{(k)}(n,t-1)$ and $\mathcal{A}_1^{(k)}(n,t-1)$, respectively, correspond to the extremal constructions (a) an (b) of the Erdős Matching Conjecture. Let $\mathcal{H}^{(k)}(n,s)$ denote the family consisting of the set $\{s+1,\dots,s+k\}$ and all k-sets that either intersect $[s-1]$, or contain s and intersect $[s+k] \setminus [s]$, i.e.

$$\left\{ e \in \binom{[n]}{k} : \min(e) < s \text{ or } \min(e) = s \text{ and } |e \cap [s+k]| \ge 2 \right\} \cup \{[s+k] \setminus [s]\}$$

Note that $\mathcal{H}^{(k)}(n,s)$ is a slight variation of $\mathcal{A}_1^{(k)}(n,s)$, $\nu(\mathcal{H}^{(k)}(n,s)) = s$, and $|\mathcal{H}^{(k)}(n,s)| = \binom{n}{k} - \binom{n-s}{k} - \binom{n-s-k}{k-1} + 1$.

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Frankl and Kupavskii [12] proposed the following conjecture.

Conjecture 20. *Let n, k, t be positive integers such that $n > kt$. Let F be a k-graph of order n. If $\nu(F) \le t$, then either $\tau(F) \le t$ or*

$$e(F) \le \max\{e(\mathcal{H}^{(k)}(n,t)), e(\mathcal{A}_2^{(k)}(n,t)), \ldots, e(\mathcal{A}_k^{(k)}(n,t))\}.$$

The case $t = 1$ of Conjecture 20 is the Hilton-Milner theorem, which is proved in [14]. Frankl and Kupavskii [12] confirmed the conjecture for $n \ge (2 + o(1))kt$. Theorem 1 extends this result to rainbow matchings.

We think the following could be true:

Conjecture 21. *Let n, k, t be three positive integers such that $n > kt$. Let $\mathcal{F} = \{F_1, \ldots, F_{t+1}\}$ be a family of k-graphs, such that, for $1 \le i \le t+1$,*

$$|F_i| > \max\{e(\mathcal{H}^{(k)}(n,t)), e(\mathcal{A}_2^{(k)}(n,t)), \ldots, e(\mathcal{A}_k^{(k)}(n,t))\}.$$

Then $\mathcal{F}$ admits a rainbow matching of size $t+1$ or there exists $W \in \binom{[n]}{t}$ such that for all $i \in [t+1]$, W is a vertex cover of F_i.

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