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# PROJECTION ALGORITHMS FOR FIXED POINT PROBLEMS

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**Abstract**

We study projection algorithms for finding fixed points of nonexpansive mappings in a Hilbert space. In previous work, the well-definedness of the iterates depended on the nonemptiness of the fixed point set. We first show that the existence of the iterates generated by a general projection algorithm is independent of the existence of fixed points. Next, we introduce a multi-step inertial projection algorithm and establish results on existence, strong convergence and iteration complexity. Our results generalise and improve projection algorithms discussed in the literature.

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## 1. Introduction

Let $\mathcal{H}$ be a real Hilbert space. A mapping $T\colon \mathcal{H} \to \mathcal{H}$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in \mathcal{H}$. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of $T$ (that is, $\mathrm{Fix}(T) = \{u \in \mathcal{H} : Tu = u\}$). The class of nonexpansive mappings includes all contractions but the existence of a fixed point of a nonexpansive mapping is not guaranteed unless some additional conditions are assumed.

We are interested in the following projection algorithm introduced by Nakajo and Takahashi [15]: $x_0 = x \in \mathcal{H}$ and

$$\begin{cases} y_k = (1 - \alpha_k)x_k + \alpha_k T x_k, \\ C_k = \{z \in \mathcal{H} : \|y_k - z\| \le \|x_k - z\|\}, \\ Q_k = \{z \in \mathcal{H} : \langle z - x_k, x - x_k\rangle \le 0\}, \\ x_{k+1} = P_{C_k \cap Q_k} x, \end{cases} \tag{1.1}$$

for $k = 0, 1, 2, \ldots$, where $\{\alpha_k\} \subset [\alpha, 1]$ for some $\alpha \in (0, 1]$ and $P_K$ denotes the metric projection from $\mathcal{H}$ onto a closed convex subset $K$ of $\mathcal{H}$. The idea of (1.1) is based on the strongly convergent inexact proximal point algorithm of Solodov and Svaiter [18, Algorithm 1]. The sequence $\{x_k\}$ generated by (1.1) converges strongly to $P_{\mathrm{Fix}(T)}x$

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[15, Theorem 3.4]. In the original result, the existence of the iterates $\{x_k\}$ was established under the assumption that $\operatorname{Fix}(T) \neq \emptyset$. In contrast, in [13], we showed that even in the absence of fixed points, it is possible to ensure that the set $C_k \cap Q_k$ is always nonempty; that is, the sequence $\{x_k\}$ is always well-defined. Furthermore, we showed that the boundedness of $\{x_k\}$ guarantees the existence of a fixed point. The projection algorithm (1.1) has subsequently been developed in various directions (see, for example, [11, 12, 14, 20]).

The inertial extrapolation technique was made immensely popular by Nesterov's accelerated gradient algorithm [16]. Subsequently, many methods have been proposed to improve the convergence speed of these algorithms (see, for example, the proximal point algorithm [1], Krasnosel'skiĭ–Mann algorithm [6, 10], forward-backward algorithm [4] and Douglas–Rachford algorithm [5]). Especially, Dong *et al.* [7] proposed the following algorithm, which incorporates a one-step inertial term into (1.1): $x, x_{-1}, x_0 \in \mathcal{H}$ and

$$\begin{cases} w_k = x_k + \lambda_k(x_k - x_{k-1}), \\ y_k = (1-\beta_k)w_k + \beta_k T w_k, \\ C_k = \{z \in \mathcal{H} : \|y_k - z\| \leq \|w_k - z\|\}, \\ Q_k = \{z \in \mathcal{H} : \langle z - x_k, x - x_k\rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k} x, \end{cases} \tag{1.2}$$

for $k = 0, 1, 2, \ldots$, where $\{\lambda_k\} \subset (a, b)$ for some $a \in (-\infty, 0]$ and $b \in [0, \infty)$, and $\{\beta_k\} \subset [\beta, 1]$ for some $\beta \in (0, 1]$. The main idea of (1.2) is to introduce an inertial effect at the current iterate $x_k$ by extrapolating $x_k$ in the direction $x_k - x_{k-1}$.

Although the algorithm incorporating the inertial term is convergent, we emphasise that algorithms with a one-step inertial term do not necessarily achieve acceleration. For instance, it has been reported in [17, Section 4] that the Douglas–Rachford algorithm with a one-step inertial term may converge more slowly than the original algorithm. Multi-step inertial techniques could help to achieve the expected improvements in the speed of convergence of algorithms.

In this paper, we analyse extensions of these projection algorithms.

- First, we introduce a projection algorithm that employs an arbitrary sequence and encompasses many existing methods. Using the proposed algorithm, we show that even in the absence of fixed points, the intersection of the half-spaces remains nonempty; that is, the existence of iterates is guaranteed, independent of the existence of a fixed point.
- Second, as an extension of algorithms such as (1.1) and (1.2), we propose a multi-step inertial projection algorithm. The idea of the multi-step inertial term is inspired by the approach applied to the forward-backward splitting algorithm in [9, Ch. 6]. We show that the boundedness of the iterates is equivalent to the existence of a fixed point of the nonexpansive mappings. Furthermore, we establish the strong convergence of the proposed algorithm under less restrictive assumptions on the inertial parameters. Finally, we present results on the iteration complexity.

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The paper is organised as follows. Section 2 presents the mathematical preliminaries. Section 3 establishes the well-definedness of the algorithm in a general setting. Section 4 provides results on the existence, strong convergence and iteration complexity of the inertial projection algorithm.

## 2. Preliminaries

This section reviews basic definitions, facts and notation that will be used throughout the paper.

Let $\mathcal{H}$ denote a real Hilbert space endowed with the *inner product* $\langle\cdot,\cdot\rangle$ and the *norm* $\|\cdot\|$. $\mathbb{R}$ and $\mathbb{N}$ denote the *set of real numbers* and the *set of positive integers*, respectively. For any $\{x_k\} \subset \mathcal{H}$ and $x \in \mathcal{H}$, we write $x_k \to x$ and $x_k \rightharpoonup x$ to denote the *strong and weak convergence* of $\{x_k\}$ to $x$, respectively. For $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{R}$,

$$\|(1-\alpha)x + \alpha y\|^2 = (1-\alpha)\|x\|^2 + \alpha\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.1}$$

(see, for example, [3, Lemma 2.12 and Corollary 2.14]).

Let $C$ be a closed convex subset of $\mathcal{H}$. Then for any $x \in \mathcal{H}$, there exists a unique point $x^* \in C$ such that

$$\|x^* - x\| = \min_{z\in C} \|z - x\|.$$

The mapping $P_C\colon \mathcal{H} \to C$ defined by $P_C x = x^*$ is called the *metric projection* from $\mathcal{H}$ onto $C$. We have $u = P_C x$ if and only if

$$\langle y - u, x - u\rangle \le 0 \quad (\text{for all } y \in C) \tag{2.2}$$

(see, for example, [3, 19]).

We use the following result of Martinez-Yanes and Xu [11] to prove the convergence result.

LEMMA 2.1 [11, Lemma 1.5]. *Let $D$ be a closed convex subset of $\mathcal{H}$ and $u \in \mathcal{H}$ and $\{x_k\}$ be a sequence in $\mathcal{H}$. Suppose that*

- *every weak cluster point of $\{x_k\}$ belongs to $D$;*
- *$\|u - x_k\| \le \|u - P_D u\|$ for all $k \ge 1$.*

*Then $x_k \to P_D u$.*

We use the following result of Khatibzadeh [8] to prove the convergence rate result.

LEMMA 2.2 [8, Lemma 5.1]. *Suppose that $\{a_k\}$ and $\{b_k\}$ are two positive real sequences such that*

- *$\{a_k\}$ is nonincreasing and convergent to zero;*
- *$\sum_{k=1}^{\infty} a_k b_k < \infty$.*

*Then $a_k = o(1/\sum_{n=1}^{k} b_n)$ (that is, $\lim_{k\to\infty}(\sum_{n=1}^{k} b_n)a_k = 0$).*

Finally, we present a technical lemma.

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LEMMA 2.3. *Let $a, b \in \mathcal{H}$. For any $z \in \mathcal{H}$, the following statements are equivalent:*

(i) $\|a - z\| \le \|b - z\|$;
(ii) $\langle \frac{1}{2}(b + a) - z, \frac{1}{2}(b + a) - a \rangle \le 0$.

PROOF.

$$\begin{aligned} \|a - z\|^2 \le \|b - z\|^2 &\Leftrightarrow \langle a - z, a - z\rangle \le \langle b - z, b - z\rangle \\ &\Leftrightarrow \langle a - b + b - z, a - z\rangle \le \langle b - a + a - z, b - z\rangle \\ &\Leftrightarrow \langle a + b - 2z, a - b\rangle \le 0 \\ &\Leftrightarrow \langle \tfrac{1}{2}(a + b) - z, \tfrac{1}{2}(a + b) - b\rangle \le 0. \end{aligned}$$

□

## 3. A projection algorithm

In this section, we first investigate the following algorithm, which generalises (1.1).

ALGORITHM 3.1. Let $x_0 \in \mathcal{H}$ and let $\{w_k\}$ be an arbitrary sequence in $\mathcal{H}$. Define

$$\begin{cases} y_k = (1 - \beta_k)w_k + \beta_k T w_k, \\ C_k = \{z \in \mathcal{H} : \|y_k - z\| \le \|w_k - z\|\}, \\ Q_k = \{z \in \mathcal{H} : \langle z - x_k, x_0 - x_k\rangle \le 0\}, \\ x_{k+1} = P_{C_k \cap Q_k} x_0, \end{cases}$$

$k = 0, 1, 2, \ldots$, where $\{\beta_k\} \subseteq [0, 1]$.

REMARK 3.2. Algorithm 3.1 includes existing algorithms. If we set $w_k \equiv x_k$, then Algorithm 3.1 becomes (1.1). Moreover, if we set $w_k \equiv x_k + \lambda_k(x_k - x_{k-1})$, Algorithm 3.1 becomes (1.2).

Simple calculations show that $C_k$ and $Q_k$ can be equivalently written as

$$\begin{aligned} C_k &= \{z \in \mathcal{H} : \langle z, y_k - w_k\rangle \le \tfrac{1}{2}\langle y_k + w_k, y_k - w_k\rangle\}, \\ Q_k &= \{z \in \mathcal{H} : \langle z, x_0 - x_k\rangle \le \langle x_k, x_0 - x_k\rangle\}. \end{aligned}$$

From Algorithm 3.1, the next iterate $x_{k+1}$ is defined by metric projection of $x_0$ onto the intersection $C_k \cap Q_k$ of the half-spaces $C_k$ and $Q_k$. The closed-form representation of the projection onto the intersection of two half-spaces is discussed in [18, Section 3] and [3, Ch. 29].

**3.1. Well-definedness of Algorithm 3.1.** In this subsection, we will prove the well-definedness of Algorithm 3.1. The following result guarantees the existence of the sequence generated by Algorithm 3.1.

LEMMA 3.3. *Consider the sequences generated by Algorithm 3.1:*

(i) *for all $k \in \mathbb{N} \cup \{0\}$, we have $\bigcap_{i=0}^{k} C_i \subset C_k \cap Q_k$ and $\bigcap_{i=0}^{k} C_i \neq \emptyset$;*
(ii) *$\{x_k\}$ is well-defined;*
(iii) *for all $k \in \mathbb{N} \cup \{0\}$, we have $\mathrm{Fix}(T) \subset C_k \cap Q_k$;*
(iv) *if $\mathrm{Fix}(T) \neq \emptyset$, then $\{x_k\}$ is bounded.*

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PROOF. ( i) For $k = 0$, the inclusion $C_0 \subset C_0 \cap Q_0$ follows from $Q_0 = \mathcal{H}$. Note that $y_0 \in C_0$ and hence $C_0 \neq \emptyset$. Let $m \in \mathbb{N}$ and assume that for all $l \in \{0, 1, \ldots, m-1\}$, $\bigcap_{i=0}^{l} C_i \subset C_l \cap Q_l$ and $\bigcap_{i=0}^{l} C_i \neq \emptyset$ are true. In this case, we can define $x_m$ by $x_m = P_{C_{m-1} \cap Q_{m-1}}(x_0)$.

We first show $\bigcap_{i=0}^{k} C_i \subset C_k \cap Q_k$. It follows from (2.2) that

$$\langle y - x_m, x_0 - x_m \rangle \leq 0 \quad (\text{for all } y \in C_{m-1} \cap Q_{m-1})$$

and hence $C_{m-1} \cap Q_{m-1} \subset Q_m$. This implies that

$$\bigcap_{i=0}^{m} C_i = C_m \cap \bigcap_{i=0}^{m-1} C_i \subset C_m \cap (C_{m-1} \cap Q_{m-1}) \subset C_m \cap Q_m.$$

Therefore, by induction, $\bigcap_{i=0}^{k} C_i \subset C_k \cap Q_k$.

We next show that $\bigcap_{i=0}^{k} C_i \neq \emptyset$. Set $D = \{u \in \mathcal{H} : \|u\| \leq \max_{0 \leq i \leq m}\{\|Tw_i\|, \|y_i\|\}\}$, let $l \in \{0, 1, \ldots, m\}$ and let $P_D$ be the metric projection onto $D$. Since $D$ is bounded, closed and convex, and the composition $P_D \circ T$ is nonexpansive on $D$, it follows from the Browder–Göhde–Kirk fixed point theorem (see, for example, [19, Corollary 3.1.7], [3, Theorem 4.20]) that there exists $z \in D$ such that $P_D \circ T(z) = z$. From (2.2),

$$\langle u - z, T(z) - z \rangle \leq 0 \quad (\text{for all } u \in D),$$

and hence

$$\langle y_l - z, z - ((1 - \beta_l)z + \beta_l Tz) \rangle = \langle y_l - z, \beta_l(z - Tz) \rangle \geq 0.$$

It follows that

$$\begin{aligned}
\|y_l - z\|^2 &\leq \langle y_l - z, y_l - ((1 - \beta_l)z + \beta_l Tz) \rangle \\
&= \langle y_l - z, (1 - \beta_l)(w_l - z) + \beta_l(Tw_l - Tz) \rangle \\
&\leq (1 - \beta_l)\|y_l - z\| \, \|w_l - z\| + \beta_l \|y_l - z\| \, \|Tw_l - Tz\| \\
&\leq \|y_l - z\| \, \|w_l - z\| \\
&\leq \|(1 - \beta_l)(w_l - z) + \beta_l(Tw_l - z)\| \, \|w_l - z\| \\
&\leq ((1 - \beta_l)\|w_l - z\| + \beta_l \|P_D \circ Tw_l - P_D \circ Tz\|)\|w_l - z\| \\
&\leq \|w_l - z\|^2.
\end{aligned}$$

Since $l$ was chosen arbitrarily in $\{0, 1, \ldots, m\}$, this shows that $z \in \bigcap_{i=0}^{m} C_i$. Therefore, $\bigcap_{i=0}^{k} C_i \neq \emptyset$ for $k = 0, 1, 2, \ldots$.

( ii) The well-definedness of $\{x_k\}$ can be obtained straightforwardly by ( i).

( iii) Let $k \in \mathbb{N} \cup \{0\}$ and let $p \in \mathrm{Fix}(T)$. For all $i \in \{0, \ldots, k\}$,

$$\begin{aligned}
\|y_i - p\| &= \|(1 - \beta_i)(w_i - p) + \beta_i(Tw_i - p)\| \\
&\leq (1 - \beta_i)\|w_i - p\| + \beta_i \|Tw_i - p\| \\
&\leq \|w_i - p\|,
\end{aligned}$$

and hence $p \in \bigcap_{i=0}^{k} C_i$. It follows that $\mathrm{Fix}(T) \subset \bigcap_{i=0}^{k} C_i \subset C_k \cap Q_k$.

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( iv) From the definition of $Q_k$, we have $x_k = P_{Q_k}(x_0)$. Fix $p \in \mathrm{Fix}(T)\,(\subset C_k \cap Q_k)$. Then

$$\|x_0 - x_k\| \leq \|x_0 - p\|$$

for all $k \in \mathbb{N} \cup \{0\}$. This implies that $\{x_k\}$ is bounded. □

REMARK 3.4. Lemma 3.3 is a generalisation of [13, Theorem 2.2], and its proof, which we presented here, follows the ideas from [13, Theorem 2.2].

In previous work (for example, [7, 12, 14, 15, 20]), the well-definedness of the projection-type algorithms was proved with the assumption that $\mathrm{Fix}(T) \neq \emptyset$.

## 4. Existence and convergence results

In this section, we propose a multi-step inertial projection algorithm and establish boundedness, existence and convergence results.

ALGORITHM 4.1. Let $l \geq 1$ and set $\mathrm{S} = \{0, 1, \ldots, l-1\}$. Assume that $x_{-i} = x_0 \in \mathcal{H}$ for each $i \in \mathrm{S}$. We consider the following multi-step inertial projection algorithm:

$$\begin{cases} w_k = x_k + \sum_{n\in\mathcal{S}} \lambda_{k,n}(x_{k-n} - x_{k-n-1}) \\ y_k = (1-\beta_k)w_k + \beta_k T w_k, \\ C_k = \{z \in \mathcal{H} : \|y_k - z\| \leq \|w_k - z\|\}, \\ Q_k = \{z \in \mathcal{H} : \langle z - x_k, x_0 - x_k\rangle \leq 0\}, \\ x_{k+1} = P_{C_k\cap Q_k} x_0, \end{cases}$$

for $k = 0, 1, 2, \ldots$, where $(\lambda_{k,0}, \lambda_{k,1}, \ldots, \lambda_{k,l-1}) \in \mathbb{R}^l$ and $\{\beta_k\} \subseteq [0, 1]$.

To establish the results, we need the following assumption.

ASSUMPTION 4.2. Assume that $\{\beta_k\}$ and $\{\lambda_{k,n}\}$ satisfy the following conditions:

(A1) $\{\beta_k\} \subset [\beta, 1]$ for some $\beta \in (0, 1]$;
(A2) $(\lambda_{k,0}, \lambda_{k,1}, \ldots, \lambda_{k,l-1}) \in (-a, a)^l$ for all $k$ for some $a \in (0, \infty)$.

**4.1. Existence result.** In this subsection, we establish a fixed point theorem for nonexpansive mappings in Hilbert spaces.

THEOREM 4.3. *Let $\{x_k\}$ be a sequence generated by Algorithm 4.1 and assume that Assumption 4.2 holds for $\{(\lambda_{k,0}, \lambda_{k,1}, \ldots, \lambda_{k,l-1})\}$ and $\{\beta_k\}$. Then the following statements are equivalent:*

- *$\{x_k\}$ is bounded;*
- *$\mathrm{Fix}(T) \neq \emptyset$.*

PROOF. From Lemma 3.3( iv), it is sufficient to consider the case when $\{x_k\}$ is bounded. Since $x_{k+1} = P_{C_k\cap Q_k} x_0 \in Q_k$,

$$\langle x_{k+1} - x_k, x_0 - x_k\rangle \leq 0.$$

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This implies that

$$\begin{aligned}\|x_{k+1}-x_0\|^2 &= \|x_{k+1}-x_k\|^2 - 2\langle x_{k+1}-x_k, x_0-x_k\rangle + \|x_0-x_k\|^2\\ &\geq \|x_{k+1}-x_k\|^2 + \|x_0-x_k\|^2\\ &\geq \|x_0-x_k\|^2.\end{aligned}$$

This together with the boundedness assumption implies that $\{\|x_k-x_0\|^2\}$ is convergent. Moreover,

$$\|x_{k+1}-x_k\|^2 \leq \|x_0-x_{k+1}\|^2 - \|x_0-x_k\|^2, \tag{4.1}$$

and hence $\lim_{k\to\infty}\|x_{k+1}-x_k\| = 0$.

On the other hand,

$$\|w_k-x_k\| = \left\|\sum_{n\in\mathcal{S}}\lambda_{k,n}(x_{k-n}-x_{k-n-1})\right\| \leq \sum_{n\in\mathcal{S}}|a|\,\|x_{k-n}-x_{k-n-1}\|$$

and it follows that $\lim_{k\to\infty}\|w_k-x_k\| = 0$. Moreover, from $T(w_k)-w_k = (1/\beta_k)(y_k-w_k)$,

$$\begin{aligned}\|T(w_k)-w_k\| = \frac{1}{\beta_k}\|y_k-w_k\| &\leq \frac{1}{\beta}(\|y_k-x_{k+1}\| + \|x_{k+1}-w_k\|)\\ &\leq \frac{2}{\beta}\|x_{k+1}-w_k\| \quad (\text{since } x_{k+1}\in C_k)\\ &\leq \frac{2}{\beta}(\|x_{k+1}-x_k\| + \|x_k-w_k\|).\end{aligned}$$

Consequently, $\lim_{k\to\infty}\|T(w_k)-w_k\| = 0$. Now, we see that

$$\begin{aligned}\|T(x_k)-x_k\| &\leq \|T(x_k)-T(w_k)\| + \|T(w_k)-w_k\| + \|w_k-x_k\|\\ &\leq 2\|x_k-w_k\| + \|T(w_k)-w_k\|,\end{aligned}$$

and hence $\lim_{k\to\infty}\|T(x_k)-x_k\| = 0$. This together with the boundedness of $\{x_k\}$ and Browder's demiclosedness principle [3, Theorem 4.27] shows that $\mathrm{Fix}(T)\neq\emptyset$. □

**4.2. Strong convergence and iteration complexity results.** In this subsection, we first establish the convergence of iterates generated by Algorithm 4.1.

THEOREM 4.4. *Let $\{x_k\}$ be a sequence generated by Algorithm 4.1. Assume that Assumption 4.2 holds for $\{(\lambda_{k,0},\lambda_{k,1},\ldots,\lambda_{k,l-1})\}$ and $\{\beta_k\}$.*

(1) *If $\mathrm{Fix}(T)\neq\emptyset$, then $x_k\to P_{\mathrm{Fix}(T)}x_0$.*
(2) *If $\{x_k\}$ is unbounded, then $\mathrm{Fix}(T)=\emptyset$.*

PROOF.

(1) From the proof of Theorem 4.3, every weak cluster point of $\{x_k\}$ belongs to $\mathrm{Fix}(T)$. Thus, the result follows from Lemma 2.1.
(2) The result follows from Theorem 4.3. □

Next, we establish iteration complexity results for Algorithm 4.1.

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THEOREM 4.5. *Let $\{x_k\}$ be a sequence generated by Algorithm 4.1 and assume that* $\mathrm{Fix}(T) \neq \emptyset$. *Then the following statements hold:*

(1) $\min_{0\le i\le k} \|x_{i+1} - x_i\| = o(1/\sqrt{k})$;
(2) *if* $\mathcal{S} = \{0\}$ *and Assumption 4.2 holds for* $\{\lambda_{k,0}\}$ *and* $\{\beta_k\}$, *then*

$$\min_{0\le i\le k} \|Tx_i - x_i\| = o(1/\sqrt{k}).$$

PROOF. (1) Using (4.1), summing from $j = 0$ to $l$ gives

$$\sum_{j=0}^{l} \|x_{j+1} - x_j\|^2 \le \|x_{l+1} - x_0\|^2 = \|P_{C_l\cap Q_l}x_0 - x_0\|^2 \le \|P_{\mathrm{Fix}(T)}x_0 - x_0\|^2, \tag{4.2}$$

where the last inequality follows from $\mathrm{Fix}(T) \subset C_l \cap Q_l$. Let $a_k := \min_{0\le i\le k} \|x_{i+1} - x_i\|^2$ and $b_k := 1$ for each $k$. Then $\{a_k\}$ is nonincreasing and converges to 0 from (4.1). Furthermore,

$$\|P_{\mathrm{Fix}(T)}x_0 - x_0\|^2 \ge \sum_{j=0}^{l} \|x_{j+1} - x_j\|^2 \ge \sum_{j=0}^{l} a_j b_j.$$

By taking $l \to \infty$, we see that $\sum_{j=0}^{\infty} a_j b_j < \infty$. It follows from Lemma 2.2 that $\min_{0\le i\le k} \|x_{i+1} - x_i\|^2 = o(1/k)$ and hence $\min_{0\le i\le k} \|x_{i+1} - x_i\| = o(1/\sqrt{k})$.

(2) In this case, Algorithm 4.1 becomes (1.2). From the definition of $w_k$,

$$w_k - x_{k+1} = x_k + \lambda_{k,0}(x_k - x_{k-1}) - x_{k+1} = (1 + \lambda_{k,0})(x_k - x_{k+1}) + \lambda_{k,0}(x_{k+1} - x_{k-1}),$$

and hence

$$x_k - x_{k+1} = \frac{1}{1 + \lambda_{k,0}}(w_k - x_{k+1}) + \frac{\lambda_{k,0}}{1 + \lambda_{k,0}}(x_{k-1} - x_{k+1}).$$

It follows from (2.1) that

$$\|x_k - x_{k+1}\|^2 = \frac{1}{1 + \lambda_{k,0}}\|w_k - x_{k+1}\|^2 + \frac{\lambda_{k,0}}{1 + \lambda_{k,0}}\|x_{k-1} - x_{k+1}\|^2 - \frac{\lambda_{k,0}}{(1 + \lambda_{k,0})^2}\|w_k - x_{k-1}\|^2$$

and hence

$$\begin{aligned}\|w_k - x_{k+1}\|^2 &= (1 + \lambda_{k,0})\|x_k - x_{k+1}\|^2 - \lambda_{k,0}\|x_{k-1} - x_{k+1}\|^2 + \frac{\lambda_{k,0}}{1 + \lambda_{k,0}}\|w_k - x_{k-1}\|^2 \\ &\le (1 + \lambda_{k,0})\|x_k - x_{k+1}\|^2 + \lambda_{k,0}(1 + \lambda_{k,0})\|x_k - x_{k-1}\|^2 \\ &\le (1 + a)(\|x_k - x_{k+1}\|^2 + a\|x_k - x_{k-1}\|^2).\end{aligned} \tag{4.3}$$

On the other hand, from the definition of $C_k$ and Lemma 2.3,

$$\|y_k - z\| \le \|w_k - z\| \quad \Leftrightarrow \quad \langle \tfrac{1}{2}(w_k + y_k) - z, \tfrac{1}{2}(w_k + y_k) - y_k\rangle \le 0$$

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for all $z \in C_k$. This implies that $\frac{1}{2}(w_k + y_k) = P_{C_k} y_k$. Since $x_{k+1} \in C_k$ and $w_k - y_k = \beta_k(w_k - Tw_k)$,

$$\begin{aligned}\frac{\beta}{2}\|w_k - Tw_k\| \le \frac{1}{2}\|w_k - y_k\| &= \left\|\frac{1}{2}(w_k + y_k) - y_k\right\| \\ &= \|P_{C_k} y_k - y_k\| \\ &\le \|x_{k+1} - y_k\| \le \|x_{k+1} - w_k\|.\end{aligned}$$

This together with (4.2) and (4.3) yields

$$\frac{\beta}{2}\sum_{j=0}^{l}\|w_k - Tw_k\|^2 \le (1+a)^2\|P_{\mathrm{Fix}(T)}x_0 - x_0\|^2.$$

The result now follows by a similar argument to that in (1). □

REMARK 4.6. Theorem 4.4 includes the corresponding results in [2, Corollary 6.6(ii)], [15, Theorem 3.4] and [7, Theorem 4.1] as special cases.

To the best of our knowledge, Theorem 4.5 is the first iteration complexity result for the projection algorithm for nonexpansive mappings.

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